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A LINEARIZED PROCEDURE FOR SOLVING INVERSE SENSITIVITY EQUATIONS OF NON-DEFECTIVE SYSTEMS

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A linearized algorithm for solving inverse sensitivity equations of non-defective systems is presented. It is based on the orthonormal decomposition of the first order directional derivatives and directional continuity along $\boldsymbol{\tau}$ of the $\boldsymbol{\tau} - \lambda$ base. The least-squares methods which minimize the trace of eigenmode matrix suggested by Pešek and Lallement, respectively, for self-adjoint systems are extended to general non-defective systems in this paper. Moreover, the new algorithm has intuitive simple geometrical significance and is consistent with the first order Taylor expansion of the $\boldsymbol{\tau} - \lambda$ base. The numerical results calculated from the aforementioned three methods are compared, respectively, with the exact solution using two simulation examples. It demonstrates that the results of the proposed algorithm are the nearest to the exact solution.

1. INTRODUCTION

Various updating techniques have been proposed for structural dynamics system identification by reconciling finite element models of structures with modal parameters identified from dynamic testing [1–5]. The inverse sensitivity method, a kind of parameter correction methods, using first order Taylor expansions of the eigenpairs was first extended to systems with repeated eigenvalues by Pešek [1] and then generalized to non-defective systems by Chen *et al*. [2]. This updating method constructs the governing inverse sensitivity equations $\mathbf{J}(\mathbf{p}, \boldsymbol{\tau}) \cdot \Delta\mathbf{p} = \Delta\mathbf{E}$ for the unknown parameter correction vector $\Delta\mathbf{p}$ from the eigendifference $\Delta\mathbf{E}$ [1]. The correction parameters Δp_k $(k = 1, 2, \ldots, n)$ are usually unknown. It leads to a set of non-linear equations for the design parametric corrections Δp_k because the coefficient Jacobian matrix $\mathbf{J}(\mathbf{p}, \boldsymbol{\tau})$ depends on the direction $\boldsymbol{\tau}$ of the unknown correction vector $\Delta\mathbf{p}$ which is essentially due to the non-differentiability of the repeated roots with respect to the parametric vector $\mathbf{p}$. It is difficult to solve analytically. Therefore, a linearized procedure is desirable.

It is well known that at a given point $P(\mathbf{p})$ in the n-dimensional parametric space, the orthonormal eigenvector matrix $\mathbf{X}_t$ corresponding to a repeated eigenvalue λ (abbr. λ-base) is termed degenerate and its derivatives with respect to design parameters are singular. That is to say, there is an infinite number of λ-base at $P(\mathbf{p})$. There also exists distinct derivable λ-base $\mathbf{Z}_t(\boldsymbol{\tau})$ (abbr. $\boldsymbol{\tau} - \lambda$ base) and there are different directional

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derivatives $\mathbf{Z}_{t,\tau}$ along different directions [6]. Usually, the $\tau-\lambda$ base $\mathbf{Z}_t(\tau)$ can be expressed in terms of an arbitrary λ-base $\mathbf{X}_t$ by the transformation matrix $\mathbf{\Gamma}_t(\mathbf{X}_t, \tau)$ which is contained in the Jacobian matrix $\mathbf{J}(\mathbf{p}, \tau)$. Hence, $\mathbf{J}(\mathbf{p}, \tau)$ depends on the direction τ. If the parameter correction vector $\Delta\mathbf{p}$ is known, the transformation matrix can be uniquely determined by solving an eigenproblem of a certain matrix with respect to direction τ [7–9]. However, in the inverse sensitivity method, the correction parameters Δp_k $(k = 1, 2, \ldots, n)$ are required and of course, the direction τ is unknown. It is impossible to determine the transformation matrix by solving the eigenproblem of an unknown matrix. Therefore, the key for linearization is to determine the transformation matrix $\mathbf{\Gamma}_t(\mathbf{X}_t, \tau)$ first. The method by minimizing the trace of a certain matrix in the sense of least squares is most often used to determine *a priori* the unknown parametric transformation matrix. For example, see reference [1]. Lallement and Kozanek [10] evaluated the transformation matrix using the trace of a certain matrix in the sense of least squares under an orthogonal condition. However, most researches have been limited to self-adjoint systems.

This paper will first generalize the method of Pešek and Lallement to general non-defective systems. Then a new method to determine the transformation matrix for multivariable non-defective systems will be proposed. This is based on the first order Taylor expansion of the $\tau-\lambda$ base $\mathbf{Z}_t(\tau)$ via the orthonormal decomposition of the directional derivatives $\mathbf{Z}_{t,\tau}$. The new expressions have intuitively simple geometric significance and are consistent with the first order Taylor expansion of the eigensolution in mathematical form. Finally, two simulation examples are given using the three methods respectively. Comparing the numerical results with the exact solution, one finds that the proposed algorithm is the nearest to the exact one.

2. THEORETICAL BACKGROUND

This section will first outline the inverse sensitivity method for parameter correction for repeated eigenvalues with multivariable. Consider the following non-defective generalized eigenvalue problem:

$$\mathbf{K}(\mathbf{p})\mathbf{\Phi}_R(\mathbf{p}) = \mathbf{M}(\mathbf{p})\mathbf{\Phi}_R(\mathbf{p})\mathbf{\Omega}(\mathbf{p}), \quad \mathbf{K}^{\mathrm{T}}(\mathbf{p})\mathbf{\Phi}_L(\mathbf{p}) = \mathbf{M}^{\mathrm{T}}(\mathbf{p})\mathbf{\Phi}_L(\mathbf{p})\mathbf{\Omega}(\mathbf{p}), \tag{1,2}$$

$$\mathbf{\Phi}_L^{\mathrm{T}}(\mathbf{p})\mathbf{M}(\mathbf{p})\mathbf{\Phi}_R(\mathbf{p}) = \mathbf{I}_N, \quad \boldsymbol{\phi}_{Ri}^{\mathrm{T}}(\mathbf{p})\mathbf{M}(\mathbf{p})\boldsymbol{\phi}_{Ri}(\mathbf{p}) = 1, \tag{3,4}$$

where the mass matrix $\mathbf{M}(\mathbf{p})$ and the stiffness matrix $\mathbf{K}(\mathbf{p})$ of an N degrees of freedom discrete vibration system are single-valued limited functions and have continuous partial derivatives of all orders with respect to multiple design parameters $\mathbf{p} = (p_1, p_2, \ldots, p_n)^{\mathrm{T}}$. $\mathbf{K}(\mathbf{p})$, $\mathbf{M}(\mathbf{p})$ are real matrices and $\mathbf{M}(\mathbf{p})$ is positive definite. For general non-defective systems, an eigenvector of equations (1), (2) and the biorthonormal condition (3) is undetermined up to a non-zero constant multiplier. An additional gauge condition (4)—every column $\boldsymbol{\phi}_{Ri}(\mathbf{p})$ $(i = 1, 2, \ldots, N)$ in right eigenvector matrix $\mathbf{\Phi}_R(\mathbf{p})$ is normalized according to the mass matrix $\mathbf{M}(\mathbf{p})$, should be imposed to result in unique eigenvectors [11]. In some special cases, for example, when $\mathbf{M}(\mathbf{p})$ is skew-symmetric, condition (4) may fail even if $\mathbf{M}(\mathbf{p})$ is real, then an alternative condition may be adopted [12]. As $\mathbf{M}(\mathbf{p})$ is positive definite in this text, condition (4) will always be valid.

It is assumed that at the computed point $\mathbf{p} = \mathbf{p}_0$ the eigenvalue problem has h repeated eigenvalues $\mathbf{\Lambda} = \lambda\mathbf{I}_h$. Let $\mathbf{Z}_t(\tau)$ $(t = R, L)$ denote the $\tau-\lambda$ base, where

$$\tau = (p_1, p_2, \ldots, p_n)^{\mathrm{T}} / \sqrt{\textstyle\sum_{i=1}^{n} (p_i)^2}.$$

The eigenproblem corresponding to $\boldsymbol{\tau}-\lambda$ base $\mathbf{Z}_t(\boldsymbol{\tau})$ can be written as

$$\mathbf{K}\mathbf{Z}_R(\boldsymbol{\tau}) = \lambda\mathbf{M}\mathbf{Z}_R(\boldsymbol{\tau}), \qquad \mathbf{K}^{\mathrm{T}}\mathbf{Z}_L(\boldsymbol{\tau}) = \lambda\mathbf{M}^{\mathrm{T}}\mathbf{Z}_L(\boldsymbol{\tau}), \tag{5,6}$$

$$\mathbf{Z}_L^{\mathrm{T}}(\boldsymbol{\tau})\mathbf{M}\mathbf{Z}_R(\boldsymbol{\tau}) = \mathbf{I}_h, \qquad \mathbf{z}_R^{\mathrm{T}}(\boldsymbol{\tau})\mathbf{M}\mathbf{z}_R(\boldsymbol{\tau}) = 1. \tag{7,8}$$

There exist transforms $\boldsymbol{\Gamma}_t(\mathbf{X}_t, \boldsymbol{\tau})$ $(t = R, L)$ between $\mathbf{Z}_t(\boldsymbol{\tau})$ and the arbitrary λ base $\mathbf{X}_t$ $(t = R, L)$, such that

$$\mathbf{Z}_R(\boldsymbol{\tau}) = \mathbf{X}_R\boldsymbol{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau}), \qquad \mathbf{Z}_L(\boldsymbol{\tau}) = \mathbf{X}_L\boldsymbol{\Gamma}_L(\mathbf{X}_t, \boldsymbol{\tau}), \tag{9,10}$$

where $\boldsymbol{\Gamma}_t(\mathbf{X}_t, \boldsymbol{\tau})$ $(t = R, L)$ can be determined by an h order eigenvalue problem of the matrix

$$\mathbf{A}(\mathbf{X}_t, \boldsymbol{\tau}) \overset{\text{def}}{=} \mathbf{X}_L^{\mathrm{T}}(\mathbf{K}_{,\boldsymbol{\tau}} - \lambda\mathbf{M}_{,\boldsymbol{\tau}})\mathbf{X}_R \tag{11}$$

and is biorthogonal, i.e.,

$$\boldsymbol{\Gamma}_L^{\mathrm{T}}(\mathbf{X}_t, \boldsymbol{\tau})\boldsymbol{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau}) = \mathbf{I}_h. \tag{12}$$

Taking the derivative of equation (5) along direction $\boldsymbol{\tau}$, one can obtain the sensitivity governing equations of the repeated roots in the eigensolutions. The solution set of the first order governing equations has the following orthonormal decomposition expression [13]:

$$\mathbf{Z}_{R,\boldsymbol{\tau}} = \mathbf{W}_R^*(\boldsymbol{\tau}) + \mathbf{Z}_R(\boldsymbol{\tau})\mathbf{D}_R, \tag{13}$$

where $(,_{\boldsymbol{\tau}})$ denotes the directional derivative with respect to direction $\boldsymbol{\tau}$, $\mathbf{D}_R$ is the coefficient matrix and $\mathbf{W}_R^*(\boldsymbol{\tau}) = \{\mathbf{w}_{R1}^*(\boldsymbol{\tau}), \mathbf{w}_{R2}^*(\boldsymbol{\tau}), \ldots, \mathbf{w}_{Rh}^*(\boldsymbol{\tau})$ is the special particular solution given by the "restrictive" generalized 1-inverse [14] set of the matrix $\boldsymbol{\kappa} = \mathbf{K} - \lambda\mathbf{M}$.

Evaluating

$$\boldsymbol{\tau} = \mathbf{e}_i = (0, \ldots, \underset{\substack{\Uparrow \\ i\text{th}}}{1}, \ldots, 0)^{\mathrm{T}} \quad (i = 1, 2, \ldots, n),$$

where $\mathbf{e}_i$ is the ith canonical vector, leads to the partial derivatives of the repeated eigenvalue and the corresponding eigenvectors with respect to p_i

$$\mathbf{Z}_R(\mathbf{e}_i) = \mathbf{X}_R\boldsymbol{\Gamma}_R(\mathbf{X}_t, \mathbf{e}_i), \qquad \mathbf{Z}_L(\mathbf{e}_i) = \mathbf{X}_L\boldsymbol{\Gamma}_L(\mathbf{X}_t, \mathbf{e}_i), \tag{14,15}$$

where $\boldsymbol{\Gamma}_t(\mathbf{X}_t, \mathbf{e}_i)$ $(t = R, L)$ can be determined obviously by the eigenvalue problem of the matrix

$$\mathbf{A}(\mathbf{X}_t, \mathbf{e}_i) \overset{\text{def}}{=} \mathbf{X}_L^{\mathrm{T}}(\mathbf{K}_{,\mathbf{e}_i} - \lambda\mathbf{M}_{,\mathbf{e}_i})\mathbf{X}_R, \tag{16}$$

i.e.,

$$\mathbf{A}(\mathbf{X}_t, \mathbf{e}_i)\boldsymbol{\Gamma}_R(\mathbf{X}_t, \mathbf{e}_i) = \boldsymbol{\Gamma}_R(\mathbf{X}_t, \mathbf{e}_i)\boldsymbol{\Lambda}_{,\mathbf{e}_i} \tag{17}$$

$$\mathbf{A}^{\mathrm{T}}(\mathbf{X}_t, \mathbf{e}_i)\boldsymbol{\Gamma}_L(\mathbf{X}_t, \mathbf{e}_i) = \boldsymbol{\Gamma}_L(\mathbf{X}_t, \mathbf{e}_i)\boldsymbol{\Lambda}_{,\mathbf{e}_i}, \qquad \boldsymbol{\Gamma}_L^{\mathrm{T}}(\mathbf{X}_t, \mathbf{e}_i)\boldsymbol{\Gamma}_R(\mathbf{X}_t, \mathbf{e}_i) = \mathbf{I}_h, \tag{18,19}$$

in which $\boldsymbol{\Lambda}_{,\mathbf{e}_i} = \mathrm{diag}(\lambda_{,\mathbf{e}_i}^1, \lambda_{,\mathbf{e}_i}^2, \ldots, \lambda_{,\mathbf{e}_i}^h)$. There exist transformations $\boldsymbol{\Theta}_t(\boldsymbol{\tau}, \mathbf{e}_i) = [\theta_t^{ij}]$ $(t = R, L)$ between $\mathbf{Z}_t(\boldsymbol{\tau})$ and $\mathbf{Z}_t(\mathbf{e}_i)$ $(t = R, L)$ such that

$$\mathbf{Z}_R(\mathbf{e}_i) = \mathbf{Z}_R(\boldsymbol{\tau})\boldsymbol{\Theta}_R(\boldsymbol{\tau}, \mathbf{e}_i), \qquad \mathbf{Z}_L(\mathbf{e}_i) = \mathbf{Z}_L(\boldsymbol{\tau})\boldsymbol{\Theta}_L(\boldsymbol{\tau}, \mathbf{e}_i). \tag{20,21}$$

Using equations (9), (10), (12), (14), (15), one has

$$\mathbf{\Theta}_R(\boldsymbol{\tau}, \mathbf{e}_i) = \mathbf{\Gamma}_L^{\mathrm{T}}(\mathbf{X}_t, \boldsymbol{\tau})\mathbf{\Gamma}_R(\mathbf{X}_t, \mathbf{e}_i), \tag{22}$$

$$\mathbf{\Theta}_L(\boldsymbol{\tau}, \mathbf{e}_i) = \mathbf{\Gamma}_R^{\mathrm{T}}(\mathbf{X}_t, \boldsymbol{\tau})\mathbf{\Gamma}_L(\mathbf{X}_t, \mathbf{e}_i), \qquad \mathbf{\Theta}_L^{\mathrm{T}}(\boldsymbol{\tau}, \mathbf{e}_i)\mathbf{\Theta}_R(\boldsymbol{\tau}, \mathbf{e}_i) = \mathbf{I}_h. \tag{23, 24}$$

The aim of the inverse sensitivity method for the parameter correction is to find the unknown changes of parameters Δp_k ($k = 1, 2, \ldots, n$) which reflect the difference between the experimental modal data and the analytical ones. This is based on the first order Taylor expansion in the $\boldsymbol{\tau} - \lambda$ bases

$$\bar{\mathbf{\Lambda}} - \mathbf{\Lambda} = \mathbf{\Lambda}_{,\boldsymbol{\tau}} \cdot \Delta\tau, \qquad \bar{\mathbf{Z}} - \mathbf{Z}_R(\boldsymbol{\tau}) = \mathbf{Z}_{R,\boldsymbol{\tau}} \cdot \Delta\tau, \tag{25, 26}$$

where $\bar{\mathbf{\Lambda}} = \mathrm{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_m)$ and $\bar{\mathbf{Z}} = \{\bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2, \ldots, \bar{\mathbf{z}}_m\}$ denote the test modal data of the structural model. $\Delta\tau$ is given by the unknown correction vector $\bar{\mathbf{p}} = \mathbf{p} + \Delta\mathbf{p}$ such that

$$\Delta\tau = |\Delta\mathbf{p}| = \sqrt{\sum_{i=1}^{n} (\Delta p_i)^2}.$$

The repeated eigenvalues and the corresponding eigenvectors for multivariable non-defective systems are not differentiable in general, although their directional derivatives exist and have been given as follows [2]

$$\mathbf{\Lambda}_{,\tau} = \sum_{i=1}^{n} \mathbf{\Theta}_R(\boldsymbol{\tau}, \mathbf{e}_i)\mathbf{\Lambda}_{,\mathbf{e}_i}\, \mathbf{\Theta}_L^{\mathrm{T}}(\boldsymbol{\tau}, \mathbf{e}_i) \cos\alpha_{\tau\mathbf{e}_i}, \tag{27}$$

$$\mathbf{Z}_{R,\boldsymbol{\tau}} = \sum_{i=1}^{n} \mathbf{W}_R^*(\mathbf{e}_i)\mathbf{\Theta}_L^{\mathrm{T}}(\boldsymbol{\tau}, \mathbf{e}_i) \cos\alpha_{\tau\mathbf{e}_i} + \mathbf{Z}_R(\boldsymbol{\tau})\mathbf{D}_R, \tag{28}$$

where $\cos\alpha_{\tau\mathbf{e}_i}$ means the angle between the direction $\boldsymbol{\tau}$ and the canonical vector $\mathbf{e}_i$ ($i = 1, 2, \ldots, n$). Substituting equation (27) into expansion (25), one has

$$\bar{\mathbf{\Lambda}} - \mathbf{\Lambda} = \sum_{\mathrm{i}=1}^{n} \mathbf{\Theta}_R(\boldsymbol{\tau}, \mathbf{e}_i)\mathbf{\Lambda}_{,\mathbf{e}_i}\, \mathbf{\Theta}_L^{\mathrm{T}}(\boldsymbol{\tau}, \mathbf{e}_i)\Delta p_i. \tag{29}$$

Then one takes the advantage of the projection operator $\mathbf{P}_L$ which has the following properties [15]:

$$\mathbf{P}_L^{\mathrm{T}}\mathbf{Z}_R(\boldsymbol{\tau}) = \mathbf{0}, \qquad \mathbf{P}_L^{\mathrm{T}}\mathbf{W}_R^*(\boldsymbol{\tau}) = \mathbf{W}_R^*(\boldsymbol{\tau}). \tag{30, 31}$$

Left-multiplying equation (26) by $\mathbf{P}_L^{\mathrm{T}}$ in order to reject the homogeneous solution part, substituting equation (28) and using properties (30) and (31) one has

$$\mathbf{P}_L^{\mathrm{T}}\overline{\mathbf{Z}} = \sum_{i=1}^{n} \mathbf{W}_R^*(\mathbf{e}_i)\mathbf{\Theta}_L^{\mathrm{T}}(\boldsymbol{\tau}, \mathbf{e}_i)\Delta p_i. \tag{32}$$

An inverse sensitivity equation results

$$\mathbf{J}(\mathbf{p}, \boldsymbol{\tau}) \cdot \Delta\mathbf{p} = \Delta\mathbf{E}, \tag{33}$$

where $\mathbf{J}(\mathbf{p}, \boldsymbol{\tau}) = \left[\begin{smallmatrix}\mathbf{J}_\lambda \\ \mathbf{J}_z\end{smallmatrix}\right]$ is the generalized Jacobian matrix [2], $\mathbf{J}_\lambda = [t_{jk}]$, $\mathbf{J}_z = [v_R^{jk}]$,

$$t_{jk} = \sum_{s=1}^{h} \theta_R^{js} \cdot \lambda_{,\mathbf{e}_k}^{s}\, \theta_L^{js}, \quad v_R^{jk} = \sum_{s=1}^{h} \mathbf{w}_{Rs,\mathbf{e}_k}\theta_L^{js} \quad (j = 1, 2, \ldots, h; \quad k = 1, 2, \ldots, n)$$

and $\Delta\mathbf{E} = \left[\begin{smallmatrix}\Delta\lambda \\ \Delta\bar{\mathbf{z}}\end{smallmatrix}\right]$ is the known eigendifference, where $\Delta\lambda = ((\bar{\lambda}_1 - \lambda), (\bar{\lambda}_2 - \lambda), \ldots, (\bar{\lambda}_h - \lambda))^{\mathrm{T}}$ and $\Delta\bar{\mathbf{z}} = (\mathbf{P}_L^{\mathrm{T}}\bar{\mathbf{z}}_1, \mathbf{P}_L^{\mathrm{T}}\bar{\mathbf{z}}_2, \ldots, \mathbf{P}_L^{\mathrm{T}}\bar{\mathbf{z}}_h)^{\mathrm{T}}$. Solving equation (33) in the sense of least squares, one

can obtain the unknown parameter correction vector

$$\Delta\mathbf{p} = \mathbf{J}(\mathbf{p}, \boldsymbol{\tau})^{+} \cdot \Delta\mathbf{E}, \tag{34}$$

where $\mathbf{J}(\mathbf{p}, \boldsymbol{\tau})^{+}$ is the Moore–Penrose generalized inverse of matrix $\mathbf{J}(\mathbf{p}, \boldsymbol{\tau})$.

This is the whole procedure of the inverse sensitivity method for parameter correction with repeated eigenvalues. Obviously, in view of equations (22) and (23), the generalized Jacobian matrix $\mathbf{J}(\mathbf{p}, \boldsymbol{\tau})$ involves the transformation matrix $\boldsymbol{\Theta}_t(\boldsymbol{\tau}, \mathbf{e}_i)$ of equations (29) and (32) which depends on the transformation matrix $\boldsymbol{\Gamma}_t(\mathbf{X}_t, \boldsymbol{\tau})$. The parameter correction vector $\Delta\mathbf{p}$ is to be determined with its direction $\boldsymbol{\tau}$ not yet known. It is impossible to determine the transformation matrix $\boldsymbol{\Gamma}_t(\mathbf{X}_t, \boldsymbol{\tau})$ by solving the eigenvalue problem of an unknown matrix $\mathbf{A}(\mathbf{X}_t, \boldsymbol{\tau})$. Hence, in order to solve the unknown correction vector $\Delta\mathbf{p}$ linearly, the transformation matrix $\boldsymbol{\Gamma}_t(\mathbf{X}_t, \boldsymbol{\tau})$ must be found first.

3. EXTENSION OF THE METHOD OF PEŠEK AND LALLEMENT

IN 1993, Lallement and Kozanek [10] presented the formula for evaluating $\boldsymbol{\Gamma}(\mathbf{X}, \boldsymbol{\tau})$ to minimise the trace of $(\mathbf{X}\boldsymbol{\Gamma}(\mathbf{X}, \boldsymbol{\tau}) - \overline{\mathbf{Z}})^{\mathrm{T}}(\mathbf{X}\boldsymbol{\Gamma}(\mathbf{X}, \boldsymbol{\tau}) - \overline{\mathbf{Z}})$ in the sense of least squares under the constraint $\boldsymbol{\Gamma}^{\mathrm{T}}(\mathbf{X}, \boldsymbol{\tau})\boldsymbol{\Gamma}(\mathbf{X}, \boldsymbol{\tau}) = \mathbf{I}_h$. This formula is

$$\boldsymbol{\Gamma}(\mathbf{X}, \boldsymbol{\tau}) = \mathbf{U}(\overline{\mathbf{Z}}^{\mathrm{T}}\mathbf{X})^{-1}, \tag{35}$$

where

$$\mathbf{U}^2 = \mathbf{B} = \mathbf{X}^{\mathrm{T}}\overline{\mathbf{Z}}\,\overline{\mathbf{Z}}^{\mathrm{T}}\mathbf{X} \tag{36}$$

and $\mathbf{B}$ is a positive-definite matrix in $R^{h\times h}$, $\mathbf{U}$ is a square root of $\mathbf{B}$. It can be shown with eigenvalue theory that a positive-definite matrix has a unique symmetric positive-definite square root.

In 1995, also for self-adjoint systems, Pešek [1] gave another formulae for the evaluation of $\boldsymbol{\Gamma}(\mathbf{X}, \boldsymbol{\tau})$. It was assumed in equation (26) that in the $\boldsymbol{\tau}-\lambda$ base, $\mathbf{Z}(\boldsymbol{\tau})$ nicely approximates the corresponding test modes $\overline{\mathbf{Z}}$. Thus, one can create $\overline{\mathbf{Z}} = \mathbf{X}\boldsymbol{\Gamma}(\mathbf{X}, \boldsymbol{\tau})$. $\boldsymbol{\Gamma}(\mathbf{X}, \boldsymbol{\tau})$ can be found by minimizing the trace of the matrix $(\overline{\mathbf{Z}} - \mathbf{Z}(\boldsymbol{\tau}))^{\mathrm{T}}(\overline{\mathbf{Z}} - \mathbf{Z}(\boldsymbol{\tau}))$ in the sense of least squares. Then $\boldsymbol{\Gamma}(\mathbf{X}, \boldsymbol{\tau})$ can be expressed as

$$\boldsymbol{\Gamma}(\mathbf{X}, \boldsymbol{\tau}) = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\overline{\mathbf{Z}}. \tag{37}$$

Since the constraint equation $\boldsymbol{\Gamma}^{\mathrm{T}}(\mathbf{X}, \boldsymbol{\tau})\boldsymbol{\Gamma}(\mathbf{X}, \boldsymbol{\tau}) = \mathbf{I}_h$ was not taken into account in the last formulae, the matrix $\mathbf{Z}(\boldsymbol{\tau})$ must be orthonormalized in additional to the matrix $\mathbf{M}$.

Now one can extend formula (35) presented by Lallement to general non-defective systems. Denote

$$\mathbf{U}_R^2 = \mathbf{B}_R = \mathbf{X}_L^{\mathrm{T}}\overline{\mathbf{Z}}\;\overline{\mathbf{Z}}^{\mathrm{T}}\mathbf{X}_R, \qquad \mathbf{U}_L^2 = \mathbf{B}_L = \mathbf{X}_R^{\mathrm{T}}\overline{\mathbf{Z}}\,\overline{\mathbf{Z}}^{\mathrm{T}}\mathbf{X}_L \tag{38, 39}$$

It can be shown that $\mathbf{B}_R = \mathbf{B}_L^{\mathrm{T}}$ yields $\mathbf{U}_R = \mathbf{U}_L^{\mathrm{T}}$. In order to satisfy the biorthonormal condition (12), one may consider

$$\boldsymbol{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau}) = \mathbf{U}_R(\overline{\mathbf{Z}}^{\mathrm{T}}\mathbf{X}_R)^{-1}, \qquad \boldsymbol{\Gamma}_L(\mathbf{X}_t, \boldsymbol{\tau}) = \mathbf{U}_L(\overline{\mathbf{Z}}^{\mathrm{T}}\mathbf{X}_L)^{-1} \tag{40, 41}$$

to get

$$\begin{aligned}\boldsymbol{\Gamma}_L^{\mathrm{T}}(\mathbf{X}_t, \boldsymbol{\tau})\boldsymbol{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau}) &= (\mathbf{X}_L^{\mathrm{T}}\overline{\mathbf{Z}})^{-1}\mathbf{U}_L^{\mathrm{T}}\mathbf{U}_R(\overline{\mathbf{Z}}^{\mathrm{T}}\mathbf{X}_R)^{-1} = (\mathbf{X}_L^{\mathrm{T}}\overline{\mathbf{Z}})^{-1}\mathbf{U}_R\mathbf{U}_R(\overline{\mathbf{Z}}^{\mathrm{T}}\mathbf{X}_R)^{-1} \\ &= (\mathbf{X}_L^{\mathrm{T}}\overline{\mathbf{Z}})^{-1}\mathbf{X}_L^{\mathrm{T}}\overline{\mathbf{Z}}\,\overline{\mathbf{Z}}^{\mathrm{T}}\mathbf{X}_R(\overline{\mathbf{Z}}^{\mathrm{T}}\mathbf{X}_R)^{-1} = \mathbf{I}_h.\end{aligned} \tag{42}$$

Similarly, one can extend formula (37) of Pešek to non-defective systems. It may be considered approximately

$$\overline{\mathbf{Z}} = \mathbf{X}_R \boldsymbol{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau}). \tag{43}$$

Premultiplying by $\mathbf{X}_L^{\mathrm{T}}$, one has

$$\boldsymbol{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau}) = (\mathbf{X}_L^{\mathrm{T}}\mathbf{X}_R)^{-1}\mathbf{X}_L^{\mathrm{T}}\overline{\mathbf{Z}}. \tag{44}$$

Similarly, $\boldsymbol{\Gamma}_L(\mathbf{X}_t, \boldsymbol{\tau})$ may be determined by the biorthonormal condition (12).

4. NEW ALGORITHM

A new method to evaluate the transformation matrix $\boldsymbol{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau})$ will be presented in this section. This is based on the orthonormal decomposition of the firstorder directional derivatives in the $\boldsymbol{\tau}-\lambda$ base $\mathbf{Z}_R(\boldsymbol{\tau})$ as equation (13).

The particular solution $\mathbf{W}_R^*(\boldsymbol{\tau}) \in \mathbf{C}_R(\boldsymbol{\kappa})$, $\mathbf{C}_R(\boldsymbol{\kappa})$ is the complementary space of the kernel of $\boldsymbol{\kappa}$. The first-order Taylor expansion of the $\boldsymbol{\tau}-\lambda$ base $\mathbf{Z}_R(\boldsymbol{\tau})$ along direction $\boldsymbol{\tau}$ is similar to equation (26). Substituting equation (13) into equation (26) one has

$$\overline{\mathbf{Z}} - \mathbf{Z}_R(\boldsymbol{\tau}) = \mathbf{W}_R^*(\boldsymbol{\tau})\cdot\Delta\tau + \mathbf{Z}_R(\boldsymbol{\tau})\mathbf{D}_R\cdot\Delta\tau. \tag{45}$$

Using the left projection

$$\mathbf{P}_L = \mathbf{I} - \mathbf{M}^{\mathrm{T}}\mathbf{X}_L\mathbf{X}_R^{\mathrm{T}} = \mathbf{I} - \mathbf{M}^{\mathrm{T}}\mathbf{Z}_L\mathbf{Z}_R^{\mathrm{T}}. \tag{46}$$

Pre-multiplying equation (45) by $\mathbf{P}_L^{\mathrm{T}}$ and substituting equations (30) and (31) one has

$$\mathbf{P}_L^{\mathrm{T}}\overline{\mathbf{Z}} = \mathbf{W}_R^*(\boldsymbol{\tau})\cdot\Delta\tau. \tag{47}$$

Substituting equations (9) (47) into equation (45) gives

$$\mathbf{X}_R\boldsymbol{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau}) = \overline{\mathbf{Z}} - \mathbf{P}_L^{\mathrm{T}}\overline{\mathbf{Z}} - \mathbf{Z}_R(\boldsymbol{\tau})\mathbf{D}_R\cdot\Delta\tau. \tag{48}$$

Premultiplying by $\mathbf{X}_L^{\mathrm{T}}$ and then by $(\mathbf{X}_L^{\mathrm{T}}\mathbf{X}_R)^{-1}$, one obtains

$$\boldsymbol{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau}) = (\mathbf{X}_L^{\mathrm{T}}\mathbf{X}_R)^{-1}\mathbf{X}_L^{\mathrm{T}}(\overline{\mathbf{Z}} - \mathbf{P}_L^{\mathrm{T}}\overline{\mathbf{Z}}) - (\mathbf{X}_L^{\mathrm{T}}\mathbf{X}_R)^{-1}\mathbf{X}_L^{\mathrm{T}}\mathbf{Z}_R(\boldsymbol{\tau})\mathbf{D}_R\cdot\Delta\tau. \tag{49}$$

Denote

$$\boldsymbol{\Gamma}_{R1} = (\mathbf{X}_L^{\mathrm{T}}\mathbf{X}_R)^{-1}\mathbf{X}_L^{\mathrm{T}}(\overline{\mathbf{Z}} - \mathbf{P}_L^{\mathrm{T}}\overline{\mathbf{Z}}), \qquad \boldsymbol{\Gamma}_{R2} = -(\mathbf{X}_L^{\mathrm{T}}\mathbf{X}_R)^{-1}\mathbf{X}_L^{\mathrm{T}}\mathbf{Z}_R(\boldsymbol{\tau})\mathbf{D}_R. \tag{50, 51}$$

Equation (49) can be rewritten as

$$\boldsymbol{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau}) = \boldsymbol{\Gamma}_{R1} + \boldsymbol{\Gamma}_{R2}\cdot\Delta\boldsymbol{\tau}. \tag{52}$$

Transposing equation (23) and then substituting equation (52), one has

$$\boldsymbol{\Theta}_L^T(\boldsymbol{\tau}, \mathbf{e}_i) = \boldsymbol{\Gamma}_L^{\mathrm{T}}(\mathbf{X}_t, \mathbf{e}_i)\boldsymbol{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau}) = \boldsymbol{\Gamma}_L^{\mathrm{T}}(\mathbf{X}_t, \mathbf{e}_i)\boldsymbol{\Gamma}_{R1} + \boldsymbol{\Gamma}_L^{\mathrm{T}}(\mathbf{X}_t, \mathbf{e}_i)\boldsymbol{\Gamma}_{R2}\cdot\Delta\tau. \tag{53}$$

Substituting equation (53) into equation (28), one obtains

$$\begin{aligned}\mathbf{Z}_{R,\tau} &= \sum_{i=1}^{n} \mathbf{W}_R^*(\mathbf{e}_i)[\boldsymbol{\Gamma}_L^{\mathrm{T}}(\mathbf{X}_t, \mathbf{e}_i)\boldsymbol{\Gamma}_{R1} + \boldsymbol{\Gamma}_L^{\mathrm{T}}(\mathbf{X}_t, \mathbf{e}_i)\boldsymbol{\Gamma}_{R2}\cdot\Delta\tau]\cos\alpha_{\boldsymbol{\tau}\mathbf{e}_i} + \mathbf{Z}_R(\boldsymbol{\tau})\mathbf{D}_R \\ &= \sum_{i=1}^{n} \mathbf{W}_R^*(\mathbf{e}_i)\boldsymbol{\Gamma}_L^{\mathrm{T}}(\mathbf{X}_t, \mathbf{e}_i)\boldsymbol{\Gamma}_{R1}\cos\alpha_{\boldsymbol{\tau}\mathbf{e}_i} + \mathbf{Z}_R(\boldsymbol{\tau})\mathbf{D}_R \\ &\quad + \sum_{i=1}^{n} \mathbf{W}_R^*(\mathbf{e}_i)\boldsymbol{\Gamma}_L^{\mathrm{T}}(\mathbf{X}_t, \mathbf{e}_i)\boldsymbol{\Gamma}_{R2}\cos\alpha_{\boldsymbol{\tau}\mathbf{e}_i}\cdot\Delta\tau.\end{aligned} \tag{54}$$

In the first order Taylor expansion (26) of the $\boldsymbol{\tau}-\lambda$ base $\mathbf{Z}_R(\boldsymbol{\tau})$, the second and higher order small quantity terms are neglected. Substituting equation (54) into the expansion

(26), one finds that the part including $\mathbf{\Gamma}_{R2}$ yields a second order term. Neglecting the first order small quantity term of the $\mathbf{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau})$ in expression (52) means the neglection of the second-order term of Taylor's expansion and will not introduce additional approximation. Therefore, when $\Delta\tau$ is a small quantity, one can neglect the term including $\mathbf{\Gamma}_{R2}$ and consider

$$\mathbf{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau}) = (\mathbf{X}_L^{\mathrm{T}}\mathbf{X}_R)^{-1}\mathbf{X}_L^{\mathrm{T}}(\overline{\mathbf{Z}} - \mathbf{P}_L^{\mathrm{T}}\overline{\mathbf{Z}}) \tag{55}$$

and $\mathbf{\Gamma}_L(\mathbf{X}_t, \boldsymbol{\tau})$ can be determined by biorthonormal condition (12).

The new algorithm (55) for $\mathbf{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau})$ is found by the orthonormal decomposition expression (13) of the first order directional derivatives $\mathbf{Z}_{R,\tau}$ of the $\boldsymbol{\tau}-\lambda$ base $\mathbf{Z}_R(\boldsymbol{\tau})$ and by neglecting the small component $\mathbf{Z}_R(\boldsymbol{\tau})\mathbf{D}_R\cdot\Delta\tau$ in the space of the kernel of $\boldsymbol{\kappa}$. Therefore, it has intuitively simple geometric significance. On the other hand, neglecting the first order small quantity term of equation (52) yields the new algorithm (55). It is consistent with the first order Taylor expansion of the $\boldsymbol{\tau}-\lambda$ base mathematically.

5. NUMERICAL EXAMPLES

In this section two simulation examples (symmetric and asymmetric) are given to demonstrate the procedure of calculating the transformation matrix $\mathbf{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau})$. The results by the three methods are compared, respectively, with the exact solution. All results are obtained by using Mathematic 3.0.

Example 1. Suppose that an asymmetric system is given as follows:

$$\mathbf{M}(\mathbf{p}) = \begin{bmatrix} 1+p^3 & 10 & 3 \\ -1+q^2 & 1+p+(1+p)^2+(1+p)^3 & 4 \\ 0 & 4 & 2 \end{bmatrix},$$

$$\mathbf{K}(\mathbf{p}) = \begin{bmatrix} 1+p+(1+p)^2+(1+p)^3 & 38 & 17 \\ (1+q)^3 & 4+p+2(2+p)^2+(2+p)^3 & 20 \\ 4 & 16 & 2+(1+p)^3 \end{bmatrix},$$

where $\mathbf{p} = (p, q)^T$ are the design parameters and at $\mathbf{p} = \mathbf{p}_0 = (1, 1)^{\mathrm{T}}$, the original system

$$\mathbf{M} = \begin{bmatrix} 2 & 10 & 3 \\ 0 & 14 & 4 \\ 0 & 4 & 2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 14 & 38 & 17 \\ 8 & 50 & 20 \\ 4 & 16 & 10 \end{bmatrix}$$

has a set of complete eigenvalues

$$\mathbf{\Omega} = \mathrm{diag}(3, 3, 6),$$

which includes a two-fold eigenvalue

$$\mathbf{\Lambda} = \lambda\mathbf{I}_2 = \mathrm{diag}(3, 3),$$

whose arbitrary biorthonormalized right and left λ base are, respectively,

$$\mathbf{X}_R = \begin{bmatrix} -1 & -0{\cdot}408248 \\ 0 & 0{\cdot}408248 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{X}_L = \begin{bmatrix} -0{\cdot}333333 & 0 \\ 0 & 0{\cdot}408248 \\ 0{\cdot}666667 & -0{\cdot}816497 \end{bmatrix}.$$

The left project matrix of the repeated eigenvalue is

$$\mathbf{P}_L = \begin{bmatrix} 0{\cdot}333333 & 0 & 0{\cdot}666667 \\ 0{\cdot}333333 & 0 & 0{\cdot}666667 \\ 0{\cdot}333333 & 0 & 0{\cdot}666667 \end{bmatrix}.$$

Another mechanical system to be considered is distinguished from the original analytical system by changing the physical parameters $\bar{\mathbf{p}} = \mathbf{p}_0 + \Delta\mathbf{p}$. Here, one uses the simulated measured (exact) data that can exclude measure errors. On the assumption that for $\bar{\mathbf{p}} = (1{\cdot}14,\ 0{\cdot}857171)^{\mathrm{T}}$, then the two-fold eigenvalue $\lambda_1 = \lambda_2 = \lambda = 3$ is split into two unrepeated eigenvalues $\bar{\lambda}_1 = 4{\cdot}21654$, $\bar{\lambda}_2 = 2{\cdot}95525$, the corresponding biorthonormalized eigenvectors (simulated measurement data) are

$$\overline{\mathbf{Z}} = \begin{bmatrix} -0{\cdot}553176 & -0{\cdot}236044 \\ -0{\cdot}161349 & 0{\cdot}32302 \\ 0{\cdot}615619 & -0{\cdot}0688866 \end{bmatrix}.$$

By expressions (40), (44) and (55), one can obtain the transformation matrix $\mathbf{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau})$, denoted as $\mathbf{\Gamma}_R(Lallement)$, $\mathbf{\Gamma}_R(Pešek)$ and $\mathbf{\Gamma}_R(new)$, respectively,

$$\mathbf{\Gamma}_R(Lallement) = \begin{bmatrix} 0{\cdot}760578 & -0{\cdot}0849182 \\ -0{\cdot}689491 & 1{\cdot}04651 \end{bmatrix},$$

$$\mathbf{\Gamma}_R(Pešek) = \begin{bmatrix} 0{\cdot}6354 & -0{\cdot}0725046 \\ -0{\cdot}298315 & 0{\cdot}77351 \end{bmatrix},$$

$$\mathbf{\Gamma}_R(new) = \begin{bmatrix} 0{\cdot}681556 & -0{\cdot}0809465 \\ -0{\cdot}395222 & 0{\cdot}791234 \end{bmatrix}.$$

In order to verify the efficiency of the mentioned methods, one can calculate the exact solution of $\mathbf{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau})$ denoted as $\bar{\mathbf{\Gamma}}_R(exact)$ by solving an eigenproblem. Here, for $\bar{\mathbf{p}} = (1{\cdot}14,\ 0{\cdot}857171)^{\mathrm{T}}$ yields $\boldsymbol{\tau} = (0{\cdot}7,\ -0{\cdot}71415)$. Along this direction, one can solve

$$\bar{\mathbf{\Gamma}}_R(exact) = \begin{bmatrix} 0{\cdot}827076 & -0{\cdot}100849 \\ -0{\cdot}56209 & 0{\cdot}994902 \end{bmatrix}.$$

The correlations between the transformation matrix of $\bar{\mathbf{\Gamma}}_R(exact)$ ($\stackrel{\text{def}}{=}[\bar{\gamma}_1 \quad \bar{\gamma}_2]$) and $\mathbf{\Gamma}_R(Lallement)$, $\mathbf{\Gamma}_R(Pešek)$, $\mathbf{\Gamma}_R(new)$ ($\stackrel{\text{def}}{=}[\gamma_1 \quad \gamma_2]$) can be evaluated by the modal assurance criterion (*mac*)

$$mac_i = \frac{(\gamma_i, \bar{\gamma}_i)(\gamma_i, \bar{\gamma}_i)}{(\gamma_i, \gamma_i)(\bar{\gamma}_i, \bar{\gamma}_i)} \quad (i = 1, 2),$$

where $(\gamma_i, \bar{\gamma}_i)$ is the inner product of vectors γ_i and $\gamma^-{}_i$. Then denote

$$u = \tfrac{1}{2}(mac_1 + mac_2).$$

It is easy to have $u(Lallement) = 0{\cdot}990131$, $u(Pešek) = 0{\cdot}987598$ and $u(new) = 0{\cdot}997454$. It should be noted that the closer u value is to 1 the better is the correlation between matrices. The cases for the other three arbitrary parameters are calculated and shown in Table 1.

The algorithm presented in this paper virtually includes all previously developed methods for self-adjoint systems as special cases. Here, a numerical example for self-adjoint system will be given to demonstrate the application.

TABLE 1

Comparison of transformation matrix $\mathbf{\Gamma}_R(\mathbf{X}_t, \boldsymbol{\tau})$ *(asymmetric system)*

	$\bar{\mathbf{p}}_1 = (0{\cdot}9, 1{\cdot}17321)^{\mathrm{T}}$	$\bar{\mathbf{p}}_2 = (1{\cdot}012, 1{\cdot}19964)^{\mathrm{T}}$	$\bar{\mathbf{p}}_3 = (1{\cdot}06, 1{\cdot}08)^{\mathrm{T}}$
$\bar{\mathbf{\Gamma}}_R(exact)$	$\begin{bmatrix} -0{\cdot}877672 & 0{\cdot}106402 \\ 0{\cdot}479261 & -0{\cdot}994323 \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}04013 & 0{\cdot}47258 \\ 0{\cdot}999194 & -0{\cdot}881288 \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}697465 & -0{\cdot}0853097 \\ -0{\cdot}716619 & 0{\cdot}996354 \end{bmatrix}$
$\mathbf{\Gamma}_R(new)$	$\begin{bmatrix} -0{\cdot}936683 & 0{\cdot}127603 \\ 0{\cdot}350165 & -1{\cdot}5098 \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}0286746 & 0{\cdot}411942 \\ 1{\cdot}11584 & -0{\cdot}78684 \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}613586 & -0{\cdot}0733859 \\ -0{\cdot}549818 & 0{\cdot}926896 \end{bmatrix}$
$\mathbf{\Gamma}_R(Pešek)$	$\begin{bmatrix} -0{\cdot}907228 & 0{\cdot}214871 \\ 0{\cdot}288324 & -1{\cdot}69303 \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}0737974 & 0{\cdot}399497 \\ 1{\cdot}21058 & -0{\cdot}76071 \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}579243 & -0{\cdot}0831492 \\ -0{\cdot}477711 & 0{\cdot}947395 \end{bmatrix}$
$\mathbf{\Gamma}_R(Lallement)$	$\begin{bmatrix} -0{\cdot}761369 & 0{\cdot}117847 \\ 0{\cdot}629473 & -1{\cdot}07923 \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}1973 & 0{\cdot}715055 \\ 0{\cdot}783804 & -0{\cdot}904482 \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}754951 & -0{\cdot}0118732 \\ -0{\cdot}750794 & 0{\cdot}991761 \end{bmatrix}$
$u(exact)$	1	1	1
$u(new)$	0·997454	0·999846	0·997651
$u(Pešek)$	0·987598	0·999748	0·994047
$u(Lallement)$	0·990131	0·944547	0·997175

Example 2. Suppose one is given the following symmetric system:

$$\mathbf{M}(\mathbf{p}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}(\mathbf{p}) = \begin{bmatrix} 3 & -2 & -1 \\ -2 & 2 + 8p + 2q & -2q \\ -1 & -2q & 1 + 2q \end{bmatrix},$$

where $\mathbf{p} = (p, q)^{\mathrm{T}}$ are the design parameters and at $\mathbf{p} = \mathbf{p}_0 = (1, 1)^{\mathrm{T}}$, the original system

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 3 & -2 & -1 \\ -2 & 12 & -2 \\ -1 & -2 & 3 \end{bmatrix}$$

has a set of complete eigenvalues

$$\mathbf{\Omega} = \mathrm{diag}(1, 4, 4),$$

which includes a two-fold eigenvalue

$$\mathbf{\Lambda} = \lambda \mathbf{I}_2 = \mathrm{diag}(4, 4),$$

whose arbitrary biorthonormalized right and left λ base are as following, respectively,

$$\mathbf{X}_R = \mathbf{X}_L = \begin{bmatrix} 0{\cdot}707107 & 0{\cdot}408248 \\ 0 & -0{\cdot}408248 \\ -0{\cdot}707107 & 0{\cdot}408248 \end{bmatrix}.$$

TABLE 2

Comparison of transformation matrix $\mathbf{\Gamma}(\mathbf{X}, \boldsymbol{\tau})$ *(symmetric system)*

	$\bar{\mathbf{p}}_1 = (0{\cdot}88, 1{\cdot}16)^{\mathrm{T}}$	$\bar{\mathbf{p}}_2 = (1{\cdot}02, 0{\cdot}801)^{\mathrm{T}}$	$\bar{\mathbf{p}}_3 = (1{\cdot}14, 1{\cdot}14283)^{\mathrm{T}}$
$\bar{\mathbf{\Gamma}}$(*exact*)	$\begin{bmatrix} 0{\cdot}799171 & 0{\cdot}601103 \\ -0{\cdot}601103 & 0{\cdot}799171 \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}676021 & 0{\cdot}736883 \\ 0{\cdot}736883 & 0{\cdot}676021 \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}45877 & 0{\cdot}888555 \\ 0{\cdot}888555 & 0{\cdot}45877 \end{bmatrix}$
$\mathbf{\Gamma}$(*new*)	$\begin{bmatrix} 0{\cdot}789638 & 0{\cdot}613343 \\ -0{\cdot}61156 & 0{\cdot}788969 \end{bmatrix}$	$\begin{bmatrix} -0.676468 & 0.735769 \\ 0.734532 & 0.677211 \end{bmatrix}$	$\begin{bmatrix} -0.457101 & 0.889181 \\ 0.889367 & 0.456718 \end{bmatrix}$
$\mathbf{\Gamma}$(*Pešek*)	$\begin{bmatrix} 0{\cdot}789638 & 0{\cdot}613343 \\ -0{\cdot}646678 & 0{\cdot}814835 \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}676468 & 0{\cdot}735769 \\ 0{\cdot}696756 & 0{\cdot}673357 \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}457101 & 0{\cdot}889181 \\ 0{\cdot}882845 & 0{\cdot}437164 \end{bmatrix}$
$\mathbf{\Gamma}$(*Lallement*)	$\begin{bmatrix} 0{\cdot}787551 & 0{\cdot}616249 \\ -0{\cdot}616249 & 0{\cdot}787551 \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}682777 & 0{\cdot}730626 \\ 0{\cdot}730626 & 0{\cdot}682777 \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}45277 & 0{\cdot}891627 \\ 0{\cdot}891627 & 0{\cdot}45277 \end{bmatrix}$
u(*exact*)	1	1	1
u(*new*)	0·999774	0·999997	0·999996
u(*Pešek*)	0·999147	0·9996	0·999806
u(*Lallement*)	0·999636	0·999915	0·999955

The project matrix of the repeated eigenvalue is

$$\mathbf{P}_L = \mathbf{P}_R = \begin{bmatrix} 0{\cdot}333333 & 0{\cdot}166667 & 0{\cdot}333333 \\ 0{\cdot}666667 & 0{\cdot}333333 & 0{\cdot}666667 \\ 0{\cdot}333333 & 0{\cdot}166667 & 0{\cdot}333333 \end{bmatrix}.$$

Similarly, the cases of three arbitrary parameters for $\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_2, \bar{\mathbf{p}}_3$ are calculated and the comparison of the transformation matrix $\mathbf{\Gamma}(\mathbf{X}, \boldsymbol{\tau})$ is shown in Table 2.

6. CONCLUSIONS

This paper gives a new algorithm to determine the transformation matrices from λ bases to $\boldsymbol{\tau}-\lambda$ base for non-defective multivariable systems. The proposed method is a key for linearization to solving the inverse sensitivity equation, which has obvious geometrical and mathematical significance. Two simulation examples illustrate the performance of the proposed method and their numerical results are compared. The result of the proposed algorithm is the nearest to the exact solution. The procedure of the method shows its potential in correcting model parameters and detecting structural damage of general non-defective systems [16].

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APPENDIXA: NOMENCLATURE

$\mathbf{e}_i$	ith canonical vector of n-dimension parametric space
h	multiplicity of the eigenvalue considered
$\mathbf{I}_h$	h order identity matrix
$\mathbf{K}, \mathbf{K}(\mathbf{p})$	structural real stiffness matrix
$\mathbf{M}, \mathbf{M}(\mathbf{p})$	structural real positive-definite mass matrix
$\mathbf{P}_t$	projection matrix
$\mathbf{p}$	n-dimension vector of structural design parameters
$P(\mathbf{p})$	computed point of the analytical model
$\mathbf{X}_t$	arbitrary $\mathbf{M}$-biorthonormalized eigenvector matrix with respect to h-fold eigenvalue λ (abbr. λ-base)
$\mathbf{Z}_t(\boldsymbol{\tau})$	derivable λ-base along direction $\boldsymbol{\tau}$ passing computed point $P(\mathbf{p})$ in the n-dimension parametric space (abbr. $\boldsymbol{\tau} - \lambda$ base)
$\overline{\mathbf{Z}}$	test mode matrix of structural model corresponding to $\mathbf{Z}_R(\boldsymbol{\tau})$

$\Delta\mathbf{p}$	parameter correction vector
$\Delta\mathbf{E}$	the differences between test modal data and computed eigensolutions, the so-called eigendifference
λ	h-fold computed eigenvalue
$\mathbf{\Lambda}$	$\lambda\mathbf{I}_h$ diagonal matrix of h-fold computed eigenvalue
$\bar{\mathbf{\Lambda}}$	diagonal matrix of test eigenvalues of structural model corresponding to $\mathbf{\Lambda}$
$\boldsymbol{\tau}$	unit vector of an arbitrary direction in n-dimension parametric space, or unit vector along correction vector $\Delta\mathbf{p}$
$\mathbf{\Gamma}_t(\mathbf{X}_t, \boldsymbol{\tau})$	transformation matrix from $\mathbf{X}_t$ to $\mathbf{Z}_t(\boldsymbol{\tau})$
$\mathbf{\Theta}_t(\boldsymbol{\tau}, \mathbf{e}_i)$	transformation matrix from $\mathbf{Z}_t(\boldsymbol{\tau})$ to $\mathbf{Z}_t(\mathbf{e}_i)$

Superscripts

T	transpose of matrix ( ) (not Hermitian transpose)

Subscripts

t	"R" or "L" denote right or left
$,\tau$	$\left.\dfrac{\partial(\,)}{\partial\boldsymbol{\tau}}\right\vert_{p=p_0}$