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THE SPATIAL AVERAGE MEAN SQUARE MOTION AS AN OBJECTIVE FUNCTION FOR OPTIMIZING DAMPING IN DAMPED MODIFIED SYSTEMS

R. G. Jacquot

Electrical and Computer Engineering Department, University of Wyoming, Box 3295 *University Station Laramie, WY* 82071, *U.S.A. E-mail: quot@uwyo.edu*

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This work considers the addition of damping to a beam system in order to limit the response due to random-in-time forcing functions. Here the optimization process for such dissipative attachments will use as an objective function the spatial average of the mean square response over the extent of the beam. It is shown here that optimal dampers and damped vibration absorbers can be found which minimize this objective function.

1. INTRODUCTION

This work considers the forced random vibration of a cantilever beam with either a viscous damper or a damped dynamic vibration absorber attached at the tip to provide energy dissipation to suppress the randomly excited motions. The theory developed is quite general and is applicable to a variety of other distributed systems to which lumped parameter dissipation has been added. There is a considerable literature of the dynamics of modified distributed parameter systems first explored in the work of Duncan [1]. Duncan coined the term admittance that was later changed to receptance in the excellent book of Bishop and Johnson [2]. In the past five decades there have been many papers on the forced and free vibration of modified systems [3–9]. More recently, several investigators have considered the random vibration of such modified systems [10–13].

Several recent papers [14, 15] have explored the incorporation of damping in distributed parameter systems to suppress random vibrations due to stationary random forcing functions. The first of these works [14] explored the application of a lumped damper and a damped dynamic absorber at the tip of a cantilever beam. The application of the tip damper was shown to limit the mean square response of the tip to any desired degree, but in doing so caused other locations on the beam to exhibit excessive vibration. As the damper became infinitely viscous, the motion of the tip became vanishingly small while beam supports approached the clamped–supported condition. A damped vibration absorber at the beam tip was shown to minimize the mean square tip vibrations at the expense of other points on the beam.

The second work [15] explores the use of damped dynamic vibration absorbers to suppress randomly forced oscillations in a simply supported rectangular plate. It was shown that for an absorber with a given mass ratio and tuning ratio an optimal damping ratio could be found so as to minimize the mean square vibration of the point at which the

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absorber is attached. As in reference [14] this optimization did not consider the motion of other points on the plate. It is clear that for any lumped parameter attachment to be truly effective in vibration suppression, points other than the point of attachment must be considered in the measure of vibration suppression effectiveness.

In this paper, the spatial average mean square motion is proposed as an objective measure of the best vibration suppression scheme. It will be shown that this measure will result in an optimal damper value or an optimal vibration absorber each applied at the tip of a cantilever beam. Although the results are presented here for a cantilever beam, the theory could be extended to other beam, plate or shell configurations.

2. DEVELOPMENT OF THE THEORY

2.1. TRANSFER FUNCTIONS

The system of interest here is the cantilever beam driven by a distributed load that is uniform in space and stationary and random in time with a damper or dynamic vibration absorber attached at the tip as illustrated in Figure 1. The transfer function between the point force applied at the tip by the damper or dynamic absorber and the motion of a point at location x [14] for a beam of length L and mass per unit length ρA is

$$G_1(s) = \frac{Y(x,s)}{P(s)} = \sum_{i=1}^{\infty} \frac{\phi_i(L)\phi_i(x)}{\rho AL(s^2 + \omega_i^2)}. \tag{1}$$

The transfer function between the uniform applied force $w(t)$ and the motion of a point x is [14]

$$G_2(s) = \frac{Y(x,s)}{W(s)} = \sum_{i=1}^{\infty} \frac{2\alpha_i \phi_i(x)}{\beta_i L\rho A(s^2 + \omega_i^2)}. \tag{2}$$

In these equations the $\phi_i(x)$ are the cantilever beam characteristic functions tabulated by Young and Felgar [16], α_i is a constant from the characteristic function and ω_i is the ith radian natural frequency of the beam and $\beta_i L$ is the ith root of the beam characteristic

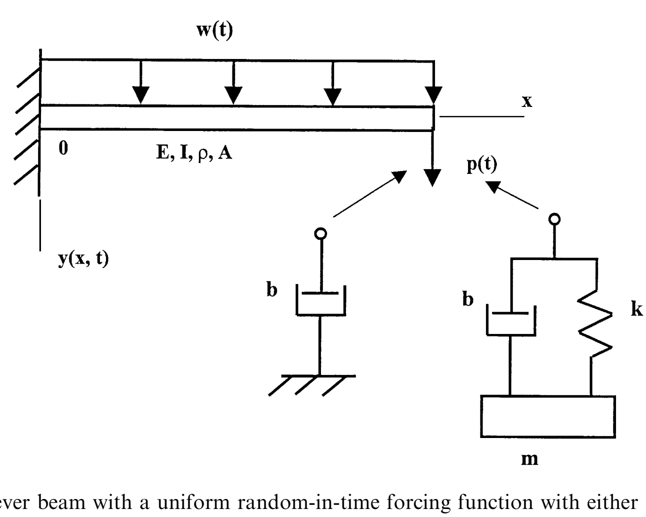

Figure 1. Cantilever beam with a uniform random-in-time forcing function with either a tip damper or a tip mounted dynamic vibration absorber.

equation. The transfer function between the driving force and the motion of the point $x = L$ in the presence of the attached damper or dynamic absorber is [14]

$$M(L,s) = \frac{Y(L,s)}{W(s)} = \frac{G_2(L,s)}{1 + G_1(L,s)H(s)}, \tag{3}$$

and that for any point x along the beam it is

$$M(x,s) = \frac{Y(x,s)}{W(s)} = G_2(x,s) - G_1(x,s)H(s)M(L,s), \tag{4}$$

where the $G_i(x,s)$ are as specified above and $H(s)$ is the displacement driving point impedance of the mechanical element(s) attached at the tip. For an attached damper, the displacement driving point impedance is

$$H(s) = bs \tag{5}$$

and for an attached dynamic absorber the driving point impedance is

$$H(s) = \frac{ms^2(bs+k)}{ms^2 + bs + k}. \tag{6}$$

2.2. SPATIAL AVERAGE MEAN SQUARE MOTION

For a zero mean stationary random forcing function, the mean square motion at a location x on the beam is given by

$$\sigma_y^2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |M(x,\mathrm{j}\omega)|^2 S_w(\omega)\,\mathrm{d}\omega. \tag{7}$$

Note that this is a function of x. If all points along the beam are of equal importance then the simplest possible gross measure of response is the spatial average of the mean square motion over the length of the beam or

$$\overline{\sigma^2} = \frac{1}{2\pi L}\int_0^L\int_{-\infty}^{\infty} |M(x,\mathrm{j}\omega)|^2 S_w(\omega)\,\mathrm{d}\omega\,\mathrm{d}x. \tag{8}$$

Once the tuning ratio (the ratio of the absorber undamped natural frequency to the first natural frequency of the beam) and mass ratio are selected it will be shown in the cases investigated here that $\overline{\sigma^2}$ can be minimized with respect to the dissipative system parameters.

2.3. FREQUENCY-SCALED TRANSFER FUNCTIONS

Let us scale the frequency variable by first scaling the complex s-domain variable in the various transfer functions by letting $s = \omega_1 p$ so the various transfer functions previously defined are

$$G_1(x,p) = \frac{1}{\rho A L \omega_1^2}\sum_{i=1}^{\infty}\frac{\phi_i(L)\phi_i(x)}{p^2 + \gamma_i^2} \tag{9}$$

and

$$G_2(x,p) = \frac{L^4}{EI}\sum_{i=1}^{\infty}\frac{2\alpha_i\phi_i(x)}{(\beta_1 L)^5\sqrt{\gamma_i}(p^2 + \gamma_i^2)}, \tag{10}$$

where the parameter $\gamma_i = \omega_i/\omega_1$ is the dimensionless ith natural frequency of the beam. Here extensive use has been made of the relation between the ith radian natural frequency and the eigenvalue β_i or

$$\omega_i^2 = \beta_i^4 \frac{EI}{\rho A}. \tag{11}$$

The respective scaled driving-point impedances are

$$H(p) = b\omega_1 p \tag{12}$$

for the damper and for the damped dynamic absorber,

$$H(p) = \frac{m\omega_1^2 p^2 (2\zeta T p + T^2)}{p^2 + 2\zeta T p + T^2}, \tag{13}$$

where $T = \omega_a/\omega_1$ is the absorber tuning ratio with $\omega_a = \sqrt{k/m}$, and the damping ratio is $\zeta = b/2\sqrt{km}$.

3. ATTACHED DAMPER

Substituting the quantities from relations (9), (10) and (12) into relation (3) the transfer function $M(x,p)$ is

$$M(L,p) = \frac{(L^4/EI)\sum_{i=1}^{\infty} 2\alpha_i \phi_i(L)/(\beta_1 L)^5 \sqrt{\gamma_i}(p^2+\gamma_i^2)}{1 + (bp/\rho A L\omega_1)\sum_{i=1}^{\infty}\phi_i^2(L)/(p^2+\gamma_i^2)} \tag{14}$$

and further substituting relations (9), (10) and (12) into relation (4), the result is

$$M(x,p) = \sum_{n=1}^{\infty} \frac{\phi_n(x)}{p^2+\gamma_n^2}\left(\frac{2\alpha_n(L^4/EI)}{(\beta_1 L)^5\sqrt{\gamma_n}} - \frac{bp\phi_n(L)}{\rho A L\omega_1} M(L,p)\right), \tag{15}$$

where $M(L,p)$ is defined by relation (14). The frequency responses associated with the transfer functions (14) and (15) are given by letting $p = \mathrm{j}f$, where f is a scaled frequency variable given by $f = \omega/\omega_1$.

The appropriate frequency response functions are

$$M(x,\mathrm{j}f) = \sum_{n=1}^{\infty} \frac{\phi_n(x)}{\gamma_n^2 - f^2}\left(\frac{2\alpha_n(L^4/EI)}{(\beta_1 L)^5\sqrt{\gamma_n}} - \frac{\mathrm{j}f\mathrm{b}\phi_n(L)}{\rho A L\omega_1} M(L,\mathrm{j}f)\right), \tag{16}$$

where $M(L,\mathrm{j}f)$ is given from relation (14)

$$M(L,\mathrm{j}f) = \frac{(L^4/EI)\sum_{i=1}^{\infty} 2\alpha_i \phi_i(L)/(\beta_1 L)^5 \sqrt{\gamma_i}(\gamma_i^2 - f^2)}{1 + (\mathrm{j}fb/\rho A L\omega_1)\sum_{i=1}^{\infty}\phi_i^2(L)/\gamma_i^2 - f^2}. \tag{17}$$

Then $M(x,\mathrm{j}f)$ can thus be written as

$$M(x,\mathrm{j}f) = \frac{L^4}{EI}\sum_{n=1}^{\infty}\frac{\varepsilon_n(\mathrm{j}f)\phi_n(x)}{\gamma_n^2 - f^2}, \tag{18}$$

where $\varepsilon_n(\mathrm{j}f)$ is from relation (16),

$$\varepsilon_n(\mathrm{j}f) = \frac{1}{(\beta_1 L)^5}\left(\frac{2\alpha_n}{\beta_n L\sqrt{\gamma_n}} - \frac{\mathrm{j}fb}{\rho A L\omega_1}\phi_n(L)\frac{\sum_{i=1}^{\infty} 2\alpha_i\phi_i(L)/\sqrt{\gamma_i}(\gamma_i^2 - f^2)}{1 + (\mathrm{j}fb/\rho A L\omega_1)\sum_{i=1}^{\nabla}\phi_i^2(L)/(\gamma_i^2 - f^2)}\right). \tag{19}$$

If the driving force is white noise with power spectral density S_w then the power spectral density of the motion of any point on the beam is

$$S_y(x,f) = S_w M(x,\mathrm{j}f)M(x,-\mathrm{j}f). \tag{20}$$

The spatial average mean square motion is given by expression (7) to be

$$\overline{\sigma^2} = \frac{S_w\omega_1}{2\pi L}\int_{-\infty}^{\infty}\int_0^L M(x,\mathrm{j}f)M(x,-\mathrm{j}f)\,\mathrm{d}x\,\mathrm{d}f, \tag{21}$$

where $M(x,\mathrm{j}f)$ is given by relation (18).

As evidenced by relation (18) both $M(x,\mathrm{j}f)$ and $M(x,-\mathrm{j}f)$ are generalized Fourier series in the beam functions $\phi_n(x)$and thus the spatial averaging integral of equation (21) will deal with the averaging of the products of the beam functions. Fortunately, the beam functions are orthogonal thus reducing the double sum to a single sum using the fact [17]

$$\int_0^L \phi_n(x)\phi_m(x)\,\mathrm{d}x = L\delta_{mn}, \tag{22}$$

where δ_{mn} is the Kronecker delta function. With all these properties noted, the objective function may be written as

$$\frac{\overline{\sigma^2}(EI)^2}{S_w\omega_1 L^8} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}\left|\frac{\varepsilon_n(\mathrm{j}f)}{\gamma_n^2-f^2}\right|^2\mathrm{d}f \tag{23}$$

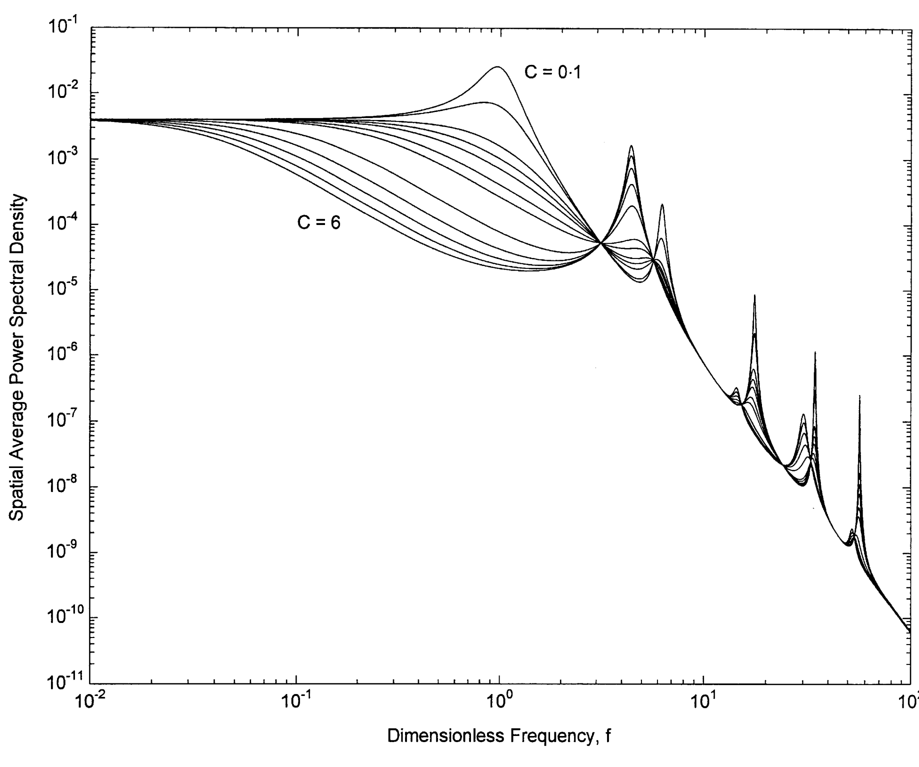

Figure 2. Dimensionless spatial average motion power spectral density for values of the dimensionless damping coefficient of $C = 0{\cdot}1$, 0·2, 0·4, 0·6, 0·8, 1, 2, 3, 4, 5, 6.

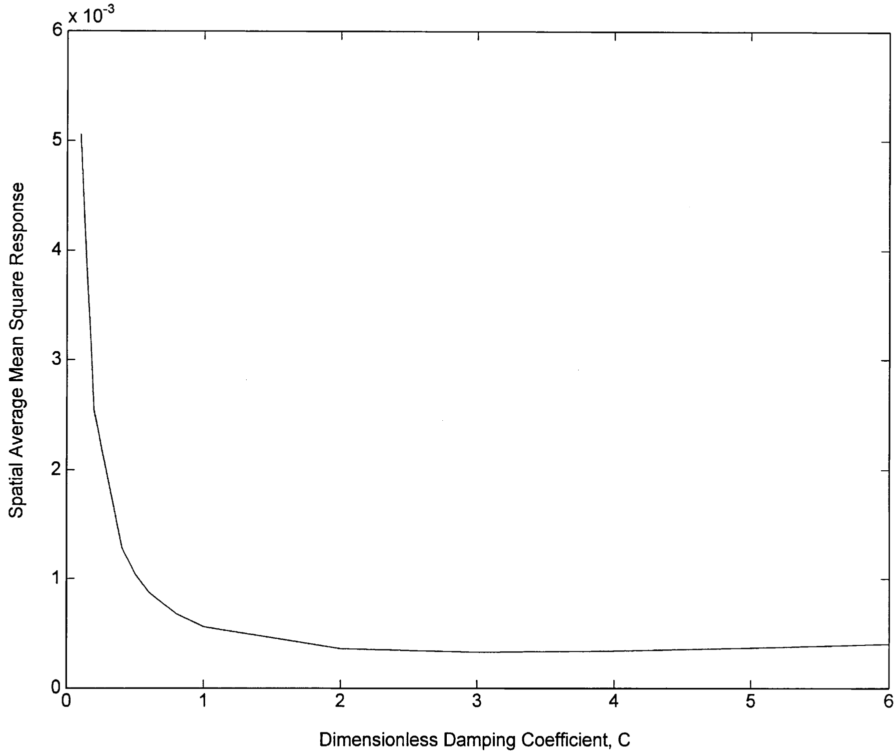

Figure 3. Dimensionless spatial average mean square motion as a function of the dimensionless damper coefficient.

and from equation (19),

$$\varepsilon_n(\mathrm{j}f) = \frac{1}{(\beta_1 L)^5}\left(\frac{2\alpha_n}{\beta_n L\sqrt{\gamma_n}} - C\mathrm{j}f\phi_n(L)\left(\sum_{i=1}^{\infty} 2\alpha_i\phi_i(L)/\sqrt{\gamma_i}(\gamma_i^2 - f^2)\right)\right)\Big/ \left(1 + C\mathrm{j}f\sum_{i=1}^{\nabla}\phi_i^2(L)/(\gamma_i^2 - f^2)\right), \tag{24}$$

where the dimensionless damper coefficient is given by

$$C = \frac{b}{\rho A L\omega_1}. \tag{25}$$

The quantity under the integral of expression (23) is the dimensionless average motion power spectral density function. This function has been evaluated for a sequence of values of the parameter C and is illustrated in Figure 2. Careful observation of this figure reveals that as the dimensionless damping coefficient C is increased the first cantilever natural frequency resonance decreases, but at some point a new resonance associated with the first mode of a clamped–supported beam begins to grow. An interesting question to be answered is whether the average mean square motion (which is proportional to the average power spectral density area) has a minimum for some value of C? The answer was given by evaluating the average mean square motion from equation (23) for a range of values

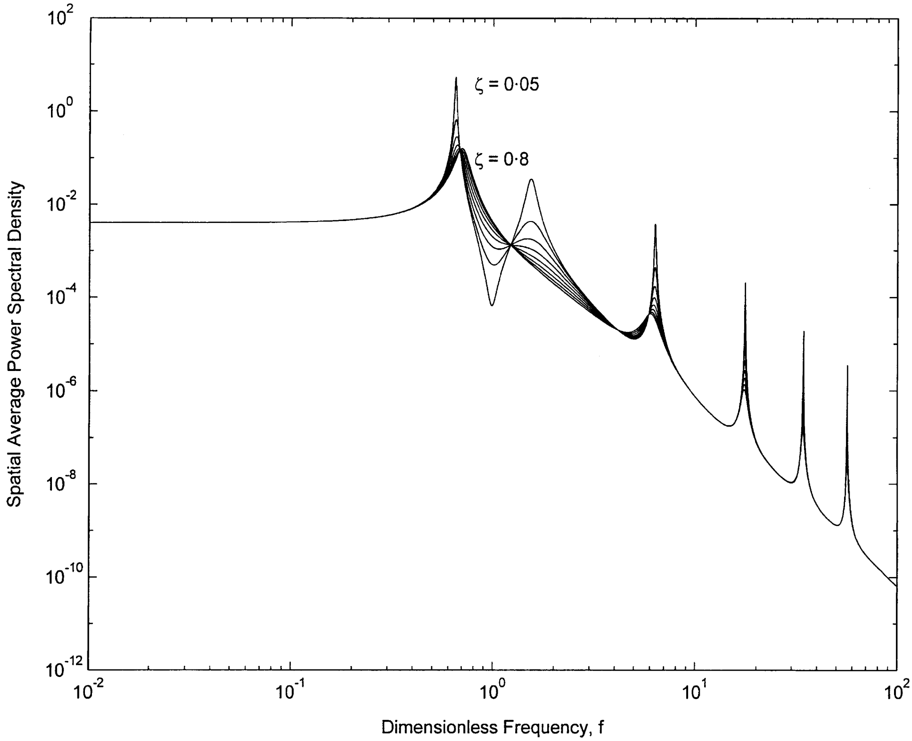

Figure 4. Dimensionless spatial average motion power spectral density for a mass ratio of $\mu = 0{\cdot}2$, a tuning ratio of unity and values of absorber damping ratio of $\zeta = 0{\cdot}05$–$0{\cdot}8$ in increments of 0·05.

of C. The spectral density integrals were evaluated using the trapezoidal rule in MATLAB and the results are given in Figure 3. The number of points and the range of integration were varied such that there was confidence in the numerical answers given. The series were also truncated at both four and five terms and the results are not sensitive to the truncation. It is now clear that a minimum does exist for a value of C near 3.

4. ATTACHED DYNAMIC VIBRATION ABSORBER

Consider the situation where a damped dynamic absorber is attached to the beam tip in order to provide vibration suppression without an attachment to ground as was the case for the damper previously considered. The needed transfer functions have already been presented so substitution of relations (9), (10) and (13) into relation (3) gives

$$M(L,p) = \frac{(L^4/EI)\sum_{i=1}^{\infty} 2\alpha_i\phi_i(L)/((\beta_1 L)^5\sqrt{\gamma_i}(p^2+\gamma_i^2))}{1+\mu(p^2(2\zeta pT+T^2)/(p^2+2\zeta Tp+T^2))\sum_{i-1}^{\infty}\phi_i^2(L)/(p^2+\gamma_i^2)}, \quad (26)$$

where the mass ratio is $\mu = m/\rho AL$. The transfer function between the applied force and the motion at any point is

$$M(x,p) = \sum_{n=1}^{\infty}\frac{\phi_n(x)}{p^2+\gamma_n^2}\left(\frac{2\alpha_n(L^4/EI)}{(\beta_1 L)^5\sqrt{\gamma_n}} - \mu\left(\frac{p^2(2\zeta Tp+T^2)}{p^2+2\zeta Tp+T^2}\right)\phi_n(L)M(L,p)\right) \quad (27)$$

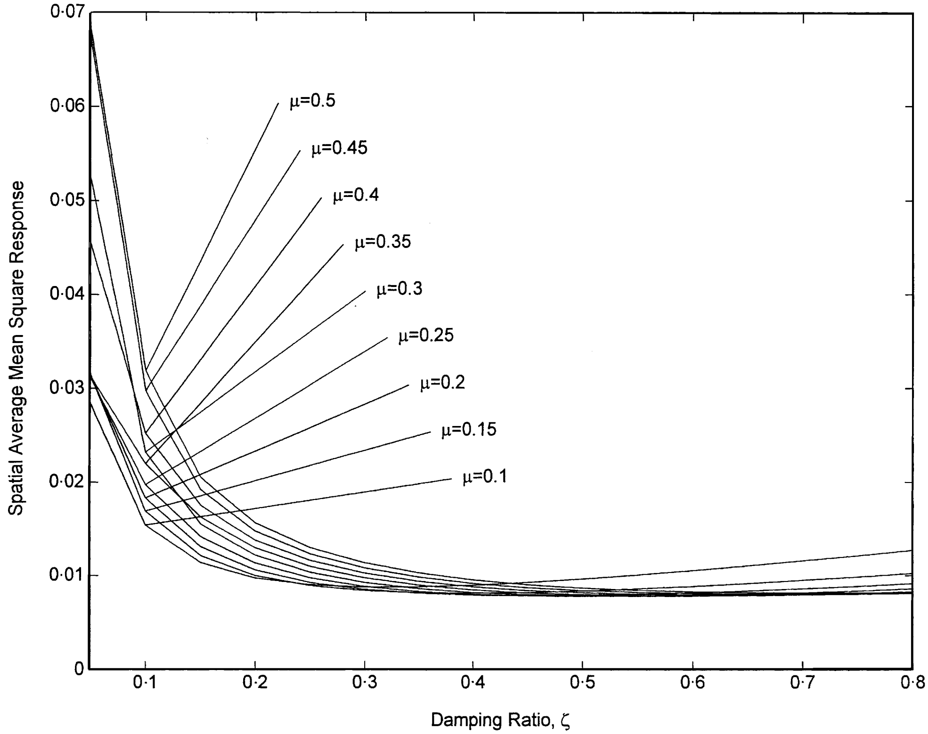

Figure 5. Dimensionless spatial average mean square motion as a function of absorber damping ratio for various mass ratios with a tuning ratio of unity.

where $M(L,p)$ is defined is defined by relation (26). The respective frequency response functions are

$$M(L,\mathrm{j}f)=\frac{(L^4/EI)\sum_{i=1}^{\infty}2\alpha_i\phi_i(L)/((\beta_1L)^5(\gamma_i^2-f^2))}{1-\mu(f^2(\mathrm{j}2\zeta Tf+T^2)/(T^2-f^2+\mathrm{j}2\zeta Tf))\sum_{i=1}^{\infty}\phi_i^2(L)/(\gamma_i^2-f^2)} \tag{28}$$

and thus as before

$$M(x,\mathrm{j}f)=\frac{L^4}{EI}\sum_{n=1}^{\infty}\frac{\varepsilon_n(\mathrm{j}f)\phi_n(x)}{\gamma_n^2-f^2}, \tag{29}$$

where the coefficient $\varepsilon_n(\mathrm{j}f)$is defined as

$$\varepsilon_n(\mathrm{j}f)=\frac{1}{(\beta_1L)^5}\left[\frac{2\alpha_n}{\sqrt{\gamma_n}}+\mu\left(\frac{f^2(\mathrm{j}2\zeta Tf+T^2)}{T^2-f^2+\mathrm{j}2\zeta Tf}\right)\phi_n(L)\left(\frac{M(L,\mathrm{j}f)(\beta_1L)^5}{L^4/EI}\right)\right] \tag{30}$$

and where $M(L,\mathrm{j}f)$ is defined in relation (28). The dimensionless spatial average motion power spectral density is then

$$\frac{\bar{S}_y(f)(EI)^2}{S_wL^8}=\sum_{n=1}^{\infty}\frac{\varepsilon_n(\mathrm{j}f)\varepsilon_n(-\mathrm{j}f)}{(\gamma_n^2-f^2)^2}. \tag{31}$$

For a mass ratio of $\mu=0{\cdot}2$ and a tuning ratio of $T=1$ the dimensionless average power spectral density of motion has been calculated and is illustrated in Figure 4 for several

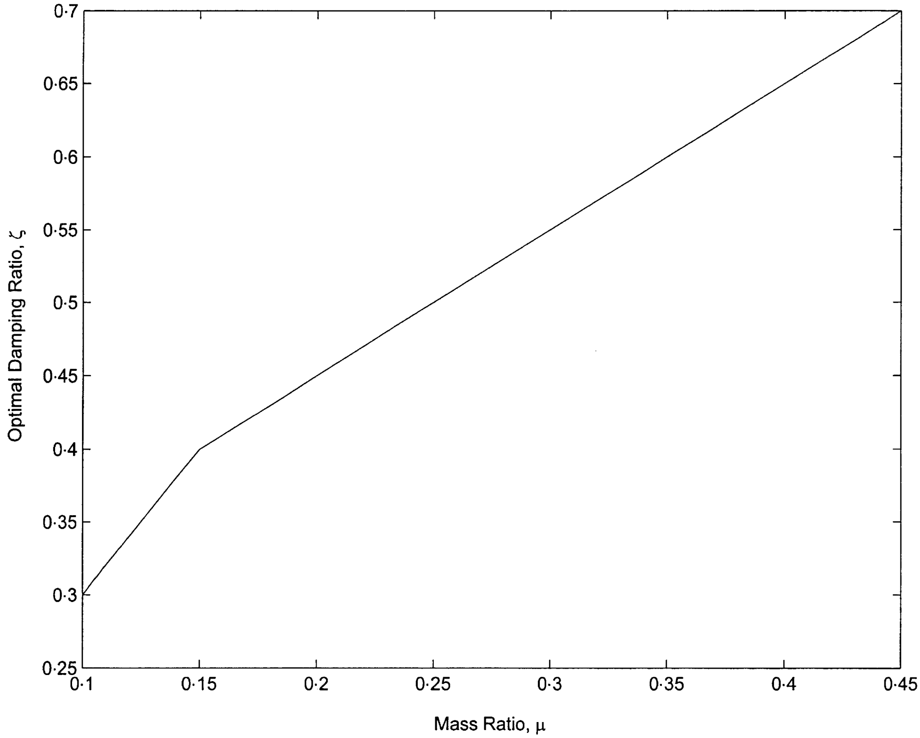

Figure 6. Optimal absorber damping ratio as a function of mass ratio for a tuning ratio of unity.

values of dynamic absorber damping ratio. The spatial average motion variance is given by integration of this function and has been accomplished for a series of values of mass ratio and damping ratio and the result is illustrated in Figure 5. It is clear that for a given mass ratio there is a damping ratio that will yield a minimum average mean square motion. The values of damping ratio that yield a minimum mean square response are shown as a function of mass ratio in Figure 6. Clearly, there is an optimal damping ratio for every mass ratio and these values are illustrated in Figure 7. If these minima are evaluated as a function of mass ratio the minimum occurs at a mass ratio of 0·25 and the optimal damping ratio for this mass ratio is 0·5. In a practical situation a smaller mass ratio would probably be employed along with an accompanying smaller damping ratio. This, of course, would result in a larger average mean square motion.

5. CONCLUSION AND SUGGESTIONS FOR FUTURE WORK

Optimal random vibration suppression in distributed parameter systems has been considered here by attachment of lumped parameter dampers and damped vibration absorbers to an undamped cantilever beam. The measure of the effectiveness of vibration suppression used here is the spatial average mean square response. It is shown that optimal damper and dynamic absorber parameters exist for minimization of the spatial average mean square motion.

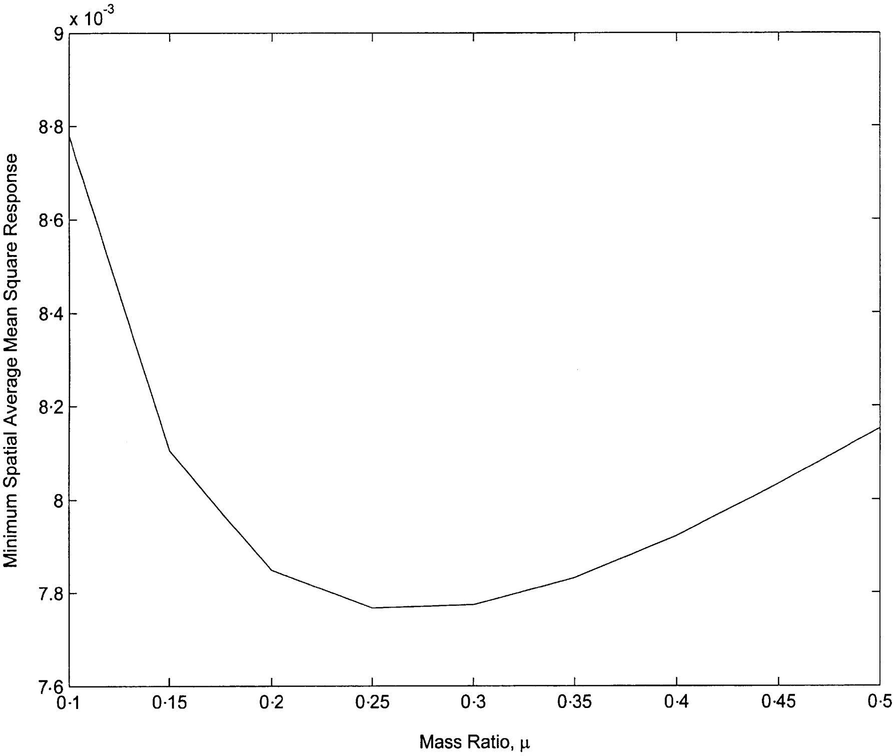

Figure 7. Minimum spatial average mean square motion as a function of mass ratio assuming optimal damping as presented in Figure 5 for a tuning ratio of unity.

In the course of this investigation it has become apparent that two additional studies need to be accomplished to bring closure to the addition of damping to suppress random vibration in distributed parameter structures. The first is to examine the effect of location of the damping element on the optimization process. The tip location for the application of damping elements for a cantilever beam is a unique situation in that the cantilever tip is an antinode for all of the beam's undamped modes, a situation not present in many other structural configurations. A second study should involve the use the weighted spatial average mean square motion as an optimization criterion for emphasis of a particular region of spatial points over others.

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