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AN ANALYSIS OF THE FREE VIBRATION OF A HERMETIC CAPSULE

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1. INTRODUCTION

A recent study by Shang [1] outlines an analytical vibration analysis for a hermetic capsule that is described as a circular cylinder closed with hemispherical caps at both ends. Shang's [1] analysis is based upon Naghdi–Reissner shell theory and includes shear deformation. The frequency of vibration results recorded by Shang [1] compare well with previous results given by Tavakoli and Singh [2] and Özakca and Hinton [3]. Tavakoli and Singh [2] studied the hermetic capsule using Love's shell equations and recorded symmetric and antisymmetric frequencies for free vibration. Later, Özakca and Hinton [3] verified the results for the same hermetic capsule using a shell finite element based upon Mindlin–Reissner theory that included shear deformation and rotary inertia effects. Özakca and Hinton [3] not only tabulated the symmetric frequencies but independently verified the vibration analysis. The capsule that was studied in references [1–3] had a radius to wall-thickness ratio of approximately 50 and would qualify as a thin shell. It would seem reasonable that theories that include shear deformation would agree with the more classic thin shell equations.

The present study extends the analysis to include a somewhat thicker hermetic capsule. A three-dimensional axisymmetric finite element based upon the equations of elasticity in cylindrical co-ordinates is used to model the capsule. The results discussed in the previous paragraph were used to verify the formulation and subsequent analysis. Additional torsional frequencies were found that were not previously reported.

Non-dimensional frequencies are tabulated for the capsule, while the thickness and length of the cylindrical section of the capsule are varied. Representative mode shapes are shown for symmetric, antisymmetric and torsional frequencies.

2. FINITE ELEMENT MODELS

A nine-node Lagrangian isoparametric finite element was used to model the cross-section of the capsule. The finite element was derived in axisymmetric two-dimensional cylindrical co-ordinates following the discussion given by Buchanan [4] for axisymmetric elasticity in r,θ,z co-ordinates. The three-dimensional strain–displacement equations and the formulation that extends the two-dimensional finite element analysis to represent a three-dimensional analysis has been outlined by Buchanan and Chua [5]. The governing three-dimensional elasticity equations are satisfied by assuming a periodic solution that

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separates θ and time dependence t from r and z. Following reference [5] assume that displacements can be represented as

$$u(r,\theta,z,t) = U(r,z)\cos n\theta \cos \omega t, \qquad v(r,\theta,z,t) = V(r,z)\sin n\theta \cos \omega t, \tag{1,2}$$

$$w(r,\theta,z,t) = W(r,z)\cos n\theta \cos \omega t, \tag{3}$$

where u,v,w and U,V,W are displacements in the r,θ,z directions respectively. The circular frequency is given by ω and n is the circumferential wave number.

The isoparametric element is used to model the spherical end caps and cylindrical mid-section as a continuous axisymmetric body. Results for complete hollow thick shells of revolution were given by McGee and Spry [6] and were used to verify the axisymmetric isoparametric formulation for a spherical shell. Those comparisons do not lead to new results and are not tabulated here.

An analysis for the cylindrical section was compared to previous studies for hollow cylinders [7] and was found to give satisfactory results.

3. FREE VIBRATION RESULTS

The model analyzed in references [1–3] had the following isotropic material properties, Young's modulus $E = 207(10^9)$ Pa, the Poisson ratio $v = 0{\cdot}3$, and density $\rho = 7800\,\mathrm{kg/m^3}$. The corresponding elasticity constants are computed in Pa as

$$C_{11} = \frac{E(1-v)}{(1+v)(1-2v)} = 287{\cdot}654\left(10^9\right), \qquad C_{12} = \frac{Ev}{(1+v)(1-2v)} = 119{\cdot}423\left(10^9\right),$$

$$C_{44} = G = \frac{E}{2(1+v)} = 79{\cdot}615\left(10^9\right). \tag{4}$$

The finite element model corresponding to references [1, 2] was defined in terms of actual dimensions, that is, the radial dimension $R = 0{\cdot}1143$ m that was assumed to correspond to the distance to the mid-surface of the shell, the shell thickness $h = 0{\cdot}00203$ m, the total length of the cylindrical section $L = 0{\cdot}343$ m, E and ρ. The frequency in reference [1] was given in Hertz (Hz) and the result was divided by 2π. The results for frequency are given in

Table 1

Comparison of frequencies in Hz with existing literature. s—symmetric mode, a—antisymmetic mode, t—torsional mode

	$n=0$			$n=1$		$n=2$		$n=3$	
m		[1]	[2]		[2]		[2]		[2]
1	3213 t	—	—	2748	2740 (0·3)	868	854 (1·6)	748	601 (24)
2	4012 s	4002·8	4011	4209	4198 (0·3)	2475	2471 (0·2)	1764	1732 (1·8)
3	5542 a	5533·5	5539	5095	5086 (0·02)	3972	3946 (0·6)	3060	3000 (2·0)
4	6299 t	—	—	5807	5790 (0·3)	5030	5015 (0·3)	4154	4099 (1·3)
5	6303 s	6296·7	6292	5947	5933 (0·2)	5826	5712 (2·0)	5029	4954 (1·5)
6	6675 a	6663·8	6671	6346	6335 (0·2)	6208	6134 (1·2)	5746	5603 (2·6)
7	6816 s	6805·9	6808	6627	—	6401	—	6669	—
8	7023 a	7012·7	7016	6786	—	6534	—	6694	—
9	7098 s	—	—	6857	—	6863	—	6944	—
10	7134 a	—	—	6972	—	6898	—	7020	—
11	7177 s	—	—	7135	—	7196	—	7373	—
12	7255 a	—	—	7157	—	7215	—	7388	—

TABLE 2

Frequencies $\Omega=\omega a[\rho/C_{44}]^{1/2}$ for a hermetic capsule with a/h=50 *and a/h*=20, ν=0·3 *and L/a*=1,2 *and* 3. *Underscored numbers indicate the order of the lowest frequencies, s—symmetric mode, a—antisymmetic mode, t—torsional mode*

n	Mode	$a/h=50$			$a/h=20$		
		$L/a=1$	$L/a=2$	$L/a=3$	$L/a=1$	$L/a=2$	$L/a=3$
0	1	1·131 s	0·934 t	0·723 t	1·154 s	0·940 t	0·727 t
	2	1·305 t	1·027 s	0·908 s	1·317 t	1·044 s	0·918 s
	3	1·356 a	1·305 a	1·259 a	1·385 a	1·331 a	1·282 a
	4	1·508 s	1·480 s	1·420 t	1·566 s	1·523 s	1·428 t
	5	1·543 a	1·530 a	1·431 s	1·615 a	1·595 a	1·460 s
	6	1·604 s	1·576 s	1·518 a	1·683 s	1·636 s	1·576 a
1	1	1·110	0·827	0·619	1·126	0·835	0·620
	2	1·142	1·042	0·952	1·160	1·056	0·961
	3	1·421	1·293	1·154	1·469	1·342	1·168
	4	1·452	1·383	1·313	1·487	1·417	1·337
	5	1·542	1·477	1·347	1·641	1·530	1·371
	6	1·577	1·488	1·440	1·720	1·541	1·475
2	1	0·561 5	0·305 4	0·190 3	0·577 3	0·320 2	0·206 1
	2	1·160	0·791	0·558	1·200	0·812	0·569 5
	3	1·408	1·144	0·893	1·450	1·187	0·916
	4	1·457	1·343	1·139	1·518	1·400	1·182
	5	1·519	1·435	1·304	1·629	1·492	1·369
	6	1·573	1·472	1·408	1·735	1·563	1·470
3	1	0·457 4	0·229 2	0·137 1	0·526 1	0·300 1	0·239 2
	2	1·056	0·617	0·389	1·160	0·680 6	0·456 4
	3	1·414	0·973	0·680	1·542	1·073	0·753
	4	1·525	1·232	0·934	1·622	1·392	1·039
	5	1·547	1·423	1·141	1·717	1·578	1·296
	6	1·610	1·526	1·307	1·871	1·638	1·517
4	1	0·395 2	0·189 1	0·163 2	0·570 2	0·426 3	0·395 3
	2	0·938	0·487	0·320 5	1·146 6	0·676 5	0·510
	3	1·350	0·815	0·540	1·603	1·032	0·727
	4	1·567	1·097	0·773	1·739	1·383	0·989
	5	1·592	1·334	0·992	1·845	1·650	1·261
	6	1·637	1·536	1·175	2·046	1·750	1·530
5	1	0·390 1	0·279 3	0·268 4	0·737 4	0·638 4	0·616 6
	2	0·861	0·455 6	0·342 6	1·224	0·810	0·693
	3	1·297	0·731	0·495	1·701	1·109	0·848
	4	1·592	1·009	0·697	1·889	1·452	1·068
	5	1·638	1·269	0·899	2·024	1·753	1·326
	6	1·674	1·514	1·087	2·268	1·898	1·603
6	1	0·445 3	0·368 5	0·376	0·988 5	0·909	0·891
	2	0·836 6	0·488	0·425	1·396	1·046	0·954
	3	1·273	0·712	0·528	1·860	1·295	1·076
	4	1·612	0·971	0·683	2·091	1·612	1·296
	5	1·689	1·237	0·869	2·263	1·917	1·493
	6	1·728	1·501	1·049	2·545	2·097	1·761

Table 1 and show good agreement with the results that are summarized in reference [1]. The first 12 frequencies are listed and those that can be compared with reference [1] are accurate to within 0·2%. New information is presented since the first and fourth frequencies are torsional modes that were not found using the shell theory formulation that was reported in references [1–3]. Additional frequencies are given in Table 1 corresponding to higher mode of motion for the circumferential wave numbers n used in reference [2]. The number in parentheses corresponds to the percent difference in the present solution and the referenced solution. There is a large discrepancy for the first mode with circumferential wave number $n = 3$.

The axisymmetric capsule is studied assuming non-dimensional co-ordinates and material properties. Co-ordinates are non-dimensional with respect to the outside radius of the capsule, a, and material constants are non-dimensional with respect C_{44} or

TABLE 3

Frequencies $\Omega = \omega a[\rho/C_{44}]^{1/2}$ for a hermetic capsule with a/h=10 and a/h=5, ν=0.3 and L/a=1,2 and 3. Underscored numbers indicate the order of the lowest frequencies, s—symmetric mode, a—antisymmetic mode, t—torsional mode

n	Mode	$a/h = 10$			$a/h = 5$		
		$L/a = 1$	$L/a = 2$	$L/a = 3$	$L/a = 1$	$L/a = 2$	$L/a = 3$
0	1	1·199 s6	0·950 t	0·732 t6	1·298 s5	0·967 t4	0·742 t4
	2	1·336 t	1·074 s	0·936 s	1·373 t6	1·136 s6	0·973 s6
	3	1·447 a	1·387 a	1·332 a	1·599 a	1·578 a	1·445 a
	4	1·670 s	1·600 s	1·440 t	1·914 s	1·756 s	1·462 t
	5	1·785 a	1·697 a	1·522 s	2·265 a	1·870 t	1·657 s
	6	1·939 s	1·772 s	1·660 a	2·388 s	1·930 a	1·813 a
1	1	1·158 4	0·865 5	0·624 4	1·231 3	0·874 2	0·633 2
	2	1·197 5	1·082	0·981	1·288 4	1·140	1·022
	3	1·579	1·362	1·200	1·782	1·491	1·278
	4	1·580	1·486	1·379	1·856	1·615	1·454
	5	1·897	1·626	1·426	2·293	1·850	1·576
	6	1·930	1·677	1·570	2·504	2·055	1·832
2	1	0·617 1	0·358 1	0·251 1	0·739 1	0·470 1	0·396 1
	2	1·272	0·865 6	0·608 3	1·437	1·010 5	0·736 3
	3	1·546	1·277	0·981	1·836	1·478	1·150
	4	1·692	1·508	1·286	2·136	1·800	1·511
	5	1·931	1·647	1·497	2·286	2·115	1·793
	6	2·082	1·824	1·620	2·647	2·191	2·056
3	1	0·691 3	0·496 2	0·445 2	1·120 2	0·954 3	0·907 5
	2	1·363	0·864 4	0·639 5	1·800	1·319	1·099
	3	1·747	1·310	0·953	2·328	1·808	1·434
	4	1·929	1·656	1·294	2·788	2·230	1·823
	5	2·227	1·845	1·608	3·375	2·604	2·190
	6	2·579	2·034	1·808	3·393	3·039	2·514
4	1	0·959 3	0·829 3	0·797	1·770	1·651	1·614
	2	1·544	1·084	0·920	2·360	1·926	1·795
	3	1·997	1·479	1·152	2·958	2·357	2·020
	4	2·254	1·863	1·459	3·502	2·814	2·367
	5	2·609	2·123	1·792	4·141	3·245	2·752
5	1	1·375	1·275	1·248	2·579	2·480	2·448
	2	1·870	1·476	1·348	3·100	2·713	2·573
	3	2·350	1·813	1·533	3·722	3·094	2·793
	4	2·680	2·190	1·796	4·319	3·546	3·100
6	1	1·894	1·808	1·783	3·491	3·401	3·371
	2	2·325	1·983	1·873	3·963	3·609	3·485
	3	2·811	2·279	2·034	4·584	3·953	3·681
	4	3·200	2·637	2·266	5·212	4·387	3·956

shear modulus G. In all cases the Poisson ratio is used as $\nu = 0{\cdot}3$. Non-dimensional variables are

$$\xi = r/a, \qquad \eta = z/a, \qquad \bar{C}_{1k} = C_{1k}/C_{44}, \qquad \Omega = \omega a\sqrt{\rho/C_{44}}. \tag{5}$$

Let $a = \rho = C_{44} = 1$ and it follows that, using equation (4), $C_{11} = 3{\cdot}5$ and $C_{12} = 1{\cdot}5$. The inside radius of the capsule is defined as, b, and the thickness of the capsule is

$$h = a - b. \tag{6}$$

Results are tabulated for values of $a/h = 50$, 20, 10 and 5 corresponding to $b = 0{\cdot}98$, 0·95, 0·9 and 0·8 respectively. The total length of the cylindrical section connecting the spherical caps is defined in terms of the radius a and results are given for $L/a = 1$, 2 and 3. The capsule with $L/a = 3$ and $a/h = 50$ is similar to the capsule of Table 1 and all other capsules that are studied have a/h less than 50. It is anticipated that as the shell becomes thicker, the elasticity finite elements will maintain the accuracy of the benchmark solutions of reference [1] indicated in Table 1.

Tables 2 and 3 show the results for non-dimensional frequency for the capsules that were studied. The tables indicate that the frequency decreases as the capsule increases in length for a given shell wall thickness and that behavior is common to all capsules that were studied. However, for a given capsule length, the frequency increases as the wall thickness increases. The higher circumferential wave numbers are less significant as the shell wall thickness increases. Similarly, as the length of the capsule increases, the circumferential wave number is less important. For instance, in Table 2, the underscored number to the right of the frequency represents the ordering of the frequencies beginning with the lowest. For $a/h = 50$, the lowest frequency occurs for $n = 5$ when $L/a = 1$, but changes to $n = 4$ when $L/a = 2$ and changes to $n = 3$ when $L/a = 3$.

The motion of the capsule can be separated into pure torsional modes or pure radial–longitudinal modes when the circumferential wave number is taken as zero. The mode

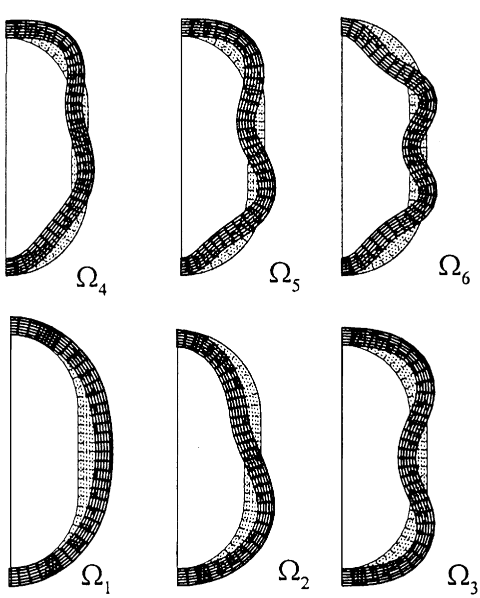

Figure 1. Radial and longitudinal motion of the cross-section of a hermetic capsule with $a/h = 5$, $L/a = 1$ and circumferential wave number $n = 2$.

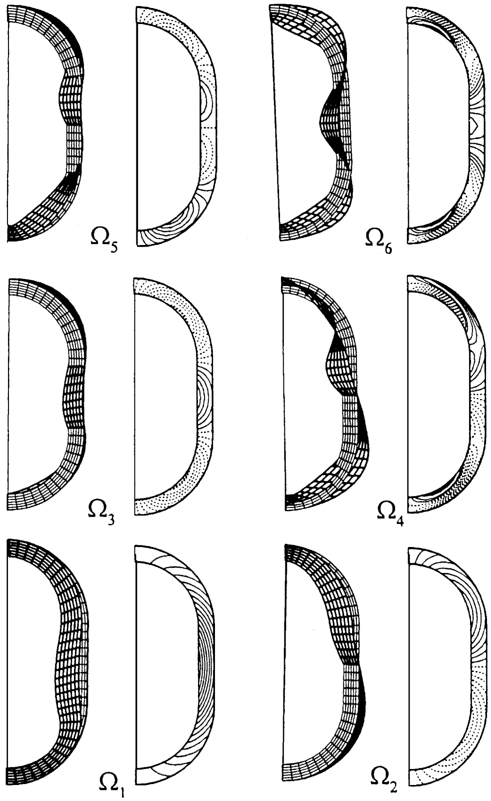

Figure 2. Torsional motion of the cross-section of a hermetic capsule with $a/h = 5$, $L/a = 1$ and circumferential wave number $n = 2$. Three-dimensional and contour plots are shown for each frequency.

shapes corresponding to $n = 0$ are designated as torsional, symmetric or antisymmetric in Tables 2 and 3. The lowest frequency occurs when the circumferential wave number n is greater than zero and for n greater than zero, the motion of the capsule is a combination of radial, longitudinal and torsional displacements. The mode shapes corresponding to the first six frequencies for $a/h = 5$, $L/a = 1$ and $n = 2$ are shown in Figures 1 and 2. The motion of the cross-section corresponding to the r–z co-ordinate directions is shown in Figure 1. The symmetric and antisymmetric motions of the cross-section are distinct. In fact, it would appear that frequencies Ω_4 and Ω_5 correspond to the same mode shape. The torsional motion for the same frequencies is shown in Figure 2. Two views are shown for each torsional motion, a three-dimensional section and a corresponding contour plot. The three-dimensional displacement is relative to a slice taken through the cross-section and the mode shape for Ω_2 is identifiable as a symmetric motion relative to the center of the capsule. The contour plot shows the same mode shape with a series of solid lines versus dashed lines for motion in the opposite directions. The three-dimensional plots for Ω_4 and Ω_5 both show a symmetrical type of motion, however, an inspection of the contour plots show that Ω_5 is a somewhat pure torsional motion symmetric with respect to the center of the capsule. The contour plot for Ω_4 shows a warping type of motion in addition to the torsional motion. The contour lines in the spherical cap section change from solid to dashed through the thickness. It follows that while modes 4 and 5 appear to have the same motion, they are quite different. The torsional motion for Ω_6 shows a similar warping type of behavior.

4. CONCLUDING REMARKS

Free vibration results for hermetic capsules have been presented in the format of tables of non-dimensional frequencies and plots of selective mode shapes. Results in the literature for a relatively thin hermetic capsule were verified and the analysis was extended to include relatively thick capsules. Finite elements, based upon three-dimensional elasticity, were used in the present analysis and compared favorably with frequencies that were computed using various shell theories.

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