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NATURAL FREQUENCIES OF ORTHOTROPIC, ELLIPTICAL AND CIRCULAR PLATES

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1. INTRODUCTION

The study of various vibration problems of elliptical and circular plates has a long history, and papers for vibration of the plates were widely reviewed by Leissa [1, 2], and Yamada and Irie [3]. Thus, the papers are not mentioned individually here. Rather, considering the fact that most of the earlier papers treated isotropic and polar orthotropic plates, more recent works for rectilinear orthotropic plates with an elliptical shape are mentioned here.

Irie and Yamada [4] treated circular and elliptical annular plates using the Rayleigh–Ritz method with spline functions as the admissible functions, Tomar and Gupta [5] studied clamped elliptical plates using the Galerkin method with two terms of admissible functions, and Narita [6] analyzed vibration problem of free elliptical plates. Young and Dickinson [7] studied plates with curved edges using the Rayleigh–Ritz method with products of simple polynomials, which is basically the same approach as in the present paper. More recently, Chakraverty and Petyt [8] studied elliptical and circular plates with seven types of orthotropic material properties for all the classical free, simply supported and clamped boundary conditions using the Rayleigh–Ritz method with two-dimensional boundary characteristic orthogonal polynomials as the admissible functions. They presented an exhaustive graphical results of the first five frequencies for various aspect ratios. Chakraverty *et al.* [9] also studied the orthotropic annular elliptic plates. Their study contains results for the first eight frequency parameters for various values of aspect ratios of the outer and inner ellipse.

In this letter, the free vibration problem of elliptical and circular plates is studied using the Rayleigh–Ritz method with products of simple polynomials as the admissible functions. The functions allow one to treat the free, simply supported and clamped boundary conditions simply taking the power of the starting polynomials as 0, 1 and 2, and the integral involved in the analysis is calculated simply by recurrence relationships. It may be worth noting that the generation of the function and the evaluation of the integral are very simple. The analysis is presented for rectilinear orthotropic material and some sample results are presented to demonstrate' the applicability and compared with existing results to show the accuracy. In addition, exhaustive numerical results are tabulated for isotropic plates.

Basically, the same method was used before by the author and some of the results for isotropic plates were presented in Korean [10]. The approach used is similar to that used by Leissa [11], who analyzed simply supported elliptical plates and by Kim *et al.* [12–14] who treated rectangular and triangular plates. In particular, the method used by Narita [6]

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for free elliptical plates can be regarded as a special case of the present approach for free, simply supported and clamped plates.

2. ANALYSIS

Consider a thin elliptical plate which lies in the $x-y$ plane, as shown in Figure 1. The periphery can be expressed as $(x/a)^2+(y/b)^2=1$. It is assumed that the thickness of the plate h is uniform and the material is rectilinear orthotropic. The direction of the orthotropy is assumed to be parallel to the co-ordinate axes. The boundary condition is assumed to be uniform and then the energy of the plate should be four times that of its first quadrant (in positive $x-y$ plane) considering the symmetry of the plate. Therefore, the maximum strain energy and kinetic energy of the plate due to simple harmonic vibration with small amplitude can be written as, respectively,

$$V_{max}=\frac{4}{2}\int_0^a\int_0^{b\sqrt{1-(x/a)^2}}\left[D_x\left(\frac{\partial^2 W}{\partial x^2}\right)^2+2v_{xy}D_y\left(\frac{\partial^2 W}{\partial x^2}\right)\left(\frac{\partial^2 W}{\partial y^2}\right)+D_y\left(\frac{\partial^2 W}{\partial y^2}\right)^2+4D_{xy}\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right]\mathrm{d}y\ \mathrm{d}x,$$

$$T_{max}=\frac{4\rho h\omega^2}{2}\int_0^a\int_0^{b\sqrt{1-(x/a)^2}}W^2\mathrm{d}y\ \mathrm{d}x, \tag{2}$$

where ω is the radian natural frequency, W the deflection amplitude normal to the plate, ρ the material density, and v_{xy} and v_{yx} the Poisson ratios. When Young's moduli in x and y directions are written as E_x and E_y, and shear modulus G_{xy}, the flexural rigidities are given as

$$D_x=E_xh^3/12(1-v_{xy}v_{yx}),\quad D_y=D_xE_y/E_x,\quad D_{xy}=G_{xy}h^3/12.$$

Introducing non-dimensional parameters $\xi=x/a$ and $\eta=y/b$, the deflection amplitude W can be expressed as

$$W=\sum_i\sum_j A_{ij}\quad \xi^{i-1}\ \eta^{j-1}\quad (\xi^2+\eta^2-1)^k, \tag{3}$$

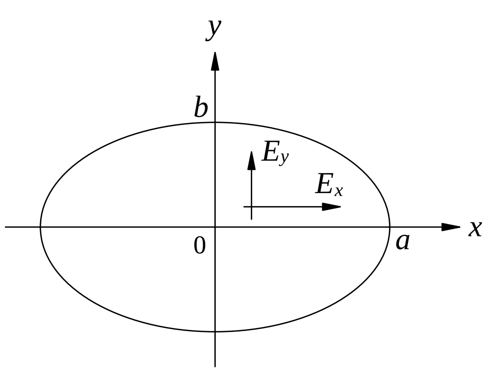

Figure 1. Orthotropic elliptical plate.

where A_{ij} are constants yet undetermined, k a constant depending upon the boundary conditions, i.e., $k=0$ for free, $k=1$ for simply supported and $k=2$ for clamped, and i and j are 1, 2, 3, $\cdots$ neglecting symmetry. When the symmetry of the plate about the co-ordinate axes is considered, symmetrical and antisymmetrical modes can be separated simply by taking odd numbers (1, 3, 5, $\cdots$) for symmetrical modes and even numbers (2, 4, 6, $\cdots$) for antisymmetrical modes.

Minimizing the frequency ω with respect to A_{ij} according to the Rayleigh–Ritz method, after substituting the non-dimensional parameters ξ and η, and equation (3) into energy expressions (1) and (2), yields the eigenvalue equation

$$\sum_m \sum_n [C_{ijmn} - \Omega^2 E_{ijmn}^{(0000)}]A_{mn} = 0,$$

where $\Omega^2 = \rho h \omega^2 a^4/H; m, n, i, j = 1, 2, 3, \cdots,$

$$E_{ijmn}^{(pqrs)} = \int_0^1 \int_0^{\sqrt{1-\xi^2}} \left[\frac{\partial^{(p+q)}}{\partial \xi^p \partial \eta^q} \left\{ \xi^{i-1} \eta^{j-1} (\xi^2+\eta^2-1)^k \right\} \times \frac{\partial^{(r+s)}}{\partial \xi^r \partial \eta^s} \left\{ \xi^{m-1} \eta^{n-1} (\xi^2+\eta^2-1)^k \right\} \right] \mathrm{d}\eta\ \mathrm{d}\xi,$$

$$c_{ijmn} = \frac{D_x}{H} E_{ijmn}^{(2020)} + \frac{D_y}{H}\left(\frac{a}{b}\right)^4 E_{ijmn}^{(0202)} + \left(1 - 2\frac{D_{xy}}{H}\right)\left(\frac{a}{b}\right)^2 \left\{ E_{ijmn}^{(2002)} + E_{ijmn}^{(0220)} \right\} + 4\frac{D_{xy}}{H}\left(\frac{a}{b}\right)^2 E_{ijmn}^{(1111)} \tag{4}$$

and $H = v_{xy}\ D_y + 2D_{xy}$. Since the isotropic plate is a special case of the orthotropic plate, equation (4) can be applied to the isotropic plate writing $v_{xy} = v_{yx} = v$, $D_x = D_y = H = D$ and $D_{xy} = (l - v)D/2$. Non-trivial solution of equation (4) yields the natural frequencies of the plate and the corresponding coefficients A_{ij}. Then, substitution of coefficients A_{ij} into equation (3) gives the mode shapes.

The integral values involved in equation (4) can be calculated easily as follows. Let

$$F(\alpha, \beta, \gamma) = \int_0^1 \int_0^{\sqrt{1-\xi^2}} \xi^{\alpha-1} \eta^{\beta-1} (\xi^2+\eta^2-1)^{\gamma-1} \quad \mathrm{d}\eta\ \ \mathrm{d}\xi, \tag{5}$$

$$\alpha, \beta, \gamma = 1, 2, 3, \cdots,$$

then

$$F(\beta, \alpha, \gamma) = \mathrm{F}(\alpha, \beta, \gamma), \tag{6}$$

$$F(\alpha, \beta, 1) = \int_0^1 \int_0^{\sqrt{1-\xi^2}} \xi^{\alpha-1} \eta^{\beta-1}\ \mathrm{d}\eta\ \ \mathrm{d}\xi = \int_0^1 \xi^{\alpha-1} \times \frac{1}{\beta} \eta^\beta \Big|_0^{\sqrt{1-\xi^2}} \mathrm{d}\xi = \frac{1}{\beta}\int_0^1 \xi^{\alpha-1}(1-\xi^2)^{\beta/2}\ \ \mathrm{d}\xi. \tag{7}$$

Substituting 1 and 2 into α, β into equation (7) yields

$$F(1,1,1) = \int_0^1 (1-\xi^2)^{1/2}\ \ \mathrm{d}\xi = \frac{\pi}{4}, \quad F(1,2,1) = \frac{1}{2}\int_0^1 (1-\xi^2)\ \ \mathrm{d}\xi = \frac{1}{3}, \tag{8a, b}$$

$$F(2,1,1) = F(1,2,1) = \frac{1}{3}, \quad F(2,2,1) = \frac{1}{2}\int_0^1 \xi(1-\xi^2)\ \ \mathrm{d}\xi = \frac{1}{8}. \tag{8c, d}$$

After some arithmetic procedure utilizing the integration by parts, the following recurrence relationships can be obtained:

$$F(\alpha+2,\beta,1)=\frac{\alpha}{\alpha+\beta+2}F(\alpha,\beta,1),\quad F(\alpha,\beta+2,1)=\frac{\beta}{\alpha+\beta+2}F(\alpha,\beta,1) \qquad (9,10)$$

$$F(\alpha,\beta,\gamma+1)=-\frac{2\gamma}{\alpha+\beta+2\gamma}F(\alpha,\beta,\gamma). \qquad (11)$$

Therefore, the integral values can be calculated using these recurrence relationships upon starting with the values in equations (8).

In addition, if the thickness of the plate is not uniform, h in kinetic energy expression (2) should be entered into the integral. Then, the approach in this letter can be applied for the case without any special difficulty when the variation of the thickness can be expressed as polynomials.

3. RESULTS AND DISCUSSION

The elliptical plate treated in this letter has four kinds of modes. They are SS, SA, AS and AA modes, where S and A denote symmetrical and antisymmetrical modes,

TABLE 1

Frequency parameters $(\rho h\omega^2 a^4/H)^{1/2}$ *for orthotropic, clamped, elliptical plates* (E_x=1·87 × 10^6 *psi*, E_y=0·60 × 10^6 *psi*, $v_{xy}E_y$ = 0·073 × 10^6 *psi*, G_{xy}, = 0·159 × 10^6 *psi*)

	No. of terms	Mode type								
b/a		SS-1	SS-2	SS-3	SA-1	SA-2	AS-1	AS-2	AA-1	AA-2
1·0	2 × 2	16·490	49·910	76·977	28·527	78·803	39·154	78·232	53·095	109·67
	3 × 3	16·488	47·191	73·213	28·508	71·936	39·121	73·176	52·987	99·356
	4 × 4	16·488	47·078	73·076	28·508	71·484	39·121	72·866	52·985	98·490
	5 × 5	16·488	47·077	73·074	28·508	71·473	39·121	72·858	52·985	98·459
	6 × 6	16·488	47·077	73·074	28·508	71·473	39·121	72·858	52·985	98·459
	2 × 1	16·650	78·167		29·305	97·726	39·834	131·87	55·208	155·58
	1 × 2	16·549	50·415		28·654	80·909	39·881	79·900	54·029	113·22
	Reference [5]	16·534	69·682							
2/3	6 × 6	23·439	78·044	95·890	52·035	110·91	45·209	116·27	76·017	154·50
	Reference [5]	23·453	98·839							
0·5	6 × 6	35·185	88·518	162·98	86·521	147·34	56·100	131·86	111·86	192·94
	Reference [5]	35·394	149·16							
2/5	6 × 6	51·165	104·02	200·53	130·97	196·27	72·210	146·81	158·92	243·10
	Reference [5]	51·931	218·85							
1/3	2 × 2	71·029	139·15	376·06	186·09	299·95	93·337	194·93	219·25	376·93
	3 × 3	70·921	125·99	270·65	185·16	262·86	92·967	169·92	216·38	317·47
	4 × 4	70·919	124·94	220·86	185·09	256·27	92·958	167·31	216·14	305·38
	5 × 5	70·919	124·94	220·50	185·09	256·24	92·958	167·31	216·14	305·31
	6 × 6	70·919	124·94	220·50	185·09	256·24	92·958	167·31	216·14	305·31
	2 × 1	71·226	139·57		189·48	303·48	93·536	195·82	222·62	381·35
	1 × 2	73·288	388·08		193·25	660·96	99·615	455·70	239·09	749·30
	Reference [5]	72·717	306·45							

respectively, and the first letters are for the symmetry about y-axis, while the second about x-axis. The symmetrical and antisymmetrical modes can be obtained separately, as mentioned before, simply taking odd or even numbers for i and j in equation (3) respectively (i, m and j, n in frequency equation (4) respectively). When the odd and even numbers are used together, the results become identical as a matter of course, and the odd number terms do not contribute to the antisymmetrical modes and even number terms do not contribute to the symmetrical modes. Hereafter, the number of terms indicated in the text and all the tables denote the number of contributing terms in x and y directions. Moreover, the Poisson ratio is taken as 0·3 for the isotropic plates.

In order to illustrate the applicability of the approach to the rectilinear orthotropic plate, circular and elliptical plates with clamped periphery are considered for several aspect ratios. The lowest three frequency parameters for the SS mode and two for each of the other kind of modes are presented in Table 1 taking the orthotropic property used by Tomar and Gupta [5] for comparison purpose. It may be mentifoned that the nine frequency parameters presented are not necessary the lowest nine frequency parameters. For instance, the frequency parameter for SA-3 mode of the circular plate ($b/a = 1$) is 90·278 which is lower than AA-2 mode (98·459) presented. The study for the rate of convergence is also included in the table using an increased number of terms for aspect ratios $b/a = 1$ and 1/3. It may be seen that the present results for SS-1 mode are in close

TABLE 2

Frequency parameters $(\rho h \omega^2 a^4/H)^{1/2}$ *for orthotropic, elliptical plates* $(D_x/H = 2{\cdot}0$, $D_y/H = 0{\cdot}5$, $\nu_{xy} = 0{\cdot}3$, $b/a = 0{\cdot}5)$

No. of terms	Mode type								
	SS-1	SS-2	SS-3	SA-1	SA-2	AS-1	AS-2	AA-1	AA-2
(1) *Free*									
2×2	10·628	23·908	68·502	29·818	72.924	28·671	47·661	11·599	55·879
3×3	9·5964	20·967	47·989	25·581	53·974	23·830	39·109	11·547	44·633
4×4	9·5451	20·772	43·198	25·266	52·382	23·453	38·318	11·547	43·530
5×5	9·5451	20·771	42·336	25·263	52·365	23·450	38·309	11·547	43·512
6×6	9·5451	20·771	42·320	25·263	52·365	23·450	38·309	11·547	43·512
7×7	9·5451	20·771	42·319	25·263	52·365	23·450	38·309	11·547	43·512
8×8	9·5451	20·771	42·319	25·263	52·365	23·450	38·309	11·547	43·512
(2) *Simply Supported*									
2×2	10·339	58·689	103·80	34·959	106·30	23·772	107·96	54·618	167·93
3×3	10·326	43·945	74·406	34·732	80·303	23·628	71·481	53·828	113·16
4×4	10·326	43·186	73·217	34·730	78·285	23·627	68·941	53·812	108·25
5×5	10·326	43·171	73·196	34·730	78·206	23·627	68·847	53·812	107·96
6×6	10·326	43·171	73·196	34·730	78·205	23·627	68·846	53·812	107·96
7×7	10·326	43·171	73·196	34·730	78·205	23·627	68·846	53·812	107·96
(3) *Clamped*									
2×2	22·199	64·480	103·43	52·965	109·07	38·157	98·997	74·519	149·71
3×3	22·195	60·688	98·468	52·926	101·79	38·125	89·849	74·360	135·09
4×4	22·195	60·513	98·272	52·926	101·29	38·125	89·199	74·358	133·74
5×5	22·195	60·510	98·270	52·926	107·27	38·125	89·181	74·358	133·68
6×6	22·195	60·510	98·270	52·926	107·27	38·125	89·181	74·358	133·68

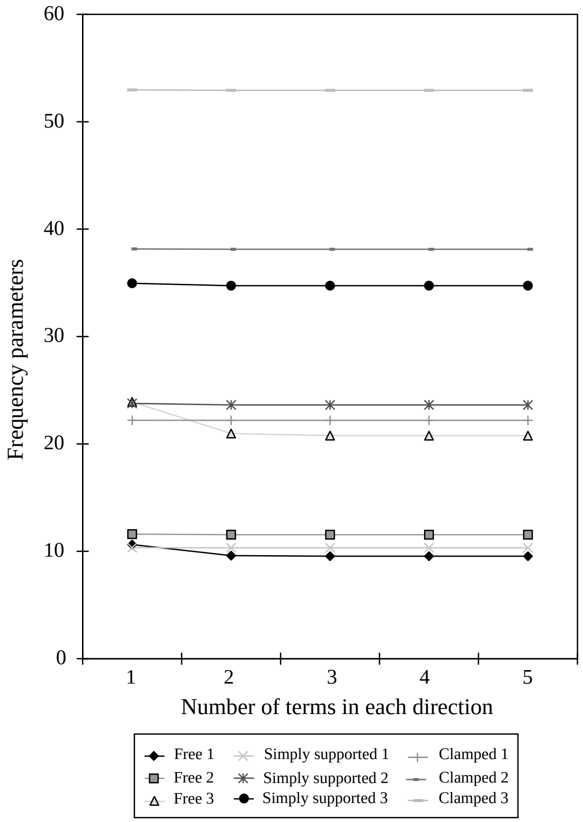

Figure 2. Convergence pattern of the lowest three frequency parameters in Table 2.

agreement with comparison values obtained from reference [5], while the results for SS-2 mode are not in accord. It is believed that the comparison values for SS-2 mode are not reasonably accurate. The inaccuracy of the values is understandable considering the fact that the values in the reference were obtained by using only two fully symmetrical terms. In order to show the inaccuracy for the second modes of two term solution, some present results for 2×1 and 1×2 term solutions are included in the table.

In Table 2, convergence test for all the classical free, simply supported and clamped boundary conditions are presented for the material with $D_x/H{=}2{\cdot}0$, $D_y/H = 0{\cdot}5$, $\nu_{xy} = 0{\cdot}3$, taking aspect ratio (b/a) 0·5. For the lowest three frequency parameters, a convergence pattern is shown in Figure 2. The pattern will be changed depending upon the material property and/or aspect ratio. The converged values for the parameters seem to be in close agreement with those in reference [8], where the convergence pattern for the cases were presented in a graphical form.

Comparison of fundamental frequency parameters for orthotropic, clamped, elliptical plates for several material properties are shown in Table 3. The values of references [15–17] in the table were those calculated by Chakraverty and Petyt [8] using the equations in the references. Close agreement may be seen to exist for all the cases, particularly with the most recent results in reference [8].

In Table 4, the frequency parameters for graphite/epoxy and carbon/epoxy are presented. The orthotropic properties of the material were obtained from reference [18]

TABLE 3

Comparison of fundamental frequency parameters $(\rho h \omega^2 a^4/H)^{1/2}$ for orthotropic, clamped elliptical plates

b/a	References				
	Present	Reference [15]	Reference [16]	Reference [17]	Reference [8]
(1) *Glass/epoxy* ($D_x/H = 3{\cdot}75$, $D_y/H = 0{\cdot}80$, $v_{xy} = 0{\cdot}26$)					
0·2	136·29	147·24	144·52	143·11	136·51
0·4	38·796	40·383	39·637	39·249	38·849
0·5	27·345	28·280	27·757	27·485	27·376
0·6	21·457	22·131	21·722	21·509	21·476
0·8	16·234	16·742	16·433	16·272	16·242
1·0	14·222	14·720	14·448	14·306	14·225
(2) *Boron/epoxy* ($D_x/H = 13{\cdot}34$, $D_y/H = 1{\cdot}21$, $v_{xy} = 0{\cdot}23$)					
0·2	168·60	180·60	177·27	175·53	168·53
0·4	50·169	51·841	50·883	50·385	50·152
0·5	37·193	38·308	37·601	37·232	37·181
0·6	30·975	31·904	31·315	31·008	30·966
0·8	25·913	26·823	26·327	26·069	25·907
1·0	24·062	25·130	24·666	24·424	24·057
(3) *Carbon/epoxy* ($D_x/H = 15{\cdot}64$, $D_y/H = 0{\cdot}91$, $v_{xy} = 0{\cdot}32$)					
0·2	148·56	158·10	155·18	153·66	148·67
0·4	46·477	47·966	47·079	46·618	46·501
0·5	35·864	36·953	36·270	35·915	35·876
0·6	30·931	31·909	31·319	31·012	30·937
0·8	26·958	28·016	27·498	27·229	26·959
1·0	25·443	26·735	26·241	25·984	25·442
(4) *Kevlar* ($D_x/H = 2{\cdot}60$, $D_y/H = 2{\cdot}60$, $v_{xy} = 0{\cdot}14$)					
0·2	236·60	261·08	256·26	253·75	236·43
0·4	63·123	67·016	65·778	65·133	63·077
0·5	42·074	44·080	43·266	42·842	42·043
0·6	30·688	31·880	31·291	30·984	30·666
0·8	19·740	20·351	19·975	19·779	19·726
1·0	15·152	15·596	15·308	15·158	15·142

and the number of terms used are 9×9 terms to obtain sufficiently accurate results for various values of aspect ratio without considering the rate of convergence.

The convergence study for isotropic, clamped plates are presented in Table 5, together with the comparison values by Shibaoka [19], McNitt [17], Tomar and Gupta [5], Singh and Chakraverty [20], Rajalingham *et al.* [21], and Carrington [22]. The most accurate values among the present results are fully converged values up to the figures. Close agreement may be seen to be achieved except the values for AS modes with those in reference [21]. For AS-1 mode, the present result is in close agreement with that in reference [20]. The author cannot explain the discrepancy between the present results and those in reference [21]. It may be noted that the rate of convergence is relatively slower for smaller values of *b/a*. The rate can be increased, with the same number of total terms, for smaller *b/a* by using more terms in the x direction than in the y direction rather than using

TABLE 4

Frequency parameters $(\rho h \omega^2 a^4/H)^{1/2}$ for orthotropic, clamped, elliptical plates

b/a	Mode type								
	SS-1	SS-2	SS-3	SA-1	SA-2	AS-1	AS-2	AA-1	AA-2
(1) *Graphite/epoxy* ($D_x/H = 10·548$, $D_y/H = 0·60023$, $D_{xy}/H = 0·41597$)									
1/3	50·252	130·09	230·58	122·66	213·58	81·654	194·55	160·56	281·58
1/2·5	38·596	119·69	164·43	88·687	177·36	70·726	182·01	125·05	243·89
1/2	29·883	106·02	116·48	61·318	148·30	63·010	146·58	97·382	203·48
1/1·5	24·221	67·351	108·38	41·202	101·82	57·935	106·12	77·872	142·41
1	21·101	40·105	74·232	28·771	55·235	54·533	78·898	65·096	95·958
1·5	19·828	29·383	44·779	23·887	36·319	52·787	66·458	58·931	75·319
2	19·320	25·566	34·919	22·045	29·849	52·022	61·418	56·327	67·246
2·5	19·046	23·633	30·175	21·082	26·662	51·592	58·711	54·896	63·001
3	18·876	22·475	27·433	20·494	24·789	51·317	57·033	53·993	60·411
(2) *Carbon/epoxy* ($D_x/H = 15·637$, $D_y/H = 0·91160$, $D_{xy}/H = 0·35642$)									
1/3	60·795	155·41	280·95	149·08	253·45	97·451	233·38	191·80	333·93
1/2·5	46·501	143·49	199·72	107·41	210·22	84·407	219·52	148·69	290·14
1/2	35·877	128·85	139·52	73·839	176·76	75·402	174·91	115·37	244·40
1/1·5	29·081	81·089	130·54	49·271	123·38	69·643	125·76	92·382	169·44
1	25·443	47·888	89·326	34·364	66·178	65·832	93·710	113·78	113·78
1·5	23·977	35·122	53·539	28·643	43·385	63·886	79·313	70·693	89·606
2	23·394	30·620	41·733	26·498	35·689	63·041	73·530	67·760	80·238
2·5	23·084	28·351	36·083	25·383	31·916	62·572	70·448	66·160	75·349
3	22·892	26·999	32·831	24·706	29·709	62·274	68·555	65·159	72·391

TABLE 5

Frequency parameters $(\rho h \omega^2 a^4/D)^{1/2}$ for isotropic, clamped, elliptical plates

No. of terms	Mode type								
	SS-1	SS-2	SS-3	SA-1	SA-2	AS-1	AS-2	AA-1	AA-2
(1) $b/a = 0·5$									
2 × 2	27·394	61·319	139·40	69·996	123·46	39·590	88·517	88·601	160·44
3 × 3	27·378	56·320	124·40	69·862	111·45	39·499	78·013	88·071	139·10
4 × 4	27·377	55·985	105·16	69·858	110·03	39·497	77·037	88·048	135·99
5 × 5	27·377	55·976	102·80	69·858	109·94	39·497	76·996	88·047	135·72
6 × 6	27·377	55·976	102·65	69·858	109·94	39·497	76·995	88·047	135·71
Reference [19]	27·5								
Reference [17]	27·746								
Reference [5]	27·746								
Reference [20]	27·377	55·985		69·858		39·497			
Reference [21]	27·377	55·976	102·65	69·858	109·94	48·077	11·27	88·047	135·71
(2) $b/a = 1$									
2 × 2	10·217	33·661	42·097	21·272	54·088	The same as SA		34·922	75·895
3 × 3	10·216	34·938	39·874	21·260	51·172			34·877	70·032
4 × 4	10·216	34·878	39·773	21·260	51·032			34·877	69·674
5 × 5	10·216	34·877	39·771	21·260	51·030			34·877	69·666
Reference [17]	10·217								
Reference [5]	10·217								
Reference [20]	10·216	34·878	39·773	21·260					
Exact [22]	10·216	34·88	39·771	21·26	51·04			34·88	69·666

the same number of terms both in the x and y directions (e.g., using 9×4 terms rather than using 6×6 terms). This is due to the fact that the length of the plate in the x direction is longer than that in the y direction and accordingly the number of waves contributed to the x direction is more than to the y direction for lower modes. For circular plates ($b/a = 1$), Carrington [22] obtained exact solutions which are given as a Bessel function as early as in 1925. The present results are the same as the exact solutions up to the figures given, with an exception of the result for SA-2 mode. The present lower value for the mode, which is the same as the upper bound in reference [23], seems to be strange because the Rayleigh–Ritz method gives an upper bound. The author calculated the exact solution using double precision to inspect the accuracy of the values presented by Carrington [22] and found that the exact solution is the same as the converged values presented here. It is believed that the value for SA-2 mode in reference [22] has a small numerical error which occurred in the calculation of Bessel function. (There was no computer in 1925.) Further, it may be noted that the values for SS-2 and AA-1 modes are the same, and in fact they can be said to be the same mode since both are modes having two nodal diameters perpendicular to each other and thus turning one mode 45° around the centre of the plate becomes the other. This kind of phenomenon can be observed not only for this case, but also for all the modes with at least one nodal diameter in axisymmetrical plates (axisymmetrical about the centre) with a uniform boundary condition.

The frequency parameters for isotropic plates were presented by previous investigators. In particular, Rajalingham *et al.* [21] presented the lowest six frequency parameters in each mode type for aspect ratio 0·5 for all the classical free, simply supported and clamped

TABLE 6

Frequency parameters $(\rho h\, \omega^2 a^4/D^{1/2}$ *for isotropic, free, elliptical plates* $(\nu = 0{\cdot}3)$

b/a	Mode type									
	SS-1	SS-2	SS-3	SA-1	SA-2	AS-1	AS-2	AS-3	AA-1	AA-2
0·2	6·7778	32·817	78·123	48·582	101·72	17·389	53·058	108·01	25·747	73·638
0·25	6·7737	32·779	77·901 101·22	39·621	86·070	17·377	52·955	107·60	20·676	61·181
0·3	6·7654	32·675	71·773 77·451	33·710	75·839	17·343	52·718	95·362	17·301	53·015
0·35	6·7518	32·503	53·775	29·520	68·592	17·284	52·345	74·353	14·891	47·243
0·4	6·7321	32·257	41·937	26·393	63·137	17·195	51·824	60·261	13·084	42·928
0·45	6·7054	31·928	33·719	23·963	58·824	17·076	50·267	51·179	11·677	39·558
0·5	6·6705	27·768	31·513	22·015	55·272	16·921	42·991	50·276	10·548	36·828
0·55	6·6264	23·325	30·987	20·411	52·241	16·726	37·480	49·183	9·6206	34·547
0·6	6·5712	19·922	30·337	19·061	49·565	16·484	33·239	47·852	8·8447	32·590
0·65	6·5029	17·264	29·551	17·901	43·290	16·185	29·938	46·289	8·1848	30·868
0·7	6·4185	15·160	28·626 43·750	16·888	37·800	15·822	27·357	44·538	7·6161	29·320
0·75	6·3144	13·478	27·580 41·595	15·988	33·366	15·386	25·345	42·669	7·1203	27·898
0·8	6·1861	12·128	26·449 39·961	15·174	29·743	14·880	23·790	40·755	6·6841	26·570
0·85	6·0286	11·048	25·277 38·695	14·427	26·755	14·311	22·599	38·852	6·2969	25·311
0·9	5·8381	10·191	24·101	13·731	24·274	13·699	21·694	36·997	5·9509	24·107
0·95	5·6135	9·5199	22·949	13·072	22·205	13·068	21·005	35·209	5·6397	22·949
1·0	5·3583	9·0031	21·835	the same as AS		12·439	20·475	33·495	5·3583	21·835

boundary conditions, and Singh and Chakraverty presented the first four frequency parameters for free [24], simply supported [25] and clamped [20] plates with various aspect ratios. Nevertheless, it may be worth presenting exhaustive accurate results for all the classical boundary conditions classifying the mode types, since the values in the literature are limited to the first several (five or less) frequency parameters without distinguishing the mode types [20, 24, 25, etc.] or to the results for each mode type for a special aspect ratio [21]. In Tables 6–8, the lowest 10 frequency parameters for various aspect ratios are presented for isotropic plates with free, simply supported and clamped boundary conditions respectively. The values are categorized with the mode types. However, the order of the mode types are changed depending upon the boundary conditions and/or the aspect ratio. Thus, when the lowest ten frequency parameters are not consisted with the mode types of the heading, additional values are attached at the bottom. For an example, the tenth frequency parameter for aspect ratio $b/a = 0{\cdot}25$ in Table 6 is 101·22 which is SS-4 mode. A sufficient number of terms depending upon the aspect ratio were used to obtain the values and the values presented are thus believed to be fully converged values up to the figures shown. The lowest four frequency parameters are in good agreement with those in references [20, 24, 25]. However, the values for AS modes of the plates with simply supported and clamped boundary conditions are fully different from those in reference [21] (the case for free boundary condition is in good agreement). In addition, it may be worth noting that the exact solution exists for circular plates ($b/a = 1$) and the values presented are the same as the exact solution available in the literature [24–27, etc.]. Further, the

TABLE 7

Frequency parameters $(\rho h\,\omega^2 a^4/D)^{1/2}$ for isotropic, simply supported, elliptical plates

b/a	Mode type									
	SS-1	SS-2	SS-3	SS-4	SA-1	SA-2	AS-1	AS-2	AS-3	AA-1
0·2	69·680	111·28	167·48	239·66	262·63	335·41	88·761	137·46	201·50	297·40
0·25	45·917	81·510	132·26	199·77	170·69	231·18	61·944	104·87	163·84	199·28
0·3	32·813	64·503	112·00	177·00	120·38	172·77	46·828	86·145	142·24	144·89
0·35	24·793	53·843	99·384	162·95	89·823	136·52	37·412	74·424	128·86	111·45
0·4	19·514	46·757	91·077	150·99	69·854	112·37	31·146	66·672	120·07	89·357
										139·09
0·45	15·853	41·845	85·338	120·61	56·071	95·406	26·782	61·320	113·96	73·952
0·5	13·213	38·326	81·190	98·787	46·150	83·013	23·641	57·482	109·47	62·764
										107·10
0·55	11·253	35·729	78·053	82·578	38·767	73·669	21·319	54·628	104·16	54·374
										96·833
0·6	9·7629	33·760	70·209	75·562	33·122	66·441	19·566	52·429	90·566	47·917
										88·843
0·65	8·6087	32·229	60·559	73·468	28·709	60·727	18·215	50·670	79·939	42·842
0·7	7·7007	31·005	52·893	71·576	25·192	56·120	17·157	49·196	71·509	38·781
0·75	6·9769	29·998	46·712	69·700	22·347	52·339	16·314	47·889	64·769	35·483
0·8	6·3935	29·139	41·677	67·650	20·012	49·177	15·634	46·633	59·393	32·769
0·85	5·9185	28·364	37·556	65·273	18·074	46·477	15·077	45·299	55·194	30·509
						63·626				
0·9	5·5282	27·600	34·208	62·573	16·448	44·115	14·615	43·750	52·076	28·608
						57·580				
0·95	5·2049	26·734	31·585	59·711	15·072	41·978	14·227	41·934	49·917	26·995
						52·576				
1·0	4·9351	25·613	29·720	56·842	the same as AS		13·898	39·957	48·479	25·613

TABLE 8

Frequency parameters $(\rho h \omega^2 a^4 / D)^{1/2}$ for isotropic, clamped, elliptical plates

b/a	Mode type									
	SS-1	SS-2	SS-3	SS-4	SA-1	SA-2	AS-1	AS-2	AS-3	AA-1
0·2	149·64	195·88	256·71	333·21 426·38	403·40	481·61	170·99	224·41 377·65	292·94 479·48	440·88
0·25	97·599	137·22	192·02	263·32	261·30	326·30	115·59	162·64	225·53 305·51	292·15
0·3	69·147	104·56	155·81	224·38	183·73	240·07	84·976	128·10	187·85	210·21
0·35	51·894	84·513	133·66	200·80	136·73	187·03	66·253	106·91	164·92	160·15
0·4	40·646	71·378	119·29	185·62	106·09	151·99	53·982	93·083	150·11	127·27 180·42
0·45	32·913	62·370	109·53	161·73	84·997	127·60	45·532	83·655	140·08	104·49 154·52
0·5	27·377	55·976	102·65	132·36	69·858	109·94	39·497	76·995	132·95	88·047
0·55	23·290	51·308	97·588	110·57	58·624	96·743	35·065	72·138	127·64	75·796 121·63
0·6	20·195	47·816	93·717	93·969	50·060	86·632	31·736	68·485	115·85	66·430 110·80
0·65	17·805	45·143	81·034	90·606	43·386	78·717	29·187	65·645	101·95	59·117
0·7	15·928	43·048	70·771	87·944	38·087	72·403	27·204	63·354	90·944	53·306
0·75	14·433	41·365	62·507	85·461	33·814	67·276	25·637	61·410	82·159	48·620
0·8	13·229	39·971	55·781	82·889	30·322	63·038	24·383	59·641	75·174	44·792
0·85	12·248	38·762	50·276	79·987	27·434	59·465 79·727	23·366	57·862	69·650	41·630
0·9	11·442	37·627	45·797	76·694	25·021	56·378 72·172	22·532	55·870	65·546	38·990
0·95	10·774	36·409	42·277	73·185	22·988	53·617 65·928	21·840	53·555	62·705	36·766
1·0	10·216	34·877	39·771	69·666	the same as AS		21·260	51·030	60·829	34·877

frequency parameters for the clamped plates do not depend upon the Poisson ratio, though the present results were obtained with $\nu = 0{\cdot}3$.

4. CONCLUDING REMARKS

The method used in this paper permits treating vibration problems of isotropic and rectilinear orthotropic elliptical plates. Using the eigenvalue equation induced, natural frequencies and mode shapes for various aspect ratio and orthotropic property can be easily obtained. The accuracy and rate of convergence were shown for both orthotropic and isotropic plates. In addition, though the equation is given for plates with uniform thickness, the approach can be extended without any special difficulty to the case with variable thickness when the variation of the thickness can be expressed as polynomials.

Except for the special case of isotropic circular plates ($b/a = 1$), the exact solution is not known even for isotropic elliptical plates and the approximate results presented in the literature are available only for the first several modes or for a special aspect ratio. In this paper, natural frequency parameters for isotropic elliptical plates with all the classical free, simply supported, and clamped boundary conditions are tabulated for various aspect ratios classifying the mode types. The values may be useful for design data.

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