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THE RESPONSE OF A DYNAMIC VIBRATION ABSORBER SYSTEM WITH A PARAMETRICALLY EXCITED PENDULUM

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The vibration response of a spring–mass–damper system with a parametrically excited pendulum hinged to the mass is investigated using the harmonic balance method. The approximate results are found to be fairly consistent with those obtained by the numerical calculation. The vibrating regions of the pendulum system are obtained which are similar to those given by Mathieu's equation. Based on the analysis of three parameters in the response equation, the characteristics of response of the system are clarified. The stabilities of the harmonic solutions are analyzed, and finally our proposed approximation is verified compared with the numerical results.

1. INTRODUCTION

There have been a few reports on the analysis of a two-degree-of-freedom dynamic vibration absorber system using a parametrically excited pendulum. Haston and Barr [1] introduced an autoparametric vibration absorber system in which the response is limited to be harmonic, and studied the vibration characteristics of the so-called model "autoparametric vibration absorber". Bajaj *et al.* [2] used the method of averaging to study forced, weakly non-linear oscillations of a two-degree-of-freedom autoparametric vibration absorber system in resonant excitation. A complete bifurcation analysis of the averaged equations is undertaken for the subharmonic case of both internal and external resonances, where the first order approximation of system response is obtained. Banerjee *et al.* [3] presented a detailed stability analysis for the same system using the second order averaging of the system. Hatwal *et al.* [4, 5] employed the harmonic balance method and direct numerical integration to study a variant of the same system at moderately higher levels of excitation, and observed that over some ranges of force frequency and amplitude, the system response presented an amplitude and phase modulated harmonic motion. For higher excitation levels, the response was found to be chaotic. Also, Hatwal *et al.* [6] studied the characteristics of the autoparametric vibration absorber system in which two types of restoring force on the pendulum were considered. Furthermore, Yabuno *et al.* [7] investigated the stability of 1/3 order subharmonic resonance of the system.

In these studies, the vibration characteristics of the system such as the steady state solutions, and conditions causing bifurcation and chaotic motion, were investigated using

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approximate methods. However, the methods used in these studies either depend on a small parameter, or are very complicated. Therefore, it is necessary to consider how to make analytical process simple.

We observed the response of a dynamic vibration absorber system using a parametrically excited pendulum by the numerical calculation in the previous study [8]. The vibration-absorber characteristics of the system for the change of the parameters are clarified. Based on the analysis of the vibration-absorber characteristics, the optimum parameters of the system are obtained. In this paper, the characteristics of responses of the primary system and pendulum are analyzed using the harmonic balance method and the third order approximation of Taylor's series. The behavior of the system is clarified, and our approximation is also verified by comparing with the numerical solutions.

2. EQUATIONS OF MOTION

Figure 1 shows the two-degree-of-freedom dynamic vibration absorber system with a parametrically excited pendulum, in which the primary system consists of the mass M, the linear spring with stiffness k, and the viscous damper represented by coefficient c. The second system comprises of a simple pendulum of mass m_p hinged at M. The distance between the supporting point and the center of gravity of the pendulum is l, J and c_θ represent the inertia moment with respect to the supporting point and the damping coefficient of the pendulum respectively. The primary system is excited directly by a harmonic force $f(t) = F\cos\omega t$.

The equations of motion of the system are

$$(M + m_p)\ddot{x} + c\dot{x} + kx + m_p l(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) = f(t), \tag{1}$$

$$J\ddot{\theta} + c_\theta\dot{\theta} + (m_p gl + m_p l\ddot{x})\sin\theta = 0, \tag{2}$$

where x is the displacement of M, θ the angle of rotation of the pendulum, and the dot denotes the derivative with respect to the time t.

Equations (1) and (2) can be written in the following non-dimensional forms:

$$u'' + \frac{2\zeta}{1+\lambda}u' + \frac{1}{1+\lambda}u + \frac{\lambda}{q(1+\lambda)}(\theta''\sin\theta + \theta'^2\cos\theta) = \frac{1}{1+\lambda}\cos z\tau, \tag{3}$$

$$\theta'' + 2\zeta_p p\theta' + (p^2 + \mu q u'')\sin\theta = 0, \tag{4}$$

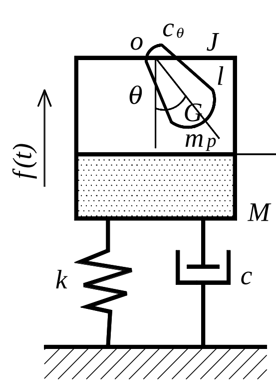

Figure 1. Autoparametric dynamic vibration absorber system.

where

$$2\varepsilon = c/M, \quad \omega_n^2 = k/M, \quad \lambda = m_p/M, \quad \omega_p^2 = m_p gl/J,$$

$$\mu = m_p l^2/J, \quad 2\varepsilon_p = c_\theta/J, \quad x_{st} = F/k,$$

$$u = x/x_{st}, \quad p = \omega_p/\omega_n, \quad z = \omega/\omega_n,$$

$$q = x_{st}/l, \quad \zeta = \varepsilon/\omega_n, \quad \zeta_p = \varepsilon_p/\omega_p, \quad \tau = \omega_n t$$

and the prime in equations (3) and (4) represents the derivative with respect to the non-dimensional time τ.

3. HARMONIC SOLUTIONS OF THE SYSTEM

Since the exciting force $f(t)$ on the right-hand side of equation (1) is harmonic, the steady state solution of the primary system is supposed to be harmonic with the same frequency z as the exciting force. On the other hand, that of the pendulum is taken as harmonic with frequency $v = z/2$ due to parametrically excited vibration. i.e.,

$$u = A\cos(z\tau + \phi), \qquad \theta = B\cos(v\tau + \varphi). \tag{5, 6}$$

Expanding $\sin\theta$ and $\cos\theta$ terms in equations (3) and (4) to the third order in Taylor series, and substituting equations (5) and (6) into equations (3) and (4), the following equations are obtained:

$$\begin{aligned}
&- Az^2\cos(z\tau+\phi) + \frac{2\zeta}{1+\lambda}(-Az\sin(z\tau+\phi) + \frac{1}{1+\lambda}A\cos(z\tau+\phi)\\
&+ \frac{\lambda}{q(1+\lambda)}\left(\left(-v^2B^2 + \frac{v^2B^4}{12}\right)\cos(z\tau+\phi)\cos(2\varphi-\phi)\right.\\
&+ \left.\left(v^2B^2 - \frac{v^2B^4}{12}\right)\sin(z\tau+\phi)\sin(2\varphi-\phi) + \frac{v^2B^4}{12}\cos 4(v\tau+\varphi)\right)\\
&= \frac{1}{1+\lambda}(\cos(z\tau+\phi)\cos\phi + \sin(z\tau+\phi)\sin\phi),
\end{aligned} \tag{7}$$

$$\begin{aligned}
&-\left(v^2 + \frac{p^2B^2}{8} - p^2\right)B\cos(v\tau+\varphi) - 2\zeta_p pvB\sin(v\tau+\varphi)\\
&-\frac{\mu qAz^2B}{2}\left(1-\frac{B^2}{6}\right)\cos(v\tau+\varphi)\cos(2\varphi-\phi) - \frac{\mu qAz^2B}{2}\\
&\times\left(1-\frac{B^2}{12}\right)\sin(v\tau+\varphi)\sin(2\varphi-\phi) - \frac{p^2B^3}{24}\cos 3(v\tau+\varphi)\\
&-\frac{\mu qAz^2B}{2}\left(1-\frac{B^2}{8}\right)\cos 3(v\tau+\varphi)\cos(2\varphi-\phi) - \frac{\mu qAz^2B}{2}\\
&\times\left(1-\frac{B^2}{8}\right)\sin 3(v\tau+\varphi)\sin(2\varphi-\phi) + \frac{\mu qAz^2B^3}{48}\cos(5v\tau+3\varphi+\phi) = 0.
\end{aligned} \tag{8}$$

By comparing coefficients of corresponding sine and cosine terms, the following four equations are obtained:

$$\left(\frac{1}{1+\lambda}-z^2\right)A-\frac{\lambda v^2B^2}{q(1+\lambda)}\left(1-\frac{B^2}{12}\right)\cos(2\varphi-\phi)=\frac{1}{1+\lambda}\cos\phi, \tag{9}$$

$$-\frac{2z\zeta}{1+\lambda}A+\frac{\lambda v^2B^2}{q(1+\lambda)}\left(1-\frac{B^2}{12}\right)\sin(2\varphi-\phi)=\frac{1}{1+\lambda}\sin\phi, \tag{10}$$

$$v^2-p^2+\frac{p^2}{8}B^2+\frac{\mu qz^2A}{2}\left(1-\frac{B^2}{6}\right)\cos(2\varphi-\phi)=0, \tag{11}$$

$$2\zeta_p pv+\frac{\mu qz^2A}{2}\left(1-\frac{B^2}{12}\right)\sin(2\varphi-\phi)=0. \tag{12}$$

Equations (9) and (10) yield

$$A^2=\frac{R^2}{T}-\frac{1}{T}\left[\left\{\frac{S^2z^4}{16}+\frac{Sp^2}{8\mu q}(R-z^2)\right\}B^4+\frac{S}{2\mu q}\{2(R-z^2)(z^2/4-p^2)+4\zeta\zeta_pRpz^2\}B^2\right], \tag{13}$$

$$\tan\phi=\left[-2Rz\zeta A+\frac{2S\zeta_pp^3zB^2}{\mu qA}\right]\bigg/\left[(R-z^2)A+\frac{2Sp^2B^2(v^2-p^2+p^2B^2/8)}{\mu qA}\right] \tag{14}$$

and equations (11) and (12) yield

$$A^2=\frac{4}{(\mu qz^2)^2}\left[(z^2/4-p^2)^2+(\zeta_ppz)^2+\frac{p^4}{64}B^4+\frac{p^2(z^2/4-p^2)}{4}B^2\right], \tag{15}$$

$$\tan(2\varphi-\phi)=\zeta_ppz\bigg/\left(z^2/4-p^2+\frac{p^2}{8}B^2\right), \tag{16}$$

where

$$R=\frac{1}{1+\lambda},\quad S=\frac{\lambda}{q(1+\lambda)},\quad T=(R-z^2)^2+(2\zeta Rz)^2.$$

Here $1-B^2/6$ and $1-B^2/12$ are regarded as one; however, their detailed description is given later.

3.1. VIBRATION REGIONS OF THE PENDULUM SYSTEM

Setting amplitude B of the pendulum in equations (13) and (15) as zero, then the following equations are obtained:

$$A_1^2=R^2/T,\qquad A_2^2=\frac{4}{(\mu qz^2)^2}[(z^2/4-p^2)^2+(\zeta_ppz)^2]. \tag{17, 18}$$

Here, A_1 is the amplitude of the primary system when the pendulum does not vibrate (locked mass), and A_2 can be considered as the amplitude of the first order vibration region of the pendulum. The reasons can be explained as follows.

When θ is so small, $\sin\theta$ is replaced by θ in equation (4) rendering it to the following form:

$$\theta''+2\zeta_pp\theta'+[p^2+\mu qu'']\theta=0. \tag{19}$$

Equation (19) is identical to Mathieu's equation [9] with additional damping term. It can be transformed into the standard form as

$$\frac{d^2\gamma}{ds^2} + (d - 2e\cos 2s)\gamma = 0, \tag{20}$$

where

$$2s = z\tau + \phi, \quad \gamma = \theta\exp(2\zeta_p ps/z), \quad d = 4p^2(1-\zeta_p^2)/z^2, \quad e = 2\mu qA,$$

where γ is the function of s and σ, where σ is a parameter with respect to d and e.

θ in equation (20) can be expressed as follows:

$$\theta = \gamma(s,\sigma)\exp(-2\zeta_p ps/z). \tag{21}$$

By considering the first order term of e, the following are obtained:

$$d \approx 1 - e\cos 2\sigma, \quad -2\zeta_p p/z \approx -\tfrac{1}{2}e\sin 2\sigma. \tag{22, 23}$$

Substituting the expressions of d and e into equations (22) and (23) yields

$$\left(\frac{z^2}{4} - p^2 + \zeta_p^2 p^2\right) - \frac{\mu qAz^2}{2}\cos 2\sigma = 0, \qquad \zeta_p pz - \frac{\mu qAz^2}{2}\sin 2\sigma = 0. \tag{24}$$

Moreover, the following equation is derived from equation (24):

$$A^2 = \frac{4}{(\mu qz^2)^2}[(z^2/4 - p^2 + \zeta_p^2 p^2)^2 + (\zeta_p pz)^2]. \tag{25}$$

It is clear that A_2 denotes the amplitude of the oscillating boundary of the pendulum system by comparing equation (25) with equation (18). Therefore, the vibration of the pendulum must occur if A_1 is larger than A_2 [10].

3.2. HARMONIC SOLUTIONS OF THE SYSTEM

The amplitude A of the primary system determined by equations (13) and (15) must coincide while the pendulum is oscillating. Hence, the equation related to the amplitude B of the pendulum is described as

$$\left[\frac{1}{T}\left\{\frac{S^2z^4}{16} + \frac{Sp^2}{8\mu q}(R - z^2)\right\} + \frac{1}{(\mu qz^2)^2}\frac{p^4}{16}\right]B^4$$
$$+\left[\frac{1}{T}\frac{S}{2\mu q}\left\{2(R-z^2)(z^2/4 - p^2) + 4\zeta\zeta_p Rpz^2\right\} + \frac{P^2(z^2/4-p^2)}{(\mu qz^2)^2}\right]B^2$$
$$+\frac{4}{(\mu qz^2)^2}[(z^2/4-p^2)^2 + (\zeta_p pz)^2] - \frac{R^2}{T} = 0 \tag{26}$$

and the amplitudes of both primary system and pendulum can be obtained by

$$B^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad A^2 = \frac{4}{(\mu qz^2)^2}\left[\left(z^2/4 - p^2 + \frac{p^2B^2}{8}\right)^2 + (\zeta_P pz)^2\right], \tag{27, 28}$$

where

$$a = \frac{1}{T}\left[\frac{S^2 z^4}{16} + \frac{Sp^2}{8\mu q}(R - z^2)\right] + \frac{p^4}{16}\frac{1}{(\mu q z^2)^2}, \tag{29}$$

$$b = \frac{1}{T}\frac{S}{2\mu q}\left[2(R - z^2)\left(\frac{z^2}{4} - p^2\right) + 4\zeta\zeta_p R p z^2\right] + \frac{(z^2/4 - p^2)p^2}{(\mu q z^2)^2}, \qquad c = A_2^2 - A_1^2. \tag{30, 31}$$

From equations (27) and (28), the relation between the amplitudes of primary system and pendulum is determined when they exhibit harmonic vibration. Thus, it is recognized that there are at most two steady state solutions at their respective frequencies for equations (5) and (6).

Setting the parameter a in equation (29) as zero, and neglecting the higher order terms of z, yields

$$\frac{p^4}{16}\left(1 + \frac{2SR\mu q}{p^2}\right)z^4 - \frac{p^4 R}{8}(1 - 2R\zeta^2)z^2 + \frac{p^4 R^2}{16} = 0. \tag{32}$$

There is no real solution for equation (32), hence the parameter a may not become zero. Also a is represented as a continuous function of z, and it becomes positive when z is so small. Therefore, a can be regarded as positive. Accordingly, the variation of the solution of B in number can be verified simply by evaluating whether the sign of b and c is positive or negative, which is summarized as follows:

Case I: $c < 0$, there is only one solution for B no matter b is positive or negative.

Case II: $c = 0$, there are two solutions for B if $b < 0$, such as 0 and $\sqrt{-b/a}$ respectively. However, there is only one solution that is equal to zero if $b > 0$.

Case III: $c > 0$, if $b < 0$ and $(b^2 - 4ac) > 0$, there are two solutions for B. On the other hand, for the case of $b > 0$ or $(b^2 - 4ac) < 0$, there is no solution.

Figure 2 shows an image about the number of solutions for amplitude B. The curve is drawn through calculating the values of b and c in terms of z, while z is determined by setting $(b^2 - 4ac) = 0$ with the condition of $\zeta_p = 0{\cdot}001 - 0{\cdot}01$.

If $1 - B^2/6$ and $1 - B^2/12$ in equation (26) are considered and they are written in the following forms:

$$\alpha = 1 - B^2/12, \qquad \beta = 1 - B^2/6.$$

Equation (26) becomes

$$\left[\frac{1}{T}\left\{\frac{S^2 z^4}{16}\alpha^2 + \frac{Sp^2}{8\mu q}(R - z^2)\frac{\alpha}{\beta}\right\} + \frac{1}{(\mu q z^2)^2}\frac{p^4}{16\beta^2}\right]B^4$$
$$+ \left[\frac{1}{T}\frac{S}{2\mu q}\left\{2(R - z^2)(z^2/4 - p^2)\frac{\alpha}{\beta} + 4\zeta\zeta_p R p z^2\right\} + \frac{P^2(z^2/4 - p^2)}{(\mu q z^2)^2\beta^2}\right]B^2$$
$$+ \frac{4}{(\mu q z^2)^2}\left[(z^2/4 - p^2)\frac{1}{\beta^2} + (\zeta_p p z)^2\frac{1}{\alpha^2}\right] - \frac{R^2}{T} = 0. \tag{33}$$

Equation (33) gives the improved solution of the harmonic balance method.

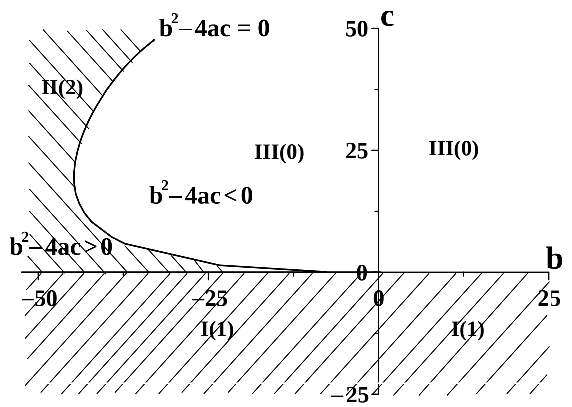

Figure 2. The number of solutions changes along with the parameter a, b, c ($\zeta = 0{\cdot}01$, $\mu = 0{\cdot}5$, $\lambda = 0{\cdot}1$, $q = 0{\cdot}01$, $\zeta_p = 0{\cdot}001{-}0{\cdot}01$) $(b^2 - 4ac) = 0$ ——; one solution I(1); two solutions II(2); no solution III(0).

3.3. STABILITY OF HARMONIC SOLUTION

In order to investigate the stability of the harmonic solutions (5) and (6), small variables from periodic state denoted by $\delta(\ )$ are given as follows:

$$A = A_0 + \delta A,\ \phi = \phi_0 + \delta\phi, \qquad B = B_0 + \delta B,\ \varphi = \varphi_0 + \delta\varphi, \tag{34}$$

where A_0, B_0, ϕ_0, φ_0 are the solutions obtained by the harmonic balance method. Substituting equations (34), (5) and (6) into equations (3) and (4), and equating the respective coefficients of sine and cosine terms the following four equations are obtained:

$$2R\zeta\delta A' + (R - z^2)\delta A - 2A_0 z\delta\phi' - 2R\zeta A_0 z\delta\phi + S[-2B_0 v\beta\sin\psi\delta B' - 2B_0 v^2\beta\cos\psi\delta B - 2B_0^2 v\alpha\cos\psi\delta\varphi' + 2B_0^2 v^2\alpha\sin\psi\delta\varphi] = 0, \tag{35}$$

$$-2z\delta A' - 2R\zeta\delta A - 2R\zeta A_0\delta\phi' + A_0(z^2 - R)\delta\phi + S[-2B_0 v\beta\cos\psi\delta B' + 2B_0 v^2\beta\sin\psi\delta B + 2B_0^2 v\alpha\sin\psi\delta\varphi' + 2B_0^2 v^2\alpha\cos\psi\delta\varphi] = 0, \tag{36}$$

$$\begin{aligned}&\mu q B_0 z\beta\sin\psi\delta A' - 0{\cdot}5\mu q B_0 z^2\beta\cos\psi\delta A\\ &- \mu q A_0 B_0 z\beta\cos\psi\delta\phi' - 0{\cdot}5\mu q A_0 B_0 z^2\beta\sin\psi\delta\phi\\ &+ 2\zeta_p p\delta B' + \left[\left(P^2 - v^2 - \frac{3}{8}p^2 B_0^2\right) - 0{\cdot}5\mu q B_0 z^2\left(1 - \frac{B_0^2}{2}\right)\cos\psi\right]\delta B - 2B_0 v\delta\varphi'\\ &+ \left[-2\zeta_p p B_0 v + 0{\cdot}5\mu q A_0 B_0 z^2\left(1 - \frac{B_0^2}{4}\right)\sin\psi\right]\delta\varphi = 0,\end{aligned} \tag{37}$$

$$\begin{aligned}&-\mu q B_0 z\alpha\cos\psi\delta A' - 0{\cdot}5\mu q B_0 z^2\alpha\sin\psi\delta A\\ &- \mu q A_0 B_0 z\alpha\sin\psi\delta\phi' + 0{\cdot}5\mu q A_0 B_0 z^2\alpha\cos\psi\delta\phi\\ &- 2v\delta B' + \left[-2\zeta_p p v - 0{\cdot}5\mu q A_0 z^2\left(1 - \frac{B_0^2}{4}\right)\sin\psi\right]\delta B\\ &- 2\zeta_p p B_0\delta\varphi' + B_0\left[\left(P^2 - v^2 + \frac{1}{8}p^2 B_0^2\right) - 0{\cdot}5\mu q A_0 z^2\cos\psi\right]\delta\varphi = 0,\end{aligned} \tag{38}$$

where $\psi = 2\varphi - \phi$. Assuming

$$\delta A = X_1 e^{\lambda_1 \tau}, \quad \delta\phi = X_2 e^{\lambda_1 \tau}, \quad \delta B = X_3 e^{\lambda_1 \tau}, \quad \delta\varphi = X_4 e^{\lambda_1 \tau}. \tag{39}$$

Substituting equation (39) into equations (35)–(38) yields

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = 0, \tag{40}$$

where

$$a_{11} = 2R\zeta\lambda_1 + (R - z^2), \quad a_{12} = -2A_0 z(\lambda_1 + R\zeta),$$

$$a_{13} = -2B_0 vS\beta(S_k\lambda_1 + vC_k), \quad a_{14} = -2B_0^2 vS\alpha(C_k\lambda_1 + vS_k)$$

$$a_{21} = -2z(\lambda_1 + R\zeta), \quad a_{22} = -A_0(2R\zeta\lambda_1 + R - z^2),$$

$$a_{23} = -2B_0 vS\beta(C_k\lambda_1 - vS_k), \quad a_{24} = 2B_0^2 vS\alpha(S_k\lambda_1 + vC_k)$$

$$a_{31} = B_z\beta(S_k\lambda_1 - 0{\cdot}5zC_k), \quad a_{32} = -A_z\beta(C_k\lambda_1 + 0{\cdot}5zS_k),$$

$$a_{33} = 2\zeta_p p\lambda_1 + p^2 - v^2 - \frac{3}{8}p^2B_0^2 - 0.5\mu q A_0 z^2\left(1 - \frac{B_0^2}{2}\right)C_k$$

$$a_{34} = -B_0\left[2v\lambda_1 - (-2\zeta_p pv + 0{\cdot}5\mu q A_0 z^2\left(1 - \frac{B_0^2}{4}\right)S_k)\right],$$

$$a_{41} = -B_z\alpha(C_k\lambda_1 + 0.5zS_k), \quad a_{42} = -A_z\alpha(S_k\lambda_1 - 0{\cdot}5zC_k)$$

$$a_{43} = -2v\lambda_1 - 2\zeta_p pv + 0{\cdot}5\mu q A_0 z^2\left(1 - \frac{B_0^2}{4}\right)S_k,$$

$$a_{44} = -B_0\left[2\zeta_p p\lambda_1 - (v^2 - p^2 + \frac{3}{8}p^2B_0^2 - 0{\cdot}5\mu q A_0 z^2 C_k)\right] \tag{41}$$

and

$$A_z = A_0 B_0 z\mu q, \quad B_z = B_0 z\mu q, \quad C_k = -\frac{v^2 - p^2 + \frac{1}{8}p^2B_0^2}{0{\cdot}5\mu q A_0 z^2}, \quad S_k = -\frac{2\zeta_p pz}{0{\cdot}5\mu q A_0 z^2} \tag{42}$$

The stability of the harmonic solutions can be investigated by extracting the root of the following characteristic equation:

$$|a_{ij}| = 0. \tag{43}$$

4. RESULTS AND DISCUSSIONS

Figure 3 shows the amplitude curves of A_1, A_2 obtained by equations (17) and (18). For example, let us divide the oscillation area as shown in Figure 3 in the case of $\zeta_p = 0{\cdot}01$ into three sections, and analyze them concretely. First, in the section between points j and k, since any value of parameter c in equation (31) is negative($A_2 < A_1$), it is seen that there is only single harmonic balance solution for amplitude B, no matter whether the parameter b is positive or negative. And the unstable motion of the system in this section can be

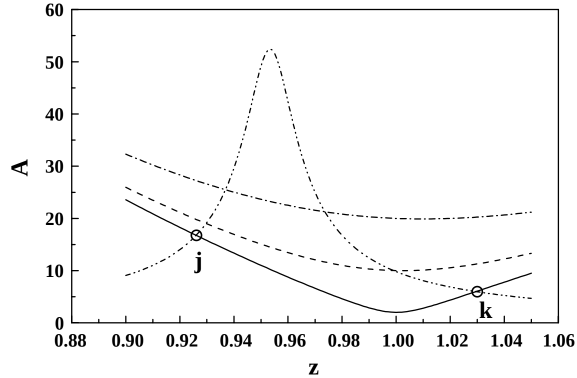

Figure 3. The vibration ranges of the pendulum ($\mu = 0{\cdot}5$, $q = 0{\cdot}01$, $\lambda = 0{\cdot}1$, $p = 0{\cdot}5$, $\zeta = 0{\cdot}01$, $\zeta_p = 0{\cdot}01$). A_1 obtained by equation (17) ----; A_2 obtained by equation (18) in the case of $\zeta_p = 0{\cdot}01$, ——; A_2 in the case of $\zeta_p = 0{\cdot}05$, - - -; A_2 in the case of $\zeta_p = 0{\cdot}1$ ---.

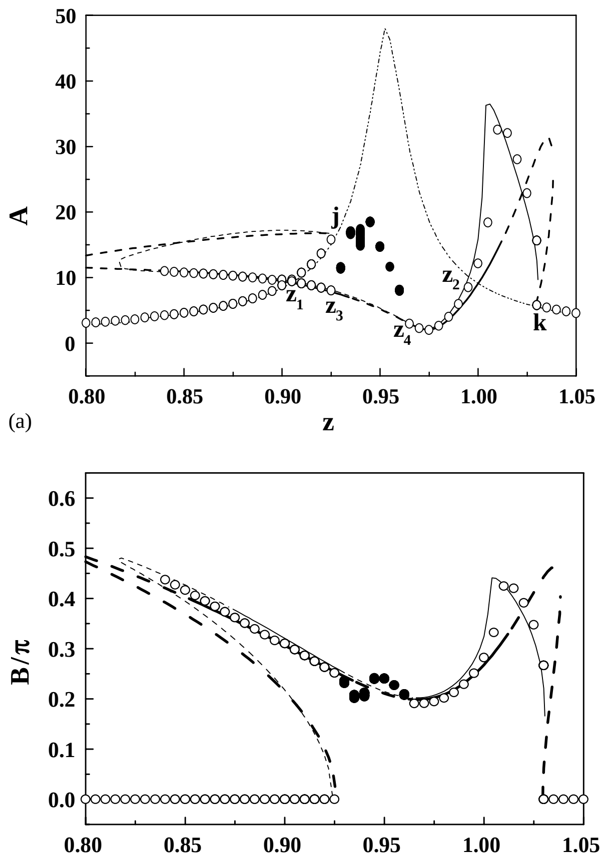

Figure 4. Responses of the system with $\zeta_p = 0{\cdot}01$ ($\mu = 0{\cdot}5$, $q = 0{\cdot}01$, $\lambda = 0{\cdot}1$, $p = 0{\cdot}5$, $\zeta = 0{\cdot}01$): (a) the primary system; (b) the pendulum. Stable solution by HM (——); unstable solution by HM (- - -); stable solution by IHM (——); unstable solution by IHM (- - -); periodic solution by RKG (○); non-periodic solution by RKG (●).

confirmed by equation (43). Second, in the section of points j and k, since parameter c equals zero $(A_2 = A_1)$, there are two solutions for B, 0 and $\sqrt{-b/a}$, if $b<0$; while there is only one solution which equals zero for the case of $b > 0$. Moreover, in the sections lower than point j and upper k, since parameter c is positive $(A_2 > A_1)$ there is no solution or exist two solutions for B, depending on whether b is positive or negative. And the oscillation of the systems can also be determined by using equation (43). If the solution is stable, the state of oscillation exists; otherwise it does not. The following results show that there are two solutions in the two sections in the case of $\zeta_p = 0{\cdot}01$. In the section lower than point j, one solution is stable for some value of z, while another is unstable for all z. But in the section upper than k, all the two solutions are completely unstable.

The responses obtained by both harmonic balance method (HM) and improved harmonic balance method (IHM), and the responses obtained by numerical calculation (Runge–Kutta–Gill) are shown in Figure 4 in the case of $\zeta_p = 0{\cdot}01$, 0·03 in Figure 5, and also $\zeta_p = 0{\cdot}05$ in Figure 6 respectively. Here, numerical results shown in these figures are the one with respect to different initial conditions ($\theta_0 = -3{\cdot}0-3{\cdot}0$, $\dot{\theta}_0 = 0$).

It is obvious from Figure 4 that the solutions of IHM are closer to the results of numerical calculation compared to those of HM. Especially, the solutions of HM are quite

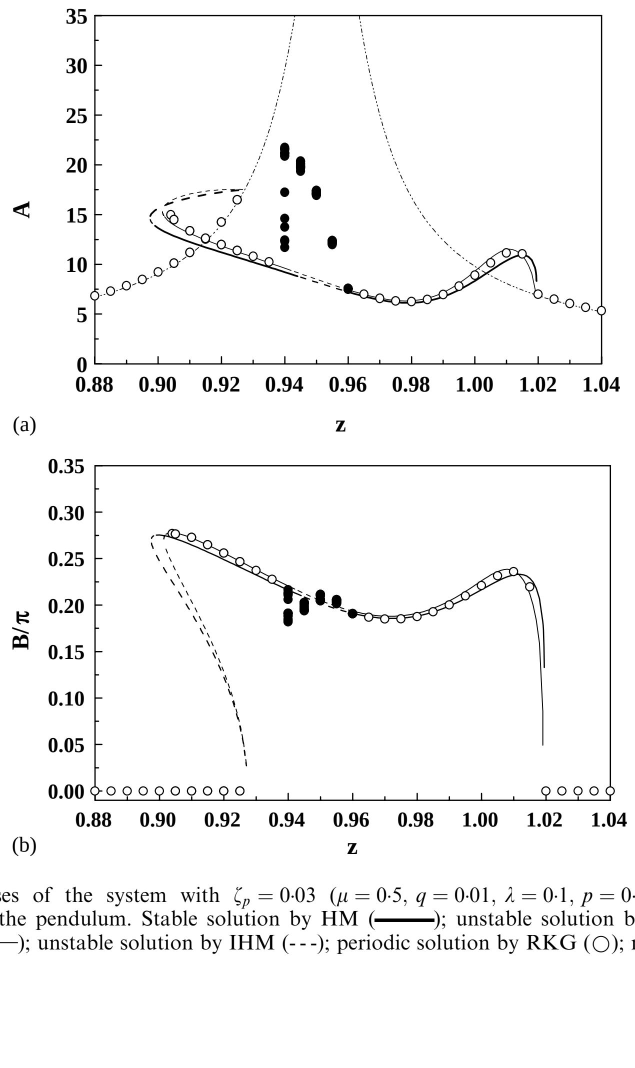

Figure 5. Responses of the system with $\zeta_p = 0{\cdot}03$ ($\mu = 0{\cdot}5$, $q = 0{\cdot}01$, $\lambda = 0{\cdot}1$, $p = 0{\cdot}5$, $\zeta = 0{\cdot}01$): (a) the primary system; (b) the pendulum. Stable solution by HM (——); unstable solution by HM (- - -); stable solution by IHM (——); unstable solution by IHM (- - -); periodic solution by RKG (○); non-periodic solution by RKG (●).

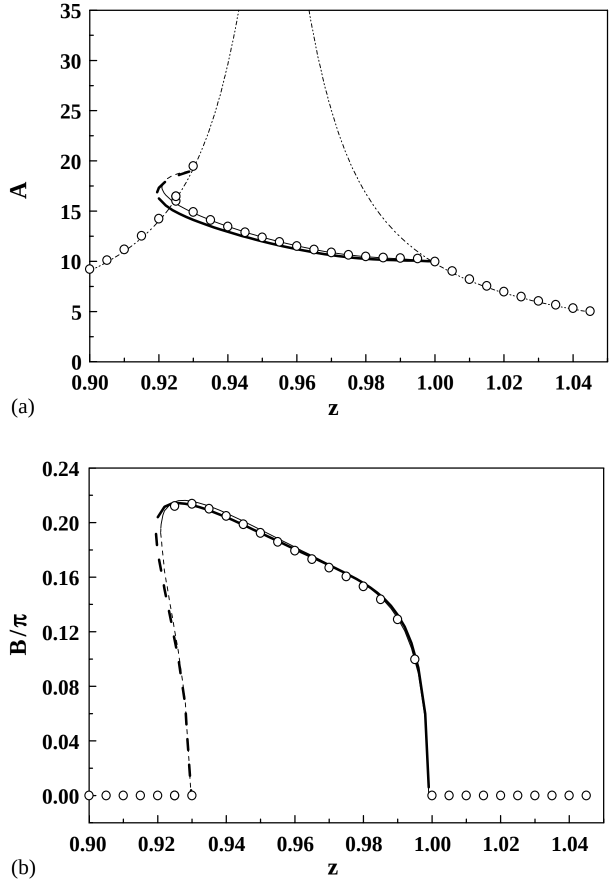

Figure 6. Responses of the system with $\zeta_p = 0{\cdot}05$ ($\mu = 0{\cdot}5$, $q = 0{\cdot}01$, $\lambda = 0{\cdot}1$, $p = 0{\cdot}5$, $\zeta = 0{\cdot}01$): (a) the primary system; (b) the pendulum. Stable solution by HM (——); unstable solution by HM (- - -); stable solution by IHM (——); unstable solution by IHM (---); periodic solution by RKG (○); non-periodic solution by RKG (●).

different from the numerical results, but the solutions of IHM show characteristics similar to the numerical results in the case of $z > 1$. In addition, there is a peak on the right-hand side of the response curve of the primary system and pendulum respectively. It shows that the vibration of the pendulum does not suppress the vibration of primary system but increase. In the area where z is small, the solutions of both HM and IHM show differences with numerical results. Due to the assumption of $1 - B^2/6 \approx 1$ and $1 - B^2/12 \approx 1$, the error of HM in calculation increases along with the increase in amplitude B. Also with the increase in amplitude B, the error of IHM in calculation becomes big. It is seen from Figure 4 that if the value of B/π is larger than 0·4, the approximate solutions and their stabilities are not consistent with the results of numerical calculation. Their inconsistencies may also result from using the third order approximation of Taylor series in equations (7) and (8).

Figure 5 shows three results of the primary system and pendulum in the case of $\zeta_p = 0{\cdot}03$. For this case the largest value of B/π is less than 0·3 and even though there are some differences between the results of HM and numerical calculation, the results of IHM are entirely consistent with those of numerical solution. It is also seen that peaks on the right-hand side of the response curves of the primary system and pendulum are shorter than those in Figure 4, which is the case $\zeta_p = 0{\cdot}01$. Moreover, Figure 6 shows three results of the primary system and pendulum in the case of $\zeta_p = 0{\cdot}05$. It is clear that the three results coincide completely in the whole area.

In addition, it is seen from Figure 4(a) that the area $z_1 - z_2$ is a vibration absorbing area, which can be obtained by solving equations (28) and (17). And the area of unstable motion (non-periodic, chaotic motion) $z_3 - z_4$ is also seen in the vibration absorbing area, which can be obtained by using equation (43). The area of unstable motion is found in Figures 4 and 5, that is, it occurs in the cases of $\zeta_p = 0{\cdot}01$ and $0{\cdot}03$, and becomes larger along with the increase in amplitude B of the pendulum. However, it does not appear in Figure 6 in the case of $\zeta_p = 0{\cdot}05$. Consequently, it is seen that the area of unstable motion obtained by three methods is fundamentally consistent.

5. CONCLUSIONS

In this paper, the vibration response of the spring–mass–damper system with a parametrically excited pendulum hinged to the mass has been investigated by using the harmonic balance method and the results verified by numerical calculation. The third order approximations are used to analyze the response characteristic and the stability of the system. The approximate results show the solutions to be not reliable for $B/\pi > 0.4$ in the lower area of z, but sufficient accuracy for $B/\pi < 0{\cdot}3$ in all area of z. And, the stability analysis shows the area of unstable motion of the system obtained from the third order approximations to be fairly consistent with that obtained from numerical calculation. That is, the third order approximation is useful in some amplitude range of the pendulum.

It is shown that the amplitude curves of A_1 and A_2 divide the oscillation area of the primary system into three sections. By evaluating whether the parameters of the quadratic equation are positive or negative, the variation of the solution of the pendulum in number in each section can be clarified simply. It is also shown that the solutions and their characteristics of the system can be obtained simply by using the quadratic equation on the response equation.

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