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# STABILITY AND HOPF BIFURCATION OF A MAGNETIC BEARING SYSTEM WITH TIME DELAYS

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The effect of time delays occurring in the feedback control loop on the linear stability of a simple magnetic bearing system is investigated by analyzing the associated characteristic transcendental equation. It is found that a Hopf bifurcation can take place when time delays pass certain values. The direction and stability of the Hopf bifurcation are determined by constructing a center manifold and by applying the normal form method. It is also found that a codimension two bifurcation can occur through a Hopf and a steady state bifurcation interaction. Finally, numerical simulations are performed to verify the analytical predictions.



## 1. INTRODUCTION

A standard magnetic bearing is basically composed of four components: sensor, controller, power amplifier, and electromagnetic actuators. Figure 1 shows the block diagram of a magnetic bearing system. In particular, the non-contact position sensor is used to measure the position of the shaft. Then the controller converts the sensor signals to the control signals. The power amplifier supplies the required currents to each of the actuator coils. Finally, the electromagnets generate the suspension and operating forces. In this closed-loop configuration, the time delays are unavoidable in the controller and actuators, even though the control decision process is carried out very quickly.

Recently, there has been significant interest in stability and bifurcations of delay differential equations, which arise in a variety of physical models, such as neutral networks [1, 2], business cycle [3], and mechanical oscillators [4, 5], to name just a few. It has been shown that delay effects cannot only lead to an oscillatory behavior but to a more complex behavior due to delay-induced instability also [6]. The time delays can render the system dynamics complicated. For example, Campbell and Belair [7] have observed codimension two bifurcations in a harmonic oscillator with a delayed forcing.

For magnetic bearing systems, much work has been done on the subject of non-linear modelling, stability and bifurcation [8–14]. In this paper, the influence of inevitable time delays on the linear stability of trivial equilibrium will be studied, and the critical length of time delay will be determined.

The outline of this paper is as follows. In section 2, the equation of motion is derived for a single-degree-of-freedom (S-d.o.f.) magnetic bearing system. In section 3, the local stability of the trivial fixed point of the system is investigated by analyzing the corresponding characteristic equation. In section 4, the nature of the Hopf bifurcation is determined by using the center manifold theory and the normal form method.

 

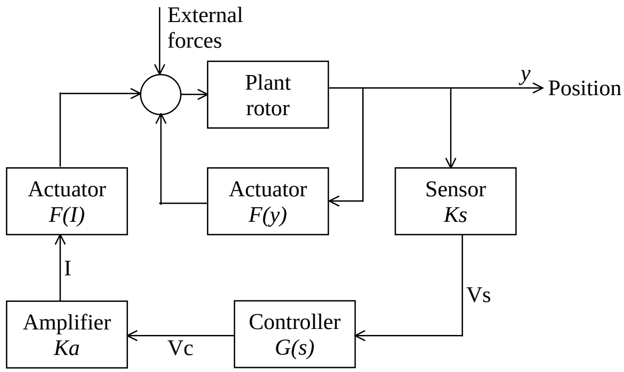


Figure 1. Block diagram of a simple magnetic bearing system.

Codimension two bifurcation is discussed in section 5. Finally, conclusions are presented in section 6.

## 2. MATHEMATICAL MODEL

The model to be discussed is a two-pole, S-d.o.f. magnetic bearing shown in Figure 2. This simple model, without unnecessary complexity, represents a fundamental structure for many more complicated magnetic bearings. The equation of motion, in Cartesian co-ordinate system for a balanced rotor in the vertical direction has a simple form

$$m\ddot{y} = mg - F(y, i_p), \tag{1}$$

where $m$ is the mass of the rotor, $F(y, i_p)$ is the magnetic force, and the dot indicates differentiation with respect to the physical time $T$. The magnetic force can be approximately expressed as [15]

$$F(y, i_p) = k\left(\frac{(I_b + i_p)^2}{(g_0 + y)^2} - \frac{(I_b - i_p)^2}{(g_0 - y)^2}\right), \tag{2}$$

where $k$ is the magnetic force constant, $I_b$ is the bias current, $i_p = i_0 + i$ is the control current, and $g_0$ is the nominal air gap between the stator and the shaft. During normal operation, the shaft is expected to levitate at the geometric center of the stator of magnetic bearing. Thus, the static component of the magnetic force $F(y, i_p)$ is required to balance the gravity of the rotor, i.e., $F(0, i_0) = mg$, where $i_0$ denotes pre-control current.

In the feedback control loop of a magnetic bearing system, the models and dynamics of sensor, controller and amplifier may be much complicated and highly non-linear. For simplicity, the feedback control system is assumed to generate a current that is proportional to the rotor displacement and velocity, i.e., a PD controller

$$i = k_p y(T - T_1) + k_d \dot{y}(T - T_1), \tag{3}$$

where $k_p$ and $k_d$ are the proportional and derivative gains respectively. $T_1$ denotes the time delay occurring in the feedback control loop. For a practical magnetic bearing system, it is obvious that the feedback gains and time delays are non-negative. To be realistic, two distinct time delays in the proportional and derivative feedback should be considered. However, this will result in greatly complicated analysis. In this work, identical time delays are considered for the sake of simplicity.

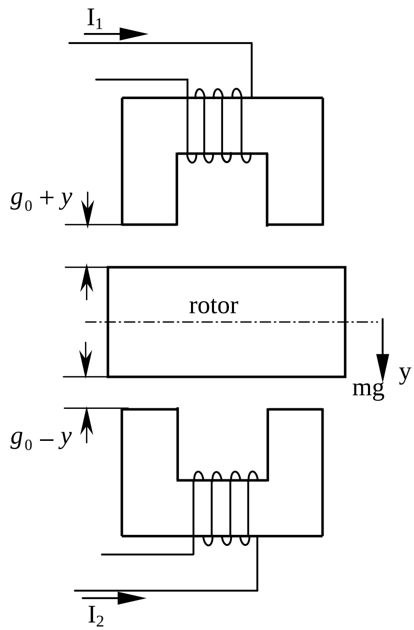


Figure 2. Schematic of a two-electromagnet magnetic bearing.

Expanding the magnetic force $F(y, i_p)$ up to the third order Taylor series about the operating point (i.e., $y = 0$, $i_p = i_0$), then substituting the resultant expression and equation (3) into equation (1), and finally introducing non-dimensional parameters $y = ug_0$, $i_p = i_c I_b$, $T = t/\Omega$, yields the equation of motion in the non-dimensional form for a balanced shaft in the direction of $y$-axis:

$$\begin{aligned} u'' &+ du'(t-\tau) - u + pu(t-\tau) + \alpha_1 u^2 - \alpha_2 uu(t-\tau) - \alpha_3 uu'(t-\tau) - 2u^3 + 3pu^2 u(t-\tau) \\ &+ 3du^2 u'(t-\tau) - q_1 uu^2(t-\tau) - q_2 uu(t-\tau)u'(t-\tau) - q_3 u[u'(t-\tau)]^2 = 0, \end{aligned} \tag{4}$$

where

$$\Omega^2 = 4kI_b^2(1+l^2)/mg_0^3, \quad l = i_0/I_b, \quad \tau = \Omega T_1, \quad d = k_d \Omega g_0/(1+l^2)I_b,$$
$$p = k_p g_0/(1+l^2)I_b, \quad \alpha_1 = 3l/(1+l^2), \quad \alpha_2 = 2lp, \quad \alpha_3 = 2ld, \quad q_1 = (1+l^2)p^2,$$
$$q_2 = 2(1+l^2)pd, \quad q_3 = (1+l^2)d^2$$

and the superscript prime denotes differentiation with respect to the non-dimensional time $t$.

## 3. LINEAR STABILITY ANALYSIS

The fixed points of equation (4), $u(t) = U = constant$, are found by solving

$$U[(3p - 2 - q_1)U^2 + (\alpha_1 - \alpha_2)U + p - 1] = 0.$$

Depending on the quantity of $\beta = (\alpha_1 - \alpha_2)^2 - 4(p-1)(3p - 2 - q_1)$, one, two, or three fixed points are possible. If $\beta < 0$, only the trivial solution $u(t) = 0$ exists. If $\beta = 0$, a trivial solution $u(t) = 0$ and a non-trivial solution $u(t) = U = -A$ exist. If $\beta > 0$, a trivial solution $u(t) = 0$ and two non-trivial solutions $u(t) = U_\pm = -A \pm \sqrt{A^2 - B}$ exist, where $A = (\alpha_1 - \alpha_2)/(6p - 4 - 2q_1)$, $B = (p-1)/(3p - 2 - q_1)$. Under normal operating condition, the rotor is expected to levitate at the center of the stator of magnetic bearing. Thus, a discussion of the local stability of the trivial fixed point appears to be of interest and of importance. In fact, a non-trivial fixed point can be easily transformed into a trivial one by a linear co-ordinate change. Then the local stability of the non-trivial fixed point can be analyzed in a similar way to that of the trivial fixed point.

To study the local stability of the trivial fixed point of equation (4), it is usual to linearize the equation around this fixed point and look for a candidate solution of the form $a\exp(\lambda t)$, where $a$ is constant. This leads to the characteristic equation

$$\lambda^2 + d\lambda \mathrm{e}^{-\lambda\tau} + p\mathrm{e}^{-\lambda\tau} - 1 = 0. \tag{5}$$

The stability of the trivial fixed point depends on the locations of the roots of characteristic equation (5).

If the time delays in the PD feedback are not taken into account, the characteristic equation (5) simplifies to a quadratic in $\lambda$

$$\lambda^2 + d\lambda + p - 1 = 0. \tag{6}$$

The local stability of the trivial fixed point requires that the characteristic roots should be in the left half-plane. A necessary condition for stable operation of the magnetic bearing system can be easily written out as $d > 0$ and $p > 1$. When $p = 1$, one of the eigenvalues is zero. In the subsequent analysis, it will be shown that this stability criterion cannot guarantee the local stability of the trivial fixed point when time delays are considered. The stability of the trivial fixed point may change through a Hopf bifurcation when time delays pass certain values.

Equation (5) is transcendental, and may have an indefinite number of roots. Unfortunately, it is impossible to get all roots explicitly. On the base of the standard results on stability of functional differential equations [16], the trivial fixed point of equation (4) is stable if and only if all roots $\lambda$ of the characteristic equation have negative real parts. Suppose the eigenvalue, $\lambda = \sigma \pm \mathrm{i}\omega$ is a solution to equation (5), without loss of generality, it is assumed that $\omega > 0$. Here, $\sigma$ and $\omega$ are real and imaginary parts of the eigenvalue respectively. Using the standard *Euler* formula, a pair of transcendental equations can be obtained as follows:

$$\begin{aligned} &\sigma^2 - \omega^2 + p\mathrm{e}^{-\sigma\tau}\cos\omega\tau + d\mathrm{e}^{-\sigma\tau}(\sigma\cos\omega\tau + \omega\sin\omega\tau) - 1 = 0,\\ &2\sigma\omega - p\mathrm{e}^{-\sigma\tau}\sin\omega\tau + d\mathrm{e}^{-\sigma\tau}(\omega\cos\omega\tau - \sigma\sin\omega\tau) = 0. \end{aligned} \tag{7}$$

The fixed point may change stability when $\mathrm{Re}(\lambda) = 0$ for some $\lambda$, where $\mathrm{Re}(\lambda)$ designates the real part of $\lambda$. This can occur in two different ways. First, a real eigenvalue passes through zero, i.e., $\lambda = 0$. This occurs when $p = 1$, where a simple steady state bifurcation generates. This condition is the same as obtained from equation (6) for presence of a zero eigenvalue. Since the trivial equilibrium is always a fixed point for equation (4) and the non-linear terms are quadratic and cubic, it could be expected pitchfork bifurcations to occur from $u(t) = 0$ at this simple zero eigenvalue [17].

The second situation can happen if a pair of complex eigenvalues crosses the imaginary axis, i.e., $\lambda = \pm\mathrm{i}\omega$, a necessary condition for Hopf bifurcation. If this is the case, equation (7) simplifies to

$$\begin{aligned} &p\cos\omega\tau + d\omega\sin\omega\tau - 1 - \omega^2 = 0,\\ &d\omega\cos\omega\tau - p\sin\omega\tau = 0. \end{aligned} \tag{8}$$

Moving the terms of trigonometric functions into the right-hand side, then squaring and adding these two equations yields a quartic equation for $\omega$

$$\omega^4 + (2 - d^2)\omega^2 + 1 - p^2 = 0, \tag{9}$$

with roots $\omega_\pm^2 = (\frac{1}{2}d^2 - 1) \pm \sqrt{p^2 - d^2 + \frac{1}{4}d^4}$.

Depending on a combination of feedback gains, the existence of positive real roots is described as follows:

(a) if $d^2 \leqslant 2$ and $p^2 < 1$, there are no real roots;

(b) if $d^2 > p^2 + \frac{1}{4}d^4$, there are no real roots;

(c) if $p^2 > 1$, there is only one positive real root $\omega_+ > 0$;

(d) if $d^2 > 2$ and $d^2 - \frac{1}{4}d^4 < p^2 < 1$, there are two positive real roots $\omega_\pm$ with $\omega_+ > \omega_- > 0$.

**Theorem** *For $p < 1$, the trivial fixed point is linearly unstable for all values of $d$ and $\tau$.*

A proof of this theorem can be obtained in a similar manner to that given in reference [7].

From the above discussion, it is easy to note that the trivial fixed point is linearly unstable for cases (a), (b) and (d). In a practical engineering system, only case (c) can occur. The trivial fixed point of system (4) is asymptotically stable only for certain finite time delay. Changes in stability may occur when a characteristic root passes through the imaginary axis. For case (c) that the characteristic equation has a pair of purely imaginary eigenvalues $\pm \mathrm{i}\omega_+$, there exists one set of time delay $\tau$, which is given by

$$\tau_n = \frac{\theta}{\omega_+} + \frac{2n\pi}{\omega_+}, \tag{10}$$

where $0 \leqslant \theta < 2\pi$, $n = 0, 1, 2, \ldots$, $\cos\theta = p(1 + \omega_+^2)/(p^2 + d^2\omega_+^2)$, $\sin\theta = d\omega_+(1 + \omega_+^2)/(p^2 + d^2\omega_+^2)$.

To make sure the existence of Hopf bifurcations, it is needed to check the transversality condition. For the sake of convenience, the time delay $\tau$ is chosen as the bifurcation parameter and $(\mathrm{d}\tau/\mathrm{d}\lambda)$ is considered instead of $(\mathrm{d}\lambda/\mathrm{d}\tau)$. The necessary condition for the existence of a Hopf bifurcation is that the critical characteristic roots cross the imaginary axis with non-zero velocity, that is,

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)_{\lambda=\mathrm{i}\omega} \neq 0. \tag{11}$$

Differentiation of equation (5) with respect to $\lambda$ yields

$$\frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \frac{d}{p\lambda + d\lambda^2} + \frac{2}{1 - \lambda^2} - \frac{\tau}{\lambda}, \tag{12}$$

thus

$$\mathrm{sign}\left\{\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\right\}_{\lambda=\mathrm{i}\omega} = \mathrm{sign}\left\{\mathrm{Re}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\right)\right\}_{\lambda=\mathrm{i}\omega} = \mathrm{sign}\{2p^2 - d^2 + d^2\omega^2\}. \tag{13}$$

It can be found that the sign is positive by inserting the expression for $\omega_+^2$ into (13). Therefore, the only crossing of the imaginary axis is from left to right as $\tau$ increases. A simple Hopf bifurcation occurs. Similar to the result given in reference [4], the local stability of the trivial fixed point can be described as follows:

**Criterion.** *For fixed feedback gains $p$ and $d$ within the region of possible existence of a pair of purely imaginary eigenvalues $\pm \mathrm{i}\omega_+$, the trivial fixed point is asymptotically stable if $\tau < \theta/\omega_+$. At the value of $\tau = \theta/\omega_+$, the characteristic equation (5) has purely imaginary*

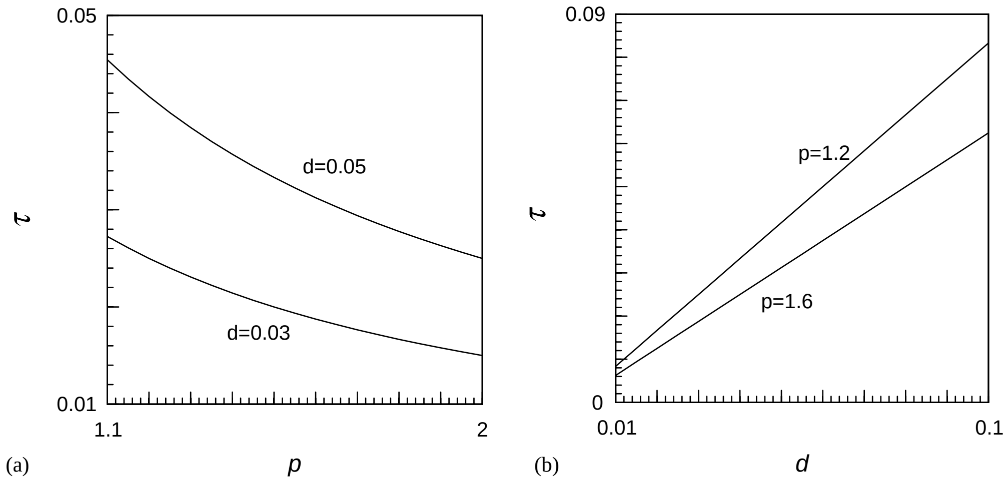


Figure 3. The variation of the smallest value of critical time delay on feedback gains.

*roots, $\pm i\omega_+$. A single Hopf bifurcation takes place. The stability of the trivial fixed point is lost and unstable thereafter.*

This criterion indicates that the linear stability of the trivial fixed point is time delay-dependent. As the length of time delay changes, the local stability of the trivial fixed point may also change. From the foregoing discussion, it is easy to see that the critical value of time delay is a function of non-dimensional feedback gains. For a fixed feedback gain, the variation of the smallest critical value of time delay on other gain is illustrated in Figure 3. The figure indicates that the smallest critical value of time delay decreases as the proportional gain increases, while it increases as the derivative gain increases. The trivial fixed point is asymptotically stable when the parameters lie in the regions below the curves. These curves also define the locations of Hopf bifurcations.

## 4. SINGLE HOPF BIFURCATION

It has been shown in section 3 that the system can undergo a single Hopf bifurcation at some values of time delay $\tau$. In this section, the effect of the non-linear terms of the original equation (4) is considered to determine the direction and stability of the single Hopf bifurcation. The analysis is based on the normal form method and the center manifold theory.

Choose the phase space as $C = C([-\tau, 0], R^2)$, then equation (4) can be rewritten, in standard notation [18], as a functional differential equation

$$\dot{\mathbf{x}} = \mathbf{L}\mathbf{x}_t + \mathbf{f}(\mathbf{x}_t), \tag{14}$$

where $\mathbf{x} = (u, u')^{\mathrm{T}}$, $\mathbf{x}_t = \mathbf{x}(t+\theta)$, $-\tau \leqslant \theta < 0$. $\mathbf{L} : C[\tau, 0] \to R^2$ is a linear operator defined by

$$\mathbf{L}\boldsymbol{\phi} = \int_{-\tau}^{0} [\mathrm{d}\boldsymbol{\eta}(\theta)]\boldsymbol{\phi}(\theta) \quad \text{for } \phi \in C, \tag{15}$$

where $\boldsymbol{\eta} : [-\tau, 0] \to R^{2\times 2}$ is a matrix-valued function of bound variation. It is given as

$$\boldsymbol{\eta}(\theta) = \begin{pmatrix} 0 & \delta(\theta) \\ \delta(\theta) - p\delta(\theta+\tau) & -d\delta(\theta+\tau) \end{pmatrix},$$

where $\delta(\theta)$ is the *Dirac* delta function.

Define the linear operator A as

$$\mathbf{A}\boldsymbol{\phi}(\theta) = \begin{cases} \mathrm{d}\boldsymbol{\phi}(\theta)/\mathrm{d}\theta, & -\tau \leqslant \theta < 0, \\ \mathbf{L}\boldsymbol{\phi}(\theta), & \theta = 0 \end{cases} \tag{16}$$

and the non-linear operator F as

$$\mathbf{F}\boldsymbol{\phi} = \begin{cases} 0, & -\tau \leqslant \theta < 0, \\ \mathbf{f}(\phi), & \theta = 0. \end{cases} \tag{17}$$

The adjoint operator $\mathbf{A}^*$ of $\mathbf{A}$ is then defined as

$$\mathbf{A}^*\boldsymbol{\psi}(s) = \begin{cases} -\mathrm{d}\boldsymbol{\psi}(s)/\mathrm{d}s, & 0 < s < \tau, \\ \mathbf{L}^{\mathrm{T}}\boldsymbol{\psi}(-s), & s = 0, \end{cases} \tag{18}$$

where $\mathbf{L}^{\mathrm{T}}$ is the transpose of operator $\mathbf{L}$.

The bilinear form is defined as

$$\langle \boldsymbol{\psi}, \boldsymbol{\phi} \rangle = (\boldsymbol{\psi}(0), \boldsymbol{\phi}(0)) - \int_{-\tau}^{0}\int_{0}^{\theta} \boldsymbol{\psi}(\xi - \theta)[\mathrm{d}\boldsymbol{\eta}(\theta)]\boldsymbol{\phi}(\xi)\,\mathrm{d}\xi, \tag{19}$$

where ( , ) represents the usual scalar product of two vectors.

For the parameter set where the linear part of equation (4) posses a pair of purely imaginary eigenvalues, there exists in the state space $C$ a two-dimensional center manifold. Then the long-term behavior of solutions to the non-linear equation is well approximated by the flow on this manifold [18]. Moreover, there exists a splitting of the space $C = P \oplus Q$, where $P$ is a two-dimensional subspace spanned by eigenvectors of the operator $\mathbf{A}$ corresponding to the purely imaginary eigenvalues, $Q$ is the complementary space of $P$. $P$ and $Q$ are invariant under the flow associated with the linear part of equation (14).

If $\mathbf{q}(\theta)$ and $\mathbf{q}^*(s)$ represent the eigenvectors of $\mathbf{A}$ and $\mathbf{A}^*$ corresponding to the eigenvalues of $\mathrm{i}\omega_+$ and $-\mathrm{i}\omega_+$, respectively, then direct computation yields

$$\mathbf{q}(\theta) = \begin{pmatrix} 1 \\ \mathrm{i}\omega \end{pmatrix} \mathrm{e}^{\mathrm{i}\omega\theta} \quad \text{and} \quad \mathbf{q}^*(\xi) = D\begin{pmatrix} -\mathrm{i}\omega \\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}\omega s},$$

where $D$ is given by its complex conjugate $\bar{D} = (2\mathrm{i}\omega + (p + \mathrm{i}\omega d)\tau\mathrm{e}^{-\mathrm{i}\omega\tau})^{-1}$, and the subscript "+" in $\omega_+$ has been omitted for the sake of brevity . $\bar{D}$ is obtained by ensuring $\langle q^*, q\rangle = 1$.

The center manifold introduced above is then given by

$$M = \{\boldsymbol{\phi} \in C;\ \boldsymbol{\phi} = \Phi z + h(z, F), z = (x, y)^{\mathrm{T}} \text{ in a neighborhood of zero in } R^2\}. \tag{20}$$

The flow on this center manifold is

$$\mathbf{x}_t = \Phi\mathbf{z}(t) + \mathbf{h}(\mathbf{z}(t), \mathbf{F}), \tag{21}$$

where $\boldsymbol{\Phi} = (\phi_1, \phi_2)$ is a basis for $P$, $h \in Q$, and $z(t)$ satisfies the ordinary differential equation

$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + \boldsymbol{\Psi}(0)\mathbf{F}(\Phi\mathbf{z}), \tag{22}$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \quad \mathbf{\Psi} = (\psi_1, \psi_2)$$

is the basis for the invariant subspace of the adjoint problem corresponding to $P$.

From the above discussion, the other elements in (22) are

$$\mathbf{\Psi}(0) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad \mathbf{\Phi} = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2) = \begin{bmatrix} \cos\omega\theta & \sin\omega\theta \\ -\omega\sin\omega\theta & \omega\cos\omega\theta \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix},$$

thus

$$\mathbf{\Phi z} = \begin{bmatrix} \cos(\omega\theta)x + \sin(\omega\theta)y \\ -\omega\sin(\omega\theta)x + \omega\cos(\omega\theta)y \end{bmatrix}. \tag{23}$$

The non-linear function $\mathbf{F}$ in equation (22) is then given as

$$\mathbf{F}(\mathbf{\Phi z}) = \begin{bmatrix} 0 \\ F_{11}x^2 + F_{12}xy + F_{111}x^3 + F_{112}x^2y + F_{122}xy^2 \end{bmatrix}, \tag{24}$$

where

$$\begin{aligned}
F_{11} &= -\alpha_1 + \alpha_2\cos(\omega\tau) + \alpha_3\omega\sin(\omega\tau),\\
F_{12} &= -\alpha_2\sin(\omega\tau) + \alpha_3\omega\cos(\omega\tau),\\
F_{111} &= 2 - 3p\cos(\omega\tau) - 3d\omega\sin(\omega\tau) + q_1\cos^2(\omega\tau) + q_2\omega\sin(\omega\tau)\cos(\omega\tau) + q_3\omega^2\sin^2(\omega\tau),\\
F_{112} &= 3p\sin(\omega\tau) - 3d\omega\cos(\omega\tau) - q_1\sin(2\omega\tau) + q_2\omega\cos(2\omega\tau) + q_3\omega^2\sin(2\omega\tau),\\
F_{122} &= q_1\sin^2(\omega\tau) - q_2\omega\sin(\omega\tau)\cos(\omega\tau) + q_3\omega^2\cos^2(\omega\tau).
\end{aligned}$$

Substituting equations (24) and (23) into (22) yields the dynamical system on the manifold

$$\begin{aligned}
\dot{x} &= -\omega y + b_{12}(F_{11}x^2 + F_{12}xy + F_{111}x^3 + F_{112}x^2y + F_{122}xy^2),\\
\dot{y} &= \omega x + b_{22}(F_{11}x^2 + F_{12}xy + F_{111}x^3 + F_{112}x^2y + F_{122}xy^2)
\end{aligned} \tag{25}$$

By smooth changes of co-ordinates, this equation can be simplified to a general normal form to third order, expressed in standard notation [17] as

$$\begin{aligned}
\dot{x} &= a(x^2 + y^2)x - (\omega + b(x^2 + y^2))y,\\
\dot{y} &= (\omega + b(x^2 + y^2))x + a(x^2 + y^2)y.
\end{aligned} \tag{26}$$

This degenerate system can be expressed in polar co-ordinate as

$$\dot{r} = ar^3, \qquad \dot{\theta} = \omega + br^2 \tag{27}$$

and its universal unfolding is [17]

$$\dot{r} = \mu r + ar^3, \qquad \dot{\theta} = \omega + br^2, \tag{28}$$

where $\mu$ is an unfolding parameter. The coefficient $a$ is given by

$$a = \frac{1}{8}(3b_{12}F_{111} + b_{22}F_{112} + b_{12}F_{122}) + \frac{1}{8\omega}(b_{12}^2F_{11}F_{12} - b_{22}^2F_{11}F_{12} - 2b_{12}b_{22}F_{11}^2), \tag{29}$$

where $b_{12} = -\omega(p\tau\cos\omega\tau + \omega d\tau\sin\omega\tau)/\Delta$, $b_{22} = (2\omega + \omega d\tau\cos\omega\tau - p\tau\sin\omega\tau)/\Delta$, $\Delta = p^2\tau^2 + d^2\omega^2\tau^2 + 4\omega^2 + 4\omega^2 d\tau\cos\omega\tau - 4\omega p\tau\sin\omega\tau$.

The sign of the cubic coefficient $a$ determines the direction and stability of the Hopf bifurcation. If $a > 0$, then the Hopf bifurcation gives rise to an unstable limit circle and it is called subcritical. If $a < 0$, the bifurcation gives rise to a stable limit circle and is called supercritical.

As an illustrative example, consider a system with $l = 0{\cdot}0$, $d = 0{\cdot}05$ and $p = 1{\cdot}5$. The time delay $\tau$ is taken as a bifurcation parameter. It is found that only trivial solution for equation (4) exists, and single Hopf bifurcation takes place at $\tau = 0{\cdot}03332$, $\omega_+ = 0{\cdot}7074$. From equation (29), it is computed that $a = 0{\cdot}00151$. Therefore, the single Hopf bifurcation for the system is subcritical. This means that an unstable limit circle exists after the trivial fixed point is unstable.

To verify these analytical results, numerical integrations are performed to the original equation (4). When numerical simulations are performed, the time delays are picked up out of the regions before and after Hopf bifurcation. They are chosen as $\tau = 0{\cdot}03$ and $0{\cdot}05$, respectively. The results are shown in Figure 4. The trivial fixed point is asymptotically stable when the time delay is less than the critical value, and unstable when larger than the critical value. The Hopf bifurcation is unstable, and thus cannot be detected by numerical integrations. The numerical results are in good agreement with the analytical predictions.

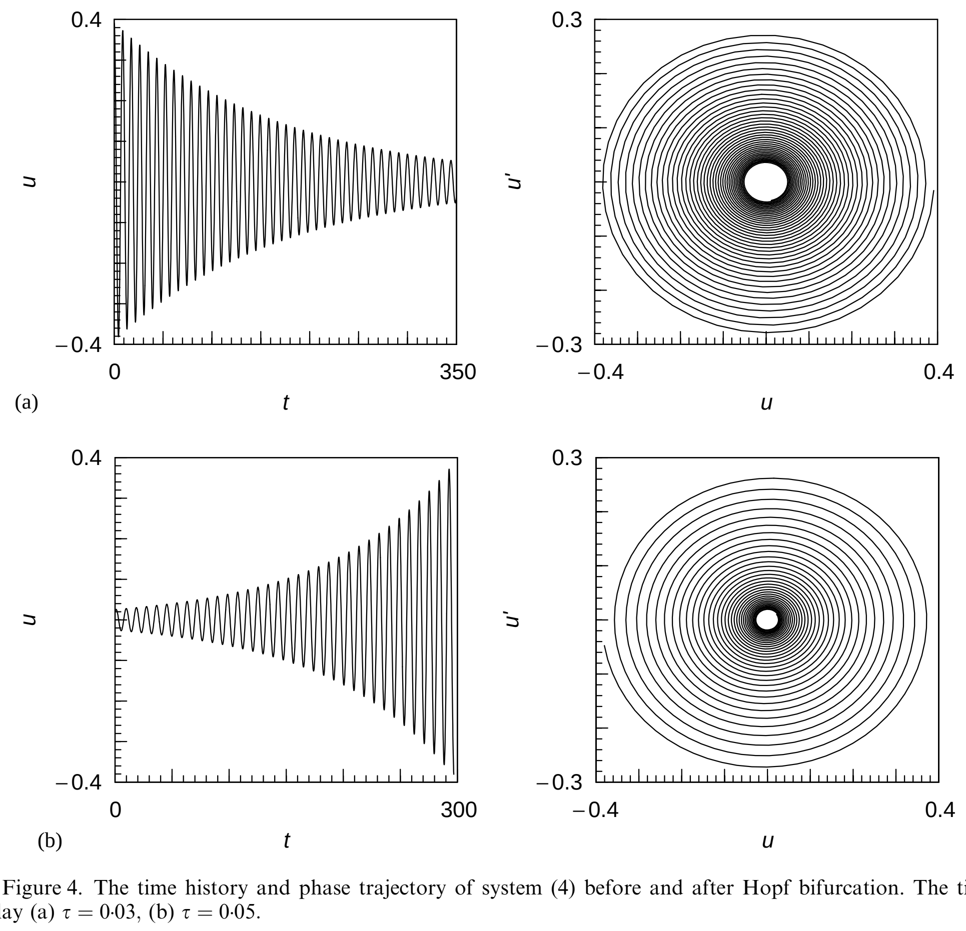


Figure 4. The time history and phase trajectory of system (4) before and after Hopf bifurcation. The time delay (a) $\tau = 0{\cdot}03$, (b) $\tau = 0{\cdot}05$.

## 5. CODIMENSION TWO BIFURCATION

As previously mentioned, the trivial fixed point may lose its stability through either a steady state bifurcation or a single Hopf bifurcation, where the characteristic equation has a zero eigenvalue or a pair of purely imaginary eigenvalues. In the parameter space these different bifurcations may have intersection points. In ordinary differential equations such points are usually associated with a codimension two bifurcation and can be a source of more complicate dynamics [17]. For this physical system, a codimension two bifurcation point corresponds to either a Hopf–Hopf intersection or a Hopf and a steady state bifurcation intersection.

A necessary condition for the Hopf–Hopf intersection is that the characteristic equation has two pairs of purely imaginary roots $\pm i\omega_1$, $\pm i\omega_2$. Points of intersection occur when the time delay $\tau$ has the same value for these two Hopf bifurcations. Further more, if the frequencies of two Hopf bifurcations obey $\omega_1 : \omega_2 = n : m$ for some $n, m \in Z$, where $Z$ is the set of positive integers, the points of intersection are resonant codimension two points. These points usually cannot be solved in closed form. They can, however, be easily computed numerically. From the expressions for $\omega_\pm$, it is known that the points of Hopf–Hopf intersections can only occur if $d^2 > 2$ and $p^2 < 1$. In this parameter region, it has been shown that the trivial fixed point is linearly unstable. Therefore, although non-resonant and resonant Hopf–Hopf interactions may occur, these points are less interesting and they will have little effect on the observable dynamics of the system.

The other codimension two bifurcation points result from the Hopf and steady state bifurcation intersections. This situation may occur when the characteristic equation has both a zero eigenvalue and a pair of purely imaginary eigenvalues. The former occurs when $p = 1$. The frequency of the Hopf bifurcation and the critical time delay can be easily obtained from equation (8) as

$$\omega = (d^2 - 2)^{1/2}, \quad \tau = \frac{1}{\omega}\arctan(d\omega), \tag{30}$$

where arctan is in the range $(-\pi/2, \pi/2)$. This is the sufficient condition for intersection points of the Hopf and steady state bifurcations. It is obvious that such points can occur only when $d^2 > 2$.

The points of Hopf and steady state bifurcation intersections locate on the boundary of the region of stability of the trivial fixed point at $p = 1$. Thus these points can be expected to have significant influence on the system dynamics. To illustrate what may occur in this

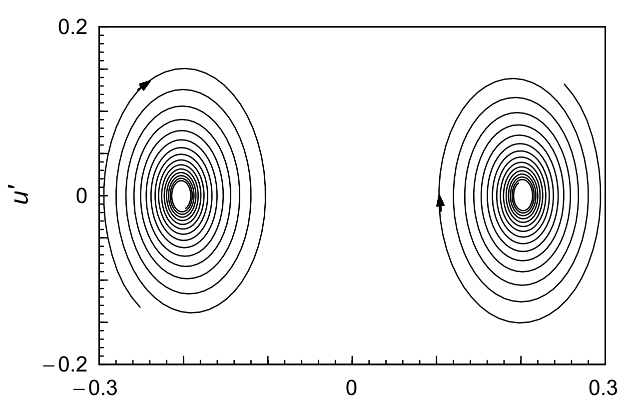


Figure 5. The coexistence of two unstable limit cycles near the Hopf–pitchfork bifurcation interaction.

system, numerical simulations are performed near the Hopf and steady state bifurcation intersection points. For $p = 1$ and $d = 2$, it is found that the Hopf–pitchfork bifurcation intersection occurs at $\omega \approx 1{\cdot}4142$, $\tau \approx 0{\cdot}8704$. Figure 5 shows the numerical simulation result near the point of Hopf–pitchfork intersection at $\tau = 0{\cdot}8$. The system exhibits a coexistence of two unstable limit cycles.

## 6. CONCLUSIONS

The stability of a simple magnetic bearing system is investigated with consideration of time delays presented in the feedback control loop. The nature of the Hopf bifurcation, depending on the non-linear terms of the original equation, is explicitly determined by the construction of a center manifold. The critical time delay is determined. If the time delay is less than this critical value, then the effect of the time delay on the local stability of the trivial fixed point can be neglected. Otherwise, the feedback gains must be redesigned in order to guarantee the system stability.

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