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FUNDAMENTAL FREQUENCY OF A BEAM ON TWO ELASTIC SUPPORTS

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1. INTRODUCTION

Consider a uniform thin beam supported only by two symmetrically placed point supports. Where do we put the supports such that the fundamental frequency, below which the beam cannot vibrate, is maximized? If the supports are rigid, Courant and Hilbert [1] showed that the optimum locations are at the interior nodes of a higher vibration mode without the supports. The situation becomes more complicated when the supports are not perfectly rigid, which often occurs in practice. Szelag and Mroz [2] showed that the optimum location may not be at a higher node, but at a bimodal location. Similar to buckling of beams with elastic support [3], Akesson and Ohloff [4] showed there exists a minimum stiffness of the support above which the fundamental frequency no longer increases. Won and Park [5] suggested a numerical method to study the optimization of multiple supports. The present note is an in-depth study of the fundamental modes of a free-vibrating beam, supported by two symmetrically placed elastic supports. For accuracy, the characteristic equations are obtained analytically.

2. FORMULATION

Let the beam be of length $2L$ and x be the distance from the middle, normalized by L. By letting the transverse displacement be $w(x)\cos(\omega t)$, the governing equation for beam vibration is [6, 7]

$$w''''(x) - \lambda^4 w = 0, \tag{1}$$

where $\lambda^4 =$ (mass per length ρ) L^4(frequency ω^2)/(flexural rigidity D) is the square of the normalized frequency. The supports are at $x = \pm b$ with spring constant c. The ends of the beam are free.

For a symmetric problem, the vibration modes can either be symmetric or antisymmetric. In either case it suffices to consider only half the beam. Let the subscript I denote the segment $0 \leqslant x \leqslant b$ and subscript II denote the segment $b \leqslant x \leqslant 1$. Thus, the solution for segment I is

$$w_{\mathrm{I}}(x) = C_1\cosh(\lambda x) + C_2\cos(\lambda x) \tag{2a}$$

for the symmetric mode, or

$$w_{\mathrm{I}}(x) = C_1 \sinh(\lambda x) + C_2 \sin(\lambda x) \tag{2b}$$

for the antisymmetric mode. For segment II, the conditions of zero moment and zero shear at the end yield the solution

$$w_{\mathrm{II}}(x) = C_3\{\sinh[\lambda(x-1)] + \sin[\lambda(x-1)]\} + C_4\{\cosh[\lambda(x-1)] + \cos[\lambda(x-1)]\}. \tag{3}$$

At the location $x=b$, we require continuity of displacement, slope, moment, but the shear is affected by the spring support.

$$w_{\mathrm{I}}(b) = w_{\mathrm{II}}(b), \qquad w'_{\mathrm{I}}(b) = w'_{\mathrm{II}}(b), \tag{4, 5}$$

$$w''_{\mathrm{I}}(b) = w''_{\mathrm{II}}(b), \qquad w'''_{\mathrm{I}}(b) - \gamma w_{\mathrm{I}}(b) = w'''_{\mathrm{II}}(b). \tag{6, 7}$$

Here, $\gamma = cL^3/D$ is the non-dimensional spring constant of the supports. Equations (2) and (3) are then substituted into equations (4)–(7). For non-trivial solutions, a 4×4 characteristic determinant is set to zero. Using a simple bisection method, the frequency factor λ can be obtained to any accuracy.

3. RESULTS AND DISCUSSIONS

Figure 1 shows the frequency as a function of support location for various constant stiffness. If the supports are rigid, $\gamma = \infty$, and the symmetric mode prevails. The characteristic equation from equations (2a), (3)–(7) is

$$[\tanh(\lambda b) + \tan(\lambda b)]\{\sinh[\lambda(b-1)]\cos[\lambda(b-1)] - \cosh[\lambda(b-1)]\sin[\lambda(b-1)]\}$$
$$- 2\{1 + \cosh[\lambda(b-1)]\cos[\lambda(b-1]\} = 0. \tag{8}$$

When $b=0$, the beam is equivalent to a cantilever with a fundamental frequency of $\lambda = 1{\cdot}8751$ which is the first root of $1 + \cosh(\lambda)\cos(\lambda) = 0$. When $b=1$, the beam is simply supported at both ends, with a frequency of $\lambda = \pi/2$. The maximum frequency of 2·36503

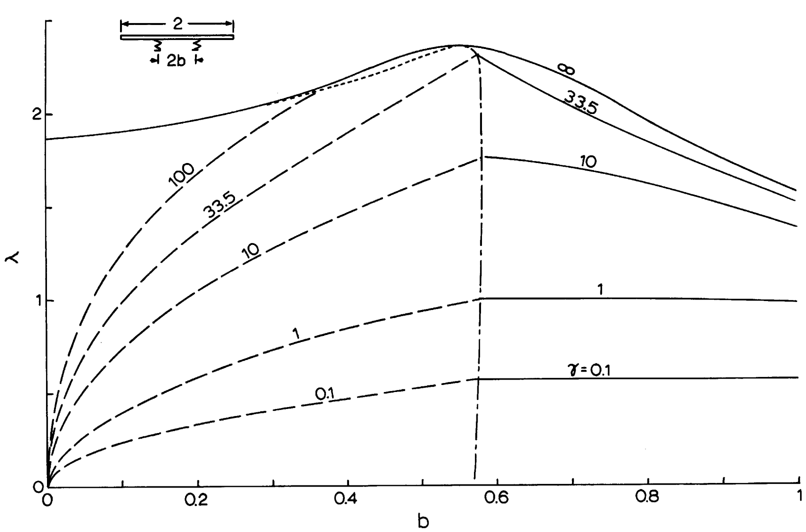

Figure 1. Fundamental frequency factor λ versus support location b for various constant support stiffness γ: ——, symmetric mode; — — —, antisymmetric mode; ---------, boundary of the two modes; —·—·—, locus of optimum location.

occurs at the optimum location of $b = 0{\cdot}55168$ which indeed is at the nodal point of the next vibration mode. But the situation is more complicated when the supports are elastic.

Now if we only consider the symmetric modes, a decrease in stiffness of the supports would decrease the fundamental frequency, except at the location $b = 0{\cdot}55168$, where there is no decrease (from 2·36503) if the stiffness γ is larger than 33·495. However, the antisymmetric mode has a lower fundamental frequency for most low b values. When $b = 0$, the fundamental frequency of the antisymmetric mode is zero, corresponding to a rigid rotation about the beam center. Figure 2 shows the frequency map in more detail. Due to the antisymmetric mode, the minimum stiffness (to maintain the fundamental frequency at 2·36503) is raised to 41·961, above which the optimum location remains at 0·55168. The frequency decreases from 2·36503, when the stiffness is decreased from 41·961. For given $\gamma < 41{\cdot}961$, the maxim frequency occurs at the boundary of symmetric and antisymmetric modes. The optimum locations are given in Table 1.

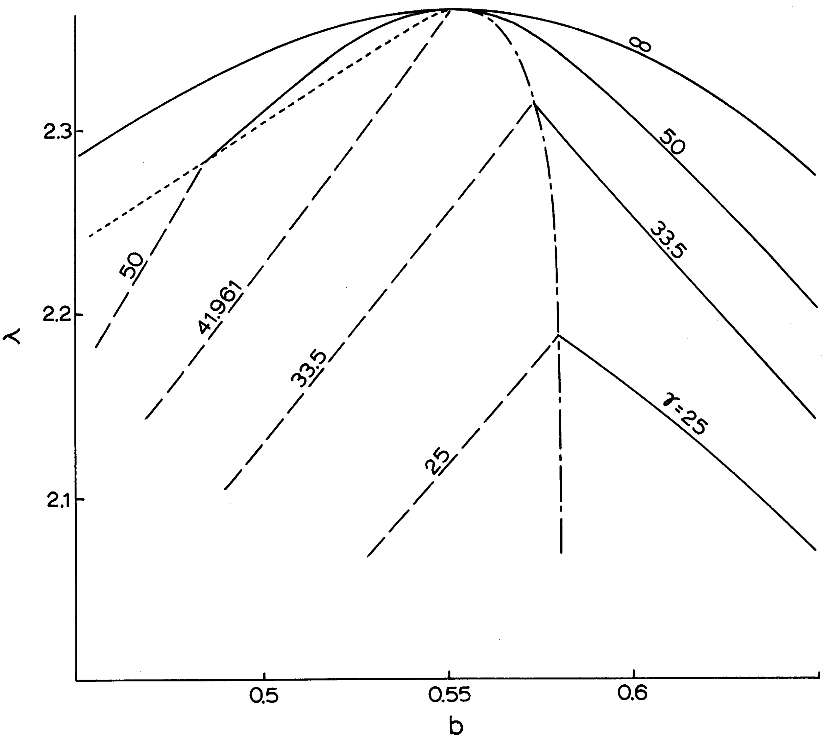

Figure 2. Detail of Figure 1, same legend.

TABLE 1

Optimum locations and the corresponding fundamental frequencies

γ	∞	41·961	33·495	20	10	1	0·1
b	0·5517	0·5517	0·5736	0·5810	0·5810	0·5777	0·5774
λ	2·3650	2·3650	2·3140	2·0667	1·7665	0·9993	0·5623

Vibration of beams with elastic supports is important and there exist a few other papers on this topic. This note shows that some care must be taken to delineate the complex interactions among the various modes and the stiffness of the supports.

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