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ESTIMATING THE COMPLEX TRANSFER FUNCTION OF A NON-LINEAR SYSTEM

M. J. Hinich

Applied Research Laboratories, The University of Texas at Austin, Austin, TX 78713-8029, *U.S.A.*
E-mail: hinich@mail.la.utexas.edu

AND

E. M. A. M. Mendes†

Department of Government, The University of Texas at Austin, Burdine Hall 536*D, Austin, TX* 78712-1087, *U.S.A. E-mail: emendes@mail.la.utexas.edu*

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1. INTRODUCTION

In the literature, the identification of non-linear systems using functional series has been approached in many different ways. Since the earlier work by Wiener [1] great attention has been devoted to find algorithms for estimating the coefficients of the so-called Volterra models [2]. These models are a direct result of the work of Frechet [3] on the theory of functionals. Examples of recent development on the field can be found in references [4, 5] and references therein.

A method is presented in this paper for estimating the linear and quadratic complex transfer function of a *weakly* non-linear system. The input to the system that is excited by a signal especially constructed for the purpose. The excitation signal is a sum of sinusoids with the same amplitude and pseudo-randomly jittered phases that are selected for the experiment and recorded.

To motivate the non-linear frequency-domain linear plus quadratic model presented in section 2, consider the following simple example. Assume that a non-linear system has only one output and the system response satisfies the homogenous non-linearly perturbed differential equation:

$$y''(t) + \lambda y'(t) + ky(t) + \delta y^2(t) = 0, \tag{1}$$

where λ and δ are small positive constants. This equation describes weakly non-linear oscillations of the type found in rotating machinery and in non-linear electric circuits [6, section 5.1]. An approximate solution for the linearized approximation of this equation

†Permanent address: Departamento de Eletricidade, Fundação de Ensino Superior de São João Del Rei, Praça Frei Orlando 170-Centro, São João Del Rei, MG-36307.904, Brazil.

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given a two frequency periodic forcing function $x(t) = \gamma_1 \sin \omega_1 t + \gamma_2 \sin \omega_2 t$ [7, p. 296] is

$$y(t) = -\frac{\delta\gamma_1^2}{2k(k-\omega_1^2)^2} - \frac{\delta\gamma_2^2}{2k(k-\omega_2^2)^2} + \frac{\gamma_1 \sin \omega_1 t}{k-\omega_1^2} + \frac{\gamma_2 \sin \omega_2 t}{k-\omega_2^2} + \frac{\lambda\gamma_1\omega_1 \cos \omega_1 t}{(k-\omega_1^2)^2}$$
$$- \frac{\lambda\gamma_2\omega_2 \cos \omega_2 t}{(k-\omega_2^2)^2} + \frac{\delta\gamma_1^2 \cos 2\omega_1 t}{2(k-\omega_1^2)^2(k-4\omega_1^2)} + \frac{\delta\gamma_2^2 \cos 2\omega_2 t}{2(k-\omega_2^2)^2(k-4\omega_2^2)}$$
$$+ \frac{\delta\gamma_1\gamma_2 \cos(\omega_1-\omega_2)t}{(k-\omega_1^2)(k-\omega_2^2)(k-(\omega_1-\omega_2)^2)} + \frac{\delta\gamma_1\gamma_2 \cos(\omega_1+\omega_2)t}{(k-\omega_1^2)(k-\omega_2^2)(k-(\omega_1+\omega_2)^2)}. \quad (2)$$

This approximate solution is a linear combination of the excitation tones with frequencies ω_1 and ω_2 and the *combination tones* $2\omega_1$, $2\omega_2$, $\omega_1 + \omega_2$, and $\omega_1 - \omega_2$ plus a constant (zero frequency) term. If the excitation has more than two tones, then all the sum and difference frequencies will be in the output signal.

This superposition of combination tones in the output of a nearly linear system is a general result when the excitation contains more than one tone. If the system satisfies an equation similar to equation (1) but with higher non-linear terms in the restoring force, the output may contain sums of all of the input tones rather than only the sum and difference tones.The above assumption is true when weakly non-linear system are investigated and will be used throughout this paper to justify the use of a method for estimating the linear and quadratic transfer functions.

The objective of this paper is two-fold: (1) the estimation of complex transfer functions using the best and simplest algorithm available on the statistical literature and (2) the design of input signals which leads to an "orthogonal design" in a statistical sense.

The combination tones approximation to equation (1) for an excitation containing a number of harmonically related tones is generalized in section 2. In section 3 an input signal is proposed so as to lead to an "orthogonal design" in the statistical sense. Section 4 presents a method for estimating the linear and quadratic transfer functions in the frequency domain. The validity of the proposed method is demonstrated by the examples given in section 5. The main points of the paper are summarized in section 6.

2. LINEAR PLUS QUADRATIC TRANSFER FUNCTIONS

Suppose that a weakly non-linear quadratic system is excited by a bounded periodic signal denoted $x(t)$ whose period is T. Let $y(t)$ denote the measured output of the system. The complex amplitudes of $x(t)$ and $y(t)$ for Fourier frequency $\omega_k = 2\pi k/T$ are the discrete-frequency Fourier transforms:

$$X(k) = \int_{-T/2}^{T/2} x(t) e^{-i\omega_k t}\, dt \quad \text{and} \quad Y(k) = \int_{-T/2}^{T/2} y(t) e^{-i\omega_k t}\, dt, \quad (3)$$

where $X(-k) = X^*(k)$ since $x(t)$ is real and similarly for $Y(-k)$. Let k_0 denote the number of the highest frequency component. The model that will be used to approximate the response of the non-linear system for the excitation $x(t)$ is

$$Y(k) = H(k)X(k) + \sum_{j=k-k_0}^{k_0} Q(j, k-j)X(j)X(k-j) + U(k), \quad (4)$$

where $H(k)$ is the discrete-frequency complex transfer function of the linear part, $Q(k_1, k_2)$ is the complex transfer function of the quadratic part of the system response and $U(k)$ is a lack of fit error term. Note that equation (4) is linear in lagged X's and products of lagged X's.

Taking the inverse double Fourier transform of equation (4) the non-linear system approximation is the quadratic Volterra system [8]

$$y(t) = \int_{s=0}^{\infty} h(s)x(t-s)\,\mathrm{d}s + \int_{r=0}^{\infty}\int_{s=0}^{\infty} q(r,s)x(t-r)x(t-s)\,\mathrm{d}r\,\mathrm{d}s. \qquad (5)$$

Bendat [9] calls the quadratic part of expression (5) a *bilinear system*. Bendat makes illusions to the connection between this "black box" type of system modelling and standard non-linear differential equations but avoids dealing with the important issues of approximation and stability, which are treated in a mathematically rigorous way by Sandberg for continuous time Volterra models [10–12] and discrete-time models [13–15].

For discrete-time measurements of $x(t)$ and $y(t)$ the model in equation (5) is a linear statistical model whose independent variables are $x(t_k - s_k)$ for lags s and $x(t_k - r_k)x(t_k - s_k)$ for lags r and s. Assume that the measurement noise in the input $x(t)$ is negligible and that the variance of the measurement noise in the output $y(t)$ is a known small quantity.

There is no need to accept this solution on blind faith since the transfer functions will be estimated using the type of least-squares statistical methodology commonly applied in economics and biostatistics. If the model is improperly specified the statistical fit will be poor, as defined later. A poor fit is an indication of model misspecification since the measurement noise in the input has been assumed to be negligible and the measurement noise of the output is small.

Assume that $Q(k_1, k_2) = Q(k_2, k_1)$ for all k_1 and k_2 in the band and that $H(k)$ and $Q(k_1, k_2)$ are bounded. Since $Y(-k) = Y^*(k)$ then $Q(k_1, k_2 - k_1) = Q^*(-k_1, -k_2 + k_1)$ or equivalently $Q(k_1, k_2) = Q^*(-k_1, -k_2)$. The symmetries of $Q(k_1, k_2)$ are shown in Figure 1(a). Figure 1(b) shows the symmetries as a function of $u_1 = k_1$ and $u_2 = k_1 + k_2$, which pertain to the Q's in the sum in expression (4).

The sum in expression (5) is twice the sum of $Q(j, k-j)X(j)X(k-j)$ from $j = k/2$ to k_0 for $k = 2, \ldots, k_0$ because of the symmetry of Q around $u_2 = 2u_1$. There are $k_0 - k_2$ such integers in the band where $k_2 = k/2$ if k is even and $k_2 = (k+1)/2$ if k is odd. The term for $k = 1$ is excluded since π/T is not a Fourier frequency.

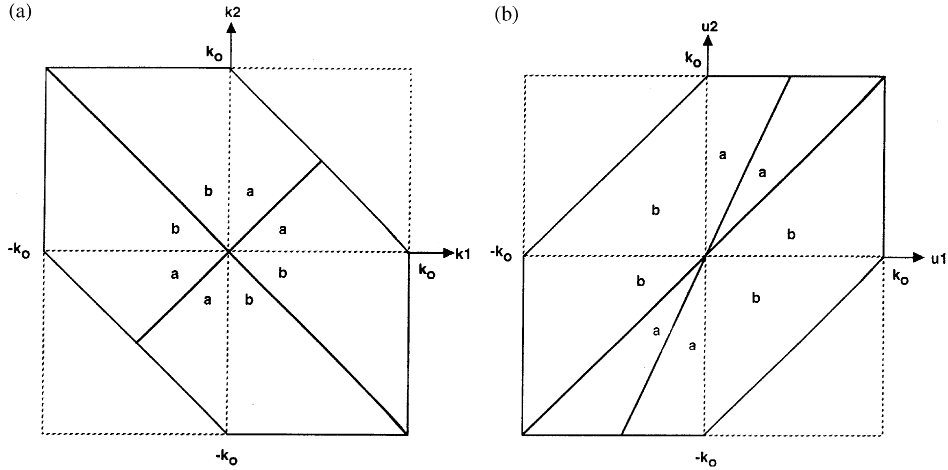

Figure 1. (a) Symmetries of $Q(k_1, k_2)$ and (b) symmetries as a function of $u_1 = k_1$ and $u_2 = k_1 + k_2$.

Let us now deal with the specification of the input signal. There is a method for designing a pseudo-random excitation signal, which will facilitate the identification and estimation of the transfer functions components.

3. CREATING AN INPUT SIGNAL

We employ the statistical theory of least squares to estimate the system's response. When one can control the input to a linear system, the best statistical method is to make the covariance matrix of the independent variables diagonal. This is called an "orthogonal design" in the statistics literature.

Designing an orthogonal excitation signal is part of a measurement methodology for studying a system's response patterns. The approach taken in this paper is to excite the system with a sequence of finite duration waveforms and sample the output during a sequence of frames of length T that is synchronized with the input. The system's response is assumed to be constant over the time span of the experiment.

Each excitation waveform excites the non-linear system. The output signal plus noise is *bandlimited* at the waveform's highest frequency $\omega_0 = 2\pi k_0/T$ and sampled at the Nyquist rate $2k_0/T$ to avoid aliasing. Thus there are $L = 2k_0$ observations in each frame of the input and output. The sampling interval is $\delta = T/2k_0$ and let $L = T/\delta$.

The true midpoint time of frame p is denoted τ_p where $p = 1, \ldots, N$. The input will be a periodic signal if the time between two adjacent midpoints is T. If the frames are taken at times when the system can be measured then the frames can be concatenated into a periodic signal with an artificial time origin. For example, each frame could be taken at a different time over a sequence of days. This is the type of experimental design that is used in biological experiments where a given subject responds to the same stimulus at different times.

The time origin is unimportant for this application. The time index $t_n = n\delta$ for frame p used in the notation from now on is relative to the midpoint time τ_p. The midpoint times will now be suppressed to simplify notation.

A simple method for generating the N excitation waveforms requires a set of pseudo-random variates. Generate $k_0 N$ pseudo-random uniform $(0, T)$ variates using any standard pseudo-random number generator. These generated variates, denoted s_{kp} for $k = 0, 1, \ldots, k_0$ and $p = 1, \ldots, N$, should have the statistical properties of $(k_0 + 1)N$ independently distributed realizations of the rectangular density $1/s$ for $0 \leqslant s \leqslant T$. The waveform for frame p is the following sum of time-shifted cosines for discrete-time points $t_n = n\delta$ for $n = 0, \ldots, L - 1$:

$$x_p(t_n|\mathbf{s}_p) = \cos(s_{0p}/T) + \frac{2}{k_0} \sum_{k=1}^{k_0} \cos \omega_{|\mathrm{k}|}(t_n + s_{kp}), \tag{6}$$

where $\mathbf{s}_p = (s_{0p}, \ldots, s_{k_0 p})$ is the vector of pseudo-random time shifts for frame p.

From now on the adjective "pseudo" will be dropped. The $k_0 N$ variates will be treated as independently distributed uniform $(0, T)$ random variables when expected values and variances are discussed. A purist notation requires the use of capital letters for random variable and lower case letters for their realizations.

The mean of $\cos \omega(t_n + S)$ is zero if S is a uniform $(0, T)$ random variable. The variance of $\cos \omega(t_n - S)$ is $\frac{1}{2}$. Thus the variance of the waveform is $2k_0^{-1} = L^{-1}$ for each frame. An example of two successive frames generated in this manner is shown in Figure 2 for $L = 50$ and $k_0 = 24$.

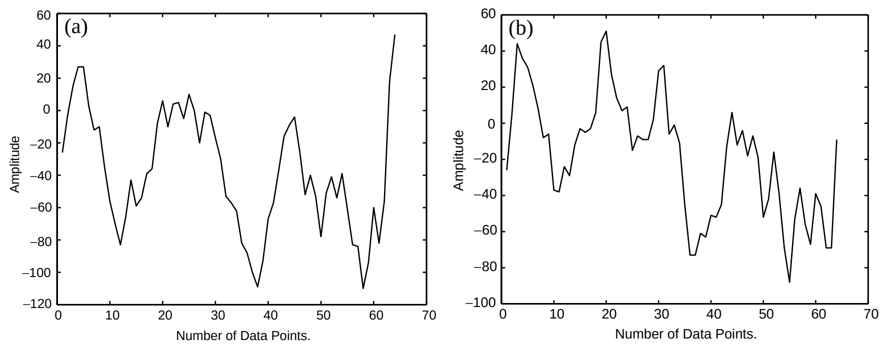

Figure 2. (a) Input and (b) output signals for the first frame of the data collected from the rotating disk experiment described in section 5.2.

The complex amplitude of the waveform $x_p(t_n|\mathbf{s}_p)$ for frequency $\omega_k = 2\pi k/T$ for $k = 1, \ldots, L/2$ frame p is

$$\sum_{n=0}^{L-1} x_p(t_n|\mathbf{s}_p)\mathrm{e}^{-\mathrm{i}\omega_k t_n} = \mathrm{e}^{(\mathrm{i}\omega_k s_{kp})}, \tag{7}$$

which only depends on the delay s_{kp}. The zero frequency complex amplitude is $\exp(\mathrm{i}s_{0p})$.

The product of the complex amplitude for frequencies ω_j and ω_{k-j}, denoted $X_p(j, k-j)$ is

$$X_p(j, k-j) = \mathrm{e}^{\mathrm{i}\omega_j s_{jp}}\mathrm{e}^{\mathrm{i}\omega_{k-j}s_{(k-j)p}} = \mathrm{e}^{\mathrm{i}(\omega_j s_{jp} + \omega_{k-j}s_{(k-j)p})}. \tag{8}$$

Each complex number $\mathrm{e}^{\mathrm{i}\omega_k s_{kp}}$ is a realization of the complex random variable $\mathrm{e}^{\mathrm{i}\omega_k S_{kp}}$, where the S_{kp} are $(k_0 + 1)N$ independently distributed uniform $(0, T)$ random variables. The expected value of $\mathrm{e}^{\mathrm{i}\omega S}$ equals $[\mathrm{e}^{\mathrm{i}\omega T} - 1]/\mathrm{i}\omega$ for a uniform $(0, T)S$, and thus $E\mathrm{e}^{(\mathrm{i}\omega_k S_{kp}} = 0$ for each k and p. The variances of $\mathrm{e}^{(\mathrm{i}\omega_k S_{kp}}$ and $\mathrm{e}^{\mathrm{i}\omega_j S_{jp}}\mathrm{e}^{\mathrm{i}\omega_{k-j}S_{(k-j)p}}$ are equal to one for each j, k and p.

In addition the joint cumulant of $\{\mathrm{e}^{\mathrm{i}\omega_{k_1}S_{k_1 p_1}}, \ldots, \mathrm{e}^{\mathrm{i}\omega_{k_n}S_{k_n p_n}}\}$ is zero for any set of integers $\{k_1, \ldots, k_n\}$ and independently distributed random variables $\{S_{k_1 p_1}, \ldots, S_{k_n p_n}\}$. This implies that (1) $\mathrm{e}^{\mathrm{i}\omega_j S_{jp}}$ is uncorrelated with $\mathrm{e}^{\mathrm{i}\omega_k S_{kp}}$ for all $j \neq k$, (2) $\mathrm{e}^{\mathrm{i}\omega_j S_{jp}}$ and $\mathrm{e}^{\mathrm{i}\omega_l S_{lp}}\mathrm{e}^{\mathrm{i}\omega_{k-l}S_{(k-l)p}}$ are uncorrelated for all j and $k \neq l$, and (3) $\mathrm{e}^{\mathrm{i}\omega_j S_{jp}}\mathrm{e}^{\mathrm{i}\omega_{k-j}S_{(k-j)p}}$ and $\mathrm{e}^{\mathrm{i}\omega_l S_{lp}}\mathrm{e}^{\mathrm{i}\omega_{k-l}S_{(k-l)p}}$ are uncorrelated.

From now on, the distinction between the random variable denoted with a capital letter and the realization denoted by a lower case letter will be dropped. The distinction between a random variable and its realizations is important but it often makes notation cumbersome, as is the case at this point of the exposition.

It then follows from the above that the covariance matrix of the vector

$$\mathbf{v}(k) = (\mathrm{e}^{\mathrm{i}\omega_k s_{kp}}, X_p(k_2, k-k_2), X_p(k_2+1, k-k_2-1), \ldots, X_p(k_0, k-k_0))^{\mathrm{T}} \tag{9}$$

is a $M_k = (k_0 - k_2 + 1)$-dimensional *identity* matrix.

These results for the complex amplitudes of the excitation waveforms are used in the next section to establish the sampling properties of least-squares estimates of the linear and quadratic transfer functions of the non-linear coupling.

4. ESTIMATING THE LINEAR AND QUADRATIC TRANSFER FUNCTIONS

Assume that the system's output contains an additive stationary noise error component denoted by $u(t)$. Recall that the pth excitation frame's complex amplitude for frequency ω_k only depends on the delay s_{kp}. Thus the complex amplitude for frequency ω_k of the pth output frame ω_k only depends on the delay s_{kp}. The complex amplitude of the pth output frame and its additive noise component are the following discrete-time, discrete frequency Fourier transforms:

$$Y_p(k) = \sum_{n=-L/2}^{L/2} y_p(t_n)\mathrm{e}^{\mathrm{i}\omega_k t_n} \quad \text{and} \quad U_p(k) = \sum_{n=-L/2}^{L/2} u(t_n)\mathrm{e}^{\mathrm{i}\omega_k t_n}. \tag{10}$$

Applying the symmetries of Q, model (2) implies that for each $k = 2, \ldots, k_0 = L/2$ and $p = 1, N$,

$$Y_p(k) = H(k)\mathrm{e}^{\mathrm{i}\omega_k s_{kp}} + 2\sum_{j=k/2}^{L/2} Q(j, k-j)X_p(j)X_p(k-j) + U_p(k). \tag{11}$$

Equation (11) for each k is a linear statistical model whose *independent* variables are $\exp(\mathrm{i}\omega_k s_{kp})$ and $2X_p(j)X_p(k-j)$ and whose *dependent* variable is $Y_p(k)$ observed for each frame p. The number of independent variable and parameters is $M_k = L/2 - k_2 + 2$. The parameters for each equation are $H(k)$ and the $Q(j, k-j)$ for $j = k_2, \ldots, L/2$.

To briefly summarize the statistical theory of least squares using standard statistical notation consider the linear system $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{u}$, where $\mathbf{X} = (x_{pj})$ is a $N \times M$ matrix whose columns are linearly independent vectors of complex valued measurements of M independent variables. The elements of vector $\mathbf{y} = (y_1, \ldots, y_N)^{\mathrm{T}}$ are complex-valued measurements of the dependent variable. The elements of the vector $\mathbf{u} = (u_1, \ldots, u_N)^{\mathrm{T}}$ are complex-valued model errors, and $\mathbf{b} = (b_1, \ldots, b_M)^{\mathrm{T}}$ is a vector of parameters. In statistical terminology N is called the *sample size*. The least-squares estimate of $\mathbf{b}$ is $\hat{\mathbf{b}} = (N^{-1}\mathbf{X}^*\mathbf{X})^{-1}(N^{-1}\mathbf{X}^*\mathbf{y})$ where $\mathbf{X}^*$ denotes the transpose and complex conjugate of the matrix $\mathbf{X}$. One standard reference on the theory of least-squares fitting of linear statistical models for real-valued variates is reference [16].

Assuming the errors u_p are uncorrelated and have the same variance denoted by σ_u^2 then $E(\hat{\mathbf{b}}) = \mathbf{b}$ and the covariance matrix of $\hat{\mathbf{b}}$ is $N^{-1}\sigma_u^2(N^{-1}\mathbf{X}^*\mathbf{X})^{-1}$ [16, section 2.3]. The distribution of $\sqrt{N}(\hat{\mathbf{b}} - \mathbf{b})$ is asymptotically multivariate Gaussian as $N \to \infty$.

If the elements of $\mathbf{X}$ are realizations from NM independently and identically distributed bounded random variables with zero means and unit variances then $E(N^{-1}\mathbf{X}^*\mathbf{X}) = \mathbf{I}_M$. Thus from the central limit theorem the large sample covariance matrix of $\hat{\mathbf{b}}$ is approximately $N^{-1}\sigma_u^2\mathbf{I}_M$ for large values of N, and is bounded [17, section 30]. The sample variances and covariances quickly converge to their expected values as N grows large since the dependent and independent variables.

The model is identified if $N \geqslant M$. The only reason to have $N > M$ is to have a non-zero vector of *residuals* $\hat{\mathbf{u}} = \mathbf{y} - \mathbf{X}\hat{\mathbf{b}}$ to estimate σ_u^2, the variance of the errors $u(t)$. The mean sum of squared residuals is the standard estimate of the error variance. If σ_u^2 is known then one can have $N = M$ but a prudent approach in this case is to check by using a few more observations than the number of parameters.

Returning to equation (11), the error term in each equation for each frame is $U_p(k)$. To simplify exposition assume that the noise $u(t)$ is a stationary *white* random process. The variance of each $U_p(k)$ is $L\sigma_u^2$. In addition, the expected value of $U_p(k)$ is zero and the correlation between $U_p(k_1)$ and $U_p(k_2)$ is of order $O(L^{-1})$. These statistical results hold even if the noise $u(t)$ is not white as long as it has a bounded spectrum and trispectrum [18].

Since L will be larger than 50 for most applications, the $L/2$ equations are approximately uncorrelated and thus the parameters for each equation can be estimated by the method of least-squares applied equation by equation. The separate fitting of each equation is both computationally and statistically efficient.

A least-squares fit of each equation yields a set of unbiased estimates $\hat{H}(k)$ and $\hat{Q}(j,k-j)$ of the parameters $H(k)$ and $Q(j,k-j)$ for $j=k_2,\ldots,L/2$. The M_k dimensional vector whose elements are $\sqrt{N}[\hat{H}(k)-H(k)]$ and the $\sqrt{N}[\hat{Q}(j,k-j)-Q(j,k-j)]$ is asymptotically multivariate Gaussian with a mean vector of zero as N goes to infinity.

The covariance matrix of the least-squares estimates $H(k)$ and the $Q(j,k-j)$ is $N^{-1}\sigma_u^2$ times the sample covariance matrix of the independent variables. Since the covariance matrix of $\mathbf{v}(k)$ was shown in section 3 to be a diagonal matrix whose diagonal elements are all L, the expected value of the covariance matrix of the independent variables is diagonal with the $(1,1)$ term equal to one and the rest equal to four. Thus the least-squares estimates are approximately uncorrelated, the variances of the $\hat{H}(k)$ are equal to $LN^{-1}\sigma_u^2+O(N^{-3/2})$, and the variances of the $\hat{Q}(j,k-j)$ are equal to $4LN^{-1}\sigma_u^2+O(N^{-3/2})$. The correlations are of order $O(N^{-3/2})$.

This simple result has important implications for identifying and estimating the transfer function terms. Since the noise variance σ_u^2 has been assumed to be small, the least-squares estimates $\hat{Q}(j,k-j)$ will be accurate for all values of L if $N\geqslant M_k=L/2-k_2+2$. The least-squares estimates $\hat{H}(k)$ will be accurate when $N\gg L\sigma_v^2$. Note that the variance of the estimates does not depend on the number of non-zero values of the parameters nor does it depend on the magnitudes of the parameters.

5. EXAMPLES

In this section two examples are given to demonstrate the proposed method.

5.1. SIMULATED EXAMPLE

To illustrate how this least-squares method can identify non-zero quadratic transfer function values, we used a simple model with many terms to generate artificial data examples. The complex-valued least-squares approach was programmed in Fortran 90. The source code is available upon request.

The excitation amplitudes were generated as described above using a standard uniform random number generator. No linear terms were used. The quadratic transfer function values for each k were $Q(j,k-j)=1+i$ for $j=k_2,\ldots,k_2+4$. The rest of the quadratic values were zero. The additive error values were generated using a standard Gaussian random number generator.

Consider the results when $L=1000$, $N=502$, and $\sigma_u^2=10^{-3}$. The noise variance is half of the signal variance since the variance of the waveforms is $1/L$. The program took 41·8 s to generate the data and fit the model and output the results using a Pentium 233 MHz pc. The standard deviation of the difference between the true coefficients and the estimated coefficients is 0·66e − 3 for the real part and 0·67e − 3 for the imaginary part. These standard errors are not statistically significant from each other. The mean differences are not statistically significant from zero, which is consistent with theory. The maximum absolute error is 1·e − 3.

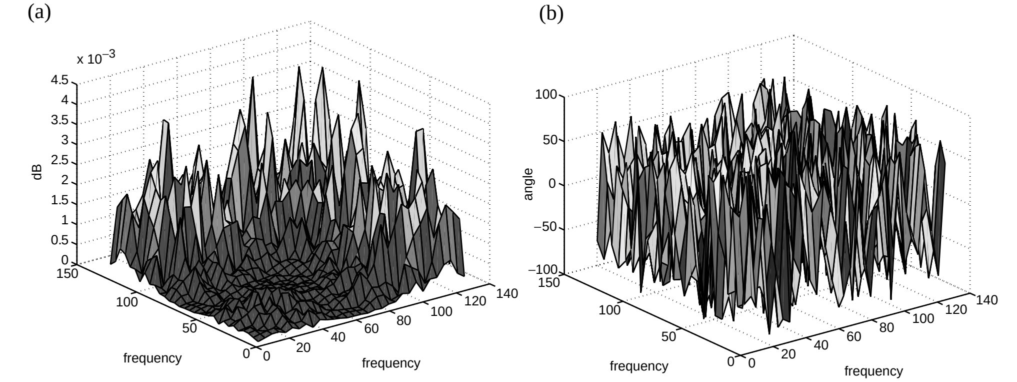

Figure 3. (a) Magnitude and (b) phase for the quadratic transfer function estimated directly from data collected from the rotating disk experiment.

The results for $L = 1000$, $N = 502$ and $\sigma_u^2 = 10^{-1}$ are as above with e − 1 instead of e − 3.

5.2. ROTATING DISK

The complete rotating disk experiment is described in reference [19]. The set-up was composed of a plexiglass tank, with a 760 900 mm^2 cross-section and a depth of 700 mm, mounted on a metal frame. This enclosure houses a disk, 500 mm diameter and 30 mm thick, placed in a horizontal plane 1·70 m from the ground and mounted on a vertical drive shaft. The complete disk/vertical drive shaft assembly is machine-tooled in stainless steel and immersed in water. Measurements in the boundary layer were performed. The data used in this paper included the measurement points corresponding to 60 disk rotations. For more details, please refer to reference [19].

In reference [20] it is shown that Volterra-type models can model the dynamics of the rotating disk for many different Reynolds (Re) numbers. The data set for Re = 380 will be used for identification purposes. The data were divided into 128 frames with 64 points each. Figure 2 shows the input and output signal for the first frame.

The complex linear and quadratic functions were estimated using the method described in section 4. The magnitude and phase for the quadratic transfer function are shown in Figure 3.

6. CONCLUSIONS

A simple least-squares methodology has been presented for estimating the linear and quadratic complex transfer function of a weakly coupled non-linear system. The input to the system is excited by a signal especially constructed for the purpose. The excitation signal is a sum of sinusoids with the same amplitude and pseudo-randomly jittered phases that are selected for the experiment and recorded. The method can easily be generalized to a non-linear coupling model with three way and higher couplings. One needs to add the appropriate products of the waveform amplitudes into the linear statistical model.

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