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LETTERS TO THE EDITOR

MULTIPLE EXTERNAL EXCITATIONS FOR TWO NON-LINEARLY COUPLED VAN DER POL OSCILLATORS

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1. INTRODUCTION

In recent years, a number of authors have studied the response of non-linear oscillators subject to N multiple resonant or non-resonant forcing terms. In particular, the Van der Pol oscillator and many other special cases have been studied extensively for $N = 2$ [1–5]. For $N > 2$ the method of multiple time scales has been used to study the van der Pol and the Duffing oscillators [6], but with no comparison with numerical results.

These particular studies have been extended to a more general class of non-linear oscillators and compared with numerical results [7]. The most important finding is that if the forcing frequencies are not close to the primary resonant frequency, the amplitude of the free oscillation will decay exponentially in time, if the amplitude of the forcing term is sufficiently large, but will otherwise approach a constant value.

When the forcing frequencies are all close to a particular frequency Ω, "quenching" is possible, but in certain cases the amplitude of the free oscillation is modulated with some frequencies determined by the detuning parameters. When the forcing frequencies are close to the resonant frequency, then both the amplitude and the phase of the free oscillation can eventually oscillate with a frequency that is determined by both the forcing amplitudes and the detuning parameters.

Another paper has considered the multiple resonant or non-resonant parametric excitations of non-linear oscillators [8] and has demonstrated that the oscillation cannot be fully quenched, because the only effect of parametric excitation is a shift in the oscillation frequency.

In this paper, two-degree-of-freedom systems with multiple external excitations are considered and in particular the transient and steady-state response of two non-linearly coupled van der Pol oscillators subject to a finite number of harmonic forcing terms are studied. The relevant system of differential equations is

$$\ddot{X} + \omega_1^2 X - \varepsilon(1 - X^2 - aY^2)\dot{X} = F(t), \tag{1}$$

$$\ddot{Y} + \omega_2^2 Y - \varepsilon(1 - bX^2 - Y^2)\dot{Y} = 0, \tag{2}$$

where the dots denote differentiation with respect to the time, ω_1 and ω_2 are incommensurable frequencies, the constants a, b are of order 1, ε is a small parameter and $F(t)$ is a finite sum of N harmonic forcing terms of the form

$$F(t) = 2\varepsilon \sum_{i=1}^{N} A_i \cos(\Omega_i t), \tag{3}$$

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where A_i is the amplitude and Ω_i is the frequency of the ith component of $F(t)$. In the following only the case $N > 1$ (multiple excitations) is considered.

The van der Pol system [9] is a basic model of self-excited oscillation in physics, mechanics, biology, electronics, chemistry and other disciplines [10–12]. For example, it appears in non-linear electrical circuits [13], electrical activity in gastrointestinal tracts of humans and animals [14] and gear systems dynamics [15], where the non-linear coupling terms in progressive damping are caused by breaking of the oil film between the meshing teeth.

The paper investigates especially the modifications induced by the non-linear terms on the solution of the linearized version ($\varepsilon = 0$) of equations (1)–(2):

$$X(t) = 2\rho_0 \cos(\omega_1 t - \vartheta_0) + \sum_{i=1}^{N} \frac{2A_i}{(1-\Omega_i^2)} \cos(\Omega_i t), \tag{4}$$

$$Y(t) = 2\chi_0 \cos(\omega_2 t - \phi_0), \tag{5}$$

where ρ_0, ϑ_0 and χ_0, ϕ_0 are fixed by the initial conditions. This solution is the sum of the free oscillation and of the forced oscillation. When the non-linear terms are added, the free oscillation, i.e., the first term of the r.h.s. of the equations (4)–(5), will persist or decay ("quenching").

The paper is organized as follows. In section 2, when the forcing frequencies are not close to each other and not close to the primary resonances of the two modes, the response of the non-linear system (1)–(2) is examined and the formal perturbation solution is carried out to the lowest order approximation. Both the conditions for the quenching of the free oscillation and the conditions for its persistence are determined and analytical results are validated by numerical integration.

In section 3 the forcing frequencies are supposed to be close to each other, but not close to the primary resonances of the two modes, while in section 4 the forcing frequencies near the primary resonance of the first mode are considered.

The conclusions are reserved for section 5.

2. FORCING FREQUENCIES NOT CLOSE TO EACH OTHER

The asymptotic perturbation method has been derived from a similar method employed in non-linear partial differential equations [16] and is based on the detailed computation of the interaction, induced by the non-linear terms, of the harmonic solutions of the linear part of the differential equation.

By means of the temporal rescaling

$$\tau = \varepsilon^q t, \tag{6}$$

with q a rational positive number, which will be fixed later on, the asymptotic behavior of the solution can be investigated: when $t \to \infty$ and $\varepsilon \to 0$, the parameter q can be chosen in such a way that τ assumes finite values.

In this section the forcing frequencies Ω_i are supposed to be not close to each other or close to the primary resonances of the two modes and the required solution is expressed as a perturbation expansion, based on the parameter ε, which is formally written

$$X(t) = \sum_{n(\mathrm{odd})=-\infty}^{+\infty} \varepsilon^{\gamma_n} \psi_n(\tau;\varepsilon) \exp(-\mathrm{i} n\omega_1 t) + \varepsilon \left(\sum_{i=1}^{N} \frac{A_i}{(1-\Omega_i^2)} (\exp(\mathrm{i}\Omega_i t) + \mathrm{c.c.}) \right), \tag{7}$$

$$Y(t) = \sum_{n(\mathrm{odd})=-\infty}^{+\infty} \varepsilon^{\gamma_n} \varphi_n(\tau;\varepsilon) \exp(-\mathrm{i} n\omega_2 t), \tag{8}$$

where c.c. stands for complex conjugate, $\gamma_n = |n| - 1$ and $\psi_n(\tau, \varepsilon) = \psi^*_{-n}(\tau, \varepsilon)$, because $X(t)$, $Y(t)$ are real (the asterisk denotes complex conjugate). The functions $\psi_n(\tau, \varepsilon)$, $\varphi_n(\tau, \varepsilon)$ depend on the parameter ε and it is supposed that their limit for $\varepsilon \to 0$ exists and is finite.

The solution is then a Fourier expansion in which the coefficients vary slowly in time and evolution equations for the amplitudes of the harmonics terms are derived by substituting the expression of solutions (7)–(8) into the original equations (1)–(2) and projecting onto each Fourier mode.

A key feature of the present method is that the advantages of the harmonic balance method (see equations (7)–(8)) and the multiple scales technique (see equation (6)) are simultaneously taken into account.

Indicating with $\psi(\tau)$, $\varphi(\tau)$ the limits of $\psi_1(\tau, \varepsilon)$, $\varphi_1(\tau, \varepsilon)$ when $\varepsilon \to 0$, the following equations are obtained for $n = 1$:

$$2\psi_\tau \varepsilon^q + \varepsilon(2A - 1 + 2a|\phi|^2 + |\psi|^2)\psi + h.o.t. = 0, \tag{9}$$

$$2\phi_\tau \varepsilon^q + \varepsilon(2A - 1 + 2b|\psi|^2 + |\phi|^2)\phi + h.o.t. = 0, \tag{10}$$

where

$$A = \sum_{i=1}^{N} \frac{A_i^2}{(1 - \Omega_i^2)^2}, \tag{11}$$

and $h.o.t.$ stands for higher order terms. For the proper balance of the various terms, the choice $q = 1$ is necessary and, by means of the standard substitutions,

$$\psi(\tau) = \rho(\tau)\exp(\mathrm{i}\vartheta(\tau)), \qquad \varphi(\tau) = \chi(\tau)\exp(\mathrm{i}\phi(\tau)), \tag{12}$$

the nonlinear equations,

$$\rho_\tau = \frac{\rho}{2}(1 - 2A - \rho^2 - 2a\chi^2), \qquad \chi_\tau = \frac{\chi}{2}(1 - 2A - \chi^2 - 2b\rho^2), \tag{13, 14}$$

$$\vartheta_\tau = \phi_\tau = 0. \tag{15}$$

can be easily obtained.

The approximate solution is then

$$X(t) = 2\rho(\varepsilon t)\cos(-\omega_1 t + \vartheta_0) + \sum_{i=1}^{N} \frac{2A_i}{(1 - \Omega_i^2)}\cos(\Omega_i t), \tag{16}$$

$$Y(t) = 2\chi(\varepsilon t)\cos(-\omega_2 t + \phi_0), \tag{17}$$

where ϑ_0 and ϕ_0 are the initial conditions for the phases ϑ and ϕ.

It is now necessary to establish the steady state solutions of the dynamical system and to perform the stability analysis. Four different constant solutions are possible:

(1) the trivial solution,

$$P_1 = (\rho_1, \chi_1) = (0; 0); \tag{18}$$

(2) a solution with one non-vanishing component,

$$P_2 = (\rho_2, \chi_2) = (\sqrt{1 - 2A}; 0), \tag{19}$$

corresponding to a $(N + 1)$-period quasi-periodic motion for the first mode;

(3) another solution with one non-vanishing component,

$$P_3 = (\rho_3, \chi_3) = (0; \sqrt{1-2A}), \tag{20}$$

corresponding to a periodic motion for the second mode;

(4) a solution with two excited modes,

$$P_4 = (\rho_4, \chi_4) = \left(\sqrt{\frac{(2a-1)(1-2A)}{(4ab-1)}}; \sqrt{\frac{(2b-1)(1-2A)}{(4ab-1)}} \right), \tag{21}$$

corresponding to a $(N+1)$-period quasi-period motion for the first mode and a periodic motion for the second mode.

Results of a stability analysis for two typical cases are given in Figures 1 and 2. They reveal the different types of behavior that occur in the parameter space (here, the (a, b) space).

There are six types of regions, which correspond to different results for the two modes competition:

(1) a blank region: no steady state solution is stable;
(2) a region filled with boxes: only the solution P_3 (20) is stable;
(3) a region filled with crosses: only the solution P_2 (19) is stable;
(4) a region filled with circles: only the solution P_4 (21) is stable;
(5) a region filled with crosses and boxes: both P_2 (19) and P_3 (20) are stable, then the survival of a specific mode depends on initial conditions;
(6) a region filled with circles, boxes and crosses: the trivial solution (18), P_2 (19) and P_3 (20) are at the same time stable, then the initial conditions determine the survival of a specific mode or the quenching;

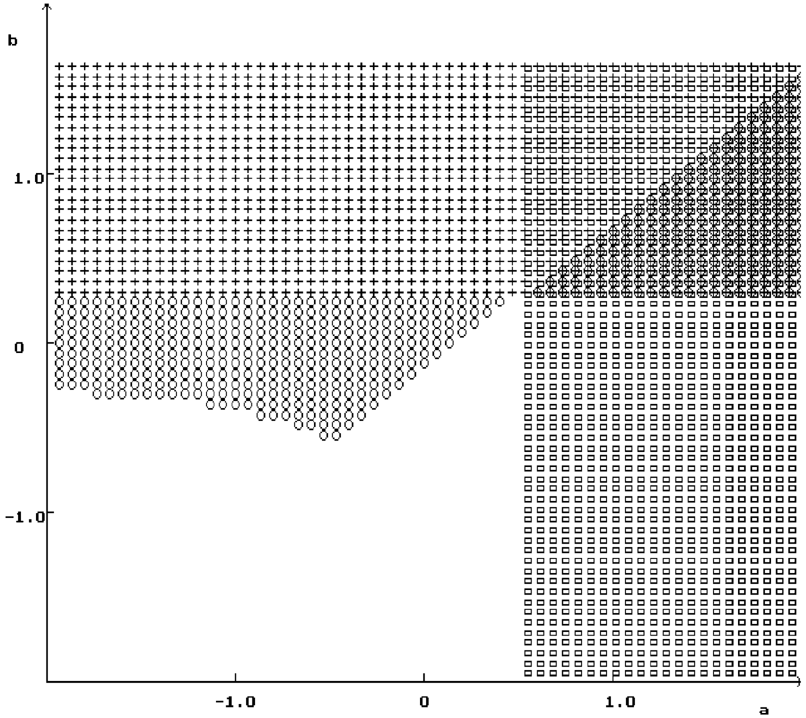

Figure 1. Results of stability analysis in the (a, b) parameter space ($A = 0{\cdot}4$).

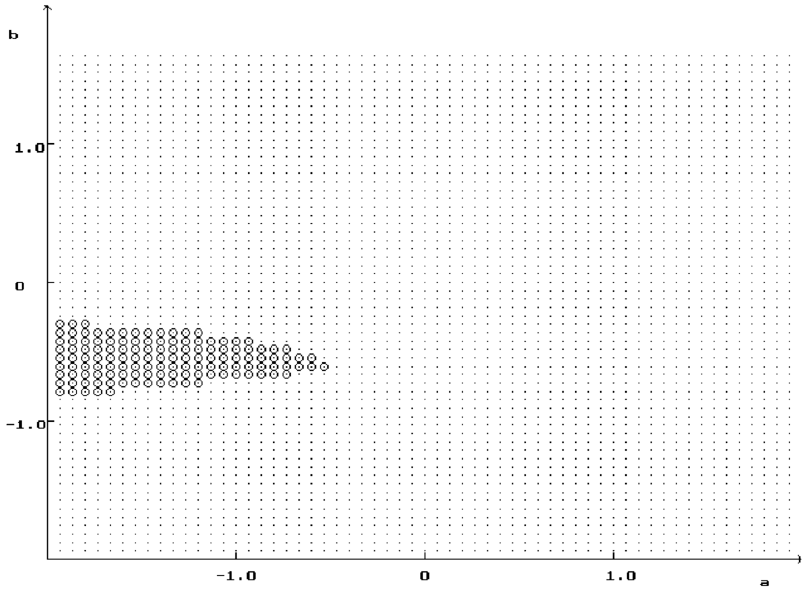

Figure 2. Results of stability analysis in the (a, b) parameter space ($A = 1{\cdot}0$).

(7) a region filled with points: only the trivial solution (18) is stable (quenching of the solution);
(8) a region filled with circles and points: the trivial solution (18) and P_4 (21) are at the same time stable, then the survival of the two modes or the quenching depend on initial conditions.

All the first six regions are present in Figure 1, while only regions (7) and (8) are present in Figure 2. In regions (2), (3) and (5) by the end of transients only one mode will be excited and if the initial conditions are two-mode amplitudes different from zero, only one mode survives and suppresses the other mode. Numerical integration of the non-linear equations (1)–(2) confirms the qualitative picture which emerges from the perturbative analysis. For example, in Figure 3, the associated map, obtained with the values $(X(0), Y(0)), (X(T), Y(T)), (X(2T), Y(2T)), \ldots,$ where $T = 2\pi/\omega_1$, has been shown and the numerical solution has been compared with the approximate solution (16,17). The mean difference between the two solutions is 0·01, i.e., of order ε^2 as expected.

3. THE APPROXIMATE SOLUTION WITH THE FREQUENCIES CLOSE TO EACH OTHER

The results of the previous section can be extended to the case when the forcing frequencies are close to each other, but not close to the primary frequencies of the two modes, i.e.,

$$\Omega_i = \Omega + \varepsilon\sigma_i, \quad i = 1, \ldots, N, \tag{22}$$

where Ω is a fixed frequency not close to one, while σ_i measures the differences of the frequencies from each other. Substituting equation (23) in equation (3), the external

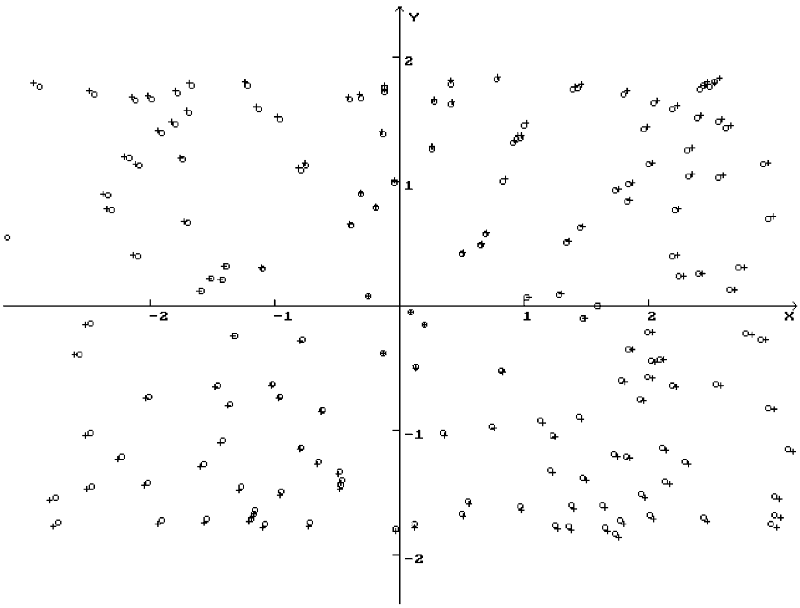

Figure 3. Comparison between numerical (crosses) and analytical (circles) solutions in the (X, Y) plane. Values of parameters: $a = -0{\cdot}5$, $b = 0{\cdot}1$, $\varepsilon = 0{\cdot}08$, $\omega_1 = 1$, $\omega_2 = \sqrt{2}$. Forcing frequencies not close to each other and not close to the primary resonance: $\Omega_1 = \sqrt{3}$, $\Omega_2 = \sqrt{5}$, $\Omega_3 = \sqrt{1{\cdot}7}$. Amplitudes of the external excitations: $A_1 = 0{\cdot}1$, $A_2 = 0{\cdot}2$, $A_3 = 0{\cdot}3$.

excitation $F(t)$ becomes

$$F(t) = \frac{\exp(\mathrm{i}\Omega t)}{(1-\Omega^2)} \sum_{i=1}^{N} \varepsilon A_i \exp(\mathrm{i}\sigma_i \tau) + \text{c.c.} + O(\varepsilon^2). \tag{23}$$

The first mode of the van der Pol system is then subject to an applied force with frequency Ω and with an amplitude that is a slowly varying function of time.

The same method as in section 2 is applied and equations (13)–(14) are newly obtained,

$$\rho_\tau = \frac{\rho}{2}(1 - 2A(\tau) - \rho^2 - 2a\chi^2), \tag{24}$$

$$\chi_\tau = \frac{\chi}{2}(1 - 2A(\tau) - \chi^2 - 2b\rho^2), \qquad \vartheta_\tau = \phi_\tau = 0, \tag{25, 26}$$

but now with the coefficient A substituted with

$$A(\tau) = B + C(\tau) = \frac{1}{(1-\Omega^2)} \sum_{i=1}^{N} A_i^2 + \frac{1}{(1-\Omega^2)} \sum_{i,j=1(i\neq j)}^{N} A_i A_j \exp(\mathrm{i}(\sigma_i - \sigma_j)\tau). \tag{27}$$

Also in this case the evolution of $\rho(\tau)$ and $\chi(\tau)$ does not depend on $\vartheta(\tau)$ and $\phi(\tau)$, but the difference is now that the coefficients of the non-linear system (13)–(14) are not constant.

Also in this case the decay of the free oscillation and the quenching of the solution are possible, but a new behavior can arise, not observable for $N = 1$: the amplitude of the free oscillation approaches an oscillatory function of time, which depends on both the amplitudes A_i as well as the detuning parameters σ_i.

The conditions for the quenching of the solution are the same as those examined in the previous section (with the coefficient A in the equation (11) substituted with B), when one or two modes are excited their amplitudes are not constant, but modulated with frequencies depending on the detuning parameters.

For example for case (4) the constant solution (21) is substituted with

$$\rho_S = \rho_4 + \sqrt{\sum_{i \neq j} \frac{\{4a\rho_4^2\chi_4^2 - 2\rho_4^2[\mathrm{i}(\sigma_i - \sigma_j) + \chi_4^2]\}A_i A_j \exp \mathrm{i}(\sigma_i - \sigma_j)\tau}{(1 - \Omega^2)\{[\mathrm{i}(\sigma_i - \sigma_j) + \rho_4^2][\mathrm{i}(\sigma_i - \sigma_j) + \chi_4^2] - 4ab\rho_4^2\chi_4^2\}}}, \tag{28}$$

$$\chi_S = \chi_4 + \sqrt{\sum_{i \neq j} \frac{\{4b\rho_4^2\chi_4^2 - 2\chi_4^2[\mathrm{i}(\sigma_i - \sigma_j) + \rho_4^2]\}A_i A_j \exp \mathrm{i}(\sigma_i - \sigma_j)\tau}{(1 - \Omega^2)\{[\mathrm{i}(\sigma_i - \sigma_j) + \rho_4^2][\mathrm{i}(\sigma_i - \sigma_j) + \chi_4^2] - 4ab\rho_4^2\chi_4^2\}}}. \tag{29}$$

The approximate solution is then

$$X(t) = 2\rho(\varepsilon t)\cos(-\omega_1 t + \vartheta_0) + \frac{1}{(1 - \Omega^2)} \sum_{i=1}^{N} 2A_i \cos(\Omega_i t), \tag{30}$$

$$Y(t) = 2\chi(\varepsilon t)\cos(-\omega_2 t + \phi_0), \tag{31}$$

where ϑ_0 and ϕ_0 are the initial conditions for the phases ϑ and ϕ.

If the initial conditions are near the point P_4 (21) then an approximate analytic solution can be easily obtained for the asymptotic solution of equations (24)–(25),

$$\rho(\varepsilon t) = \rho_0 \left[1 - \frac{2}{(1 - \Omega^2)} \sum_{i,j=1 (i \neq j, i > j)}^{N} \frac{A_i A_j}{(\sigma_i - \sigma_j)} \sin[(\sigma_i - \sigma_j)\tau] \right], \tag{32}$$

$$\chi(\varepsilon t) = \chi_0 \left[1 - \frac{2}{(1 - \Omega^2)} \sum_{i,j=1 (i \neq j, i > j)}^{N} \frac{A_i A_j}{(\sigma_i - \sigma_j)} \sin[(\sigma_i - \sigma_j)\tau] \right]. \tag{33}$$

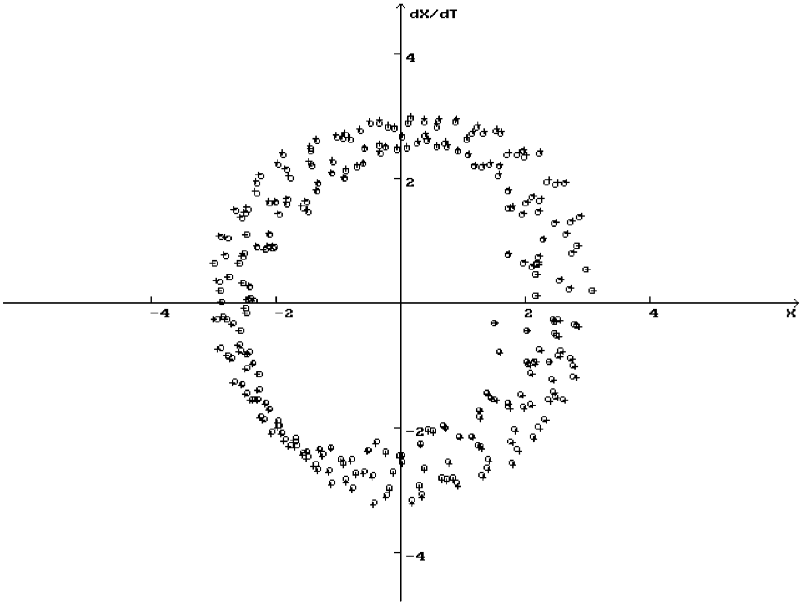

Figure 4. Comparison between numerical (crosses) and analytical (circles) solutions in the $(X, \mathrm{d}X/\mathrm{d}T)$ plane. Values of parameters: $a = -0{\cdot}5$, $b = 0{\cdot}1$, $\varepsilon = 0{\cdot}08$, $\omega_1 = 1$, $\omega_2 = \sqrt{2}$. Forcing frequencies close to each other but not close to the primary resonance: $\Omega_1 = \sqrt{3}$, $\Omega_2 = \sqrt{3{\cdot}02}$, $\Omega_3 = \sqrt{2{\cdot}99}$. Amplitudes of the external excitations: $A_1 = 0{\cdot}1$, $A_2 = 0{\cdot}2$, $A_3 = 0{\cdot}3$.

The resulting motion is slowly modulated with frequencies depending on the detuning parameters.

In Figure 4 the associated map obtained with the values $(X(0), \dot{X}(0))$, $(X(T), \dot{X}(T))$, $(X(2T), \dot{X}(2T))$,..., where $T = 2\pi/\omega_1$, has been shown and the numerical solution has been compared with the approximate solution (30)–(31). The mean difference between the two solutions is 0·02, i.e., of order ε^2 as expected.

4. FORCING FREQUENCIES NEAR PRIMARY RESONANCE

The case when the frequency of each component of the forcing term is near the primary resonant frequency of the first mode is now considered, i.e.,

$$\Omega_i = 1 + \varepsilon\sigma_i, \qquad i = 1, \dots, N, \tag{34}$$

where σ_i measures the differences of the frequencies from the natural frequency of the oscillator. Substituting equation (34) into equation (3), the external excitation $F(t)$ becomes

$$F(t) = \varepsilon \exp(\mathrm{i}t) \sum_{i=1}^{N} A_i \exp(\mathrm{i}\sigma_i\tau) + \text{c.c.} \tag{35}$$

The non-linear oscillator is then subject to an applied force with N different frequencies and amplitudes, which are supposed to be of order ε (the primary resonance zone).

The solution can be expressed in the form

$$X(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\gamma_n} \psi_n(\tau; \varepsilon) \exp(-\mathrm{i}n\omega_1 t) = 2\rho\cos(\omega_1 t - \vartheta) + O(\varepsilon), \tag{36}$$

$$Y(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\gamma_n} \varphi_n(\tau; \varepsilon) \exp(-\mathrm{i}n\omega_2 t) = 2\chi\cos(\omega_2 t - \phi) + O(\varepsilon), \tag{37}$$

with the same conventions as in equations (7)–(8).

Substituting equations (36)–(37) in equations (1)–(2) so as to obtain different equations for each n and equating coefficients of like powers of ε yield

$$-2\mathrm{i}\omega_1\psi_\tau = \mathrm{i}\omega_1(|\psi|^2\psi - \psi + 2a\psi|\varphi|^2) + \sum_{i=1}^{N} A_i \exp(-\mathrm{i}\sigma_i\tau) = 0, \tag{38}$$

$$-2\mathrm{i}\omega_1\psi_\tau = \mathrm{i}\omega_1(|\psi|^2\psi - \psi + 2a\psi|\varphi|^2) + \sum_{i=1}^{N} A_i \exp(-\mathrm{i}\sigma_i\tau) = 0. \tag{39}$$

The details of the calculation are not given and only the final results are furnished. By means of the substitution (12), the equations for the amplitude and the phase of the free oscillation become

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \frac{\rho}{2}(1 - \rho^2 - 2a\chi^2) + \frac{1}{2\omega_1}\sum_{i=1}^{N} A_i \sin(\sigma_i\tau + \vartheta), \tag{40}$$

$$\frac{\mathrm{d}\chi}{\mathrm{d}\tau} = \frac{\chi}{2}(1 - \chi^2 - 2b\rho^2), \tag{41}$$

$$\rho\frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} = \frac{1}{2\omega_1}\sum_{i=1}^{N} A_i \cos(\sigma_i\tau + \vartheta), \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = 0. \tag{42}$$

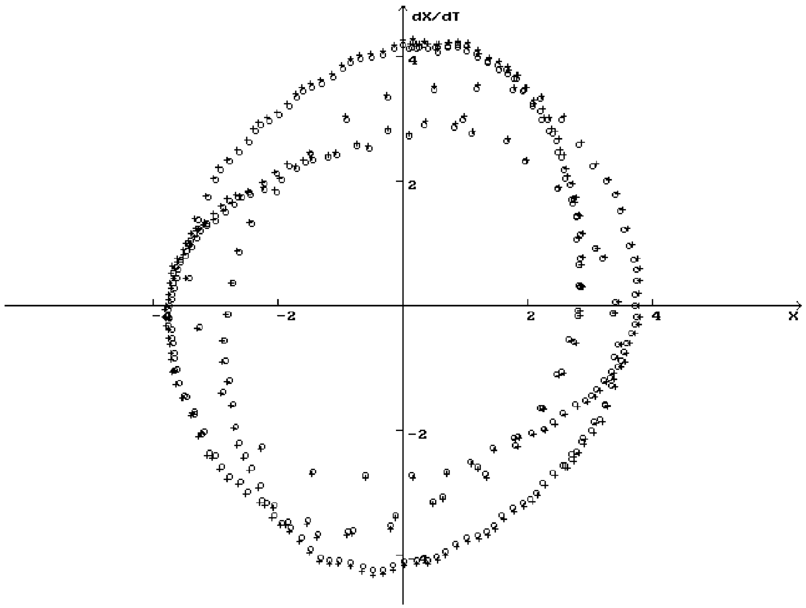

Figure 5. Comparison between numerical (crosses) and analytical (circles) solutions in the (X, dX/dT) plane. Values of parameters: $a = -0{\cdot}5$, $b = 0{\cdot}1$, $\varepsilon = 0{\cdot}08$, $\omega_1 = 1$, $\omega_2 = \sqrt{2}$. Forcing frequencies close to each other and close to the primary resonance: $\Omega_1 = \sqrt{1{\cdot}01}$, $\Omega_2 = \sqrt{0{\cdot}99}$, $\Omega_3 = \sqrt{1{\cdot}03}$. Amplitudes of the external excitations: $A_1 = 0{\cdot}1$, $A_2 = 0{\cdot}2$, $A_3 = 0{\cdot}3$.

The difference with the preceding cases is that now equations (40)–(42) are three coupled non-linear differential equations, which must be integrated numerically.

Both the amplitude and the phase of the free oscillation can eventually oscillate with a frequency that is determined by both the forcing amplitudes A_i and the detuning parameters σ_i.

In Figure 5 the associated map obtained with the values $(X(0), \dot{X}(0))$, $(X(T), \dot{X}(T))$, $(X(2T), \dot{X}(2T))$,…, where $T = 2\pi/\omega_1$, has been shown and the numerical solution has been compared with the approximate solution. The mean difference between the two solutions is 0·03, i.e., of order ε^2 as expected.

5. CONCLUSION

The problem studied in this paper clearly demonstrates the power of the asymptotic perturbation method, because an important feature of this method is that it provides quantitative results regarding dynamic behavior, in contrast to much of the current work in dynamical systems theory, which is concerned with qualitative behavior.

The asymptotic perturbation method has been used to analyze the transient and steady state response of two non-linearly coupled van der Pol oscillators under a finite number of harmonic forcing terms. Three cases of different forcing frequencies are investigated and the corresponding analytical results are compared to numerical simulations. If the forcing frequencies are not close to each other or close to the resonant frequencies of the two modes, then the original free oscillation can vanish (“quenching”) or maintain a finite value.

When the forcing frequencies are all close to a particular frequency Ω, “quenching” is possible, but in certain cases a new behavior arises, because the amplitude of the free oscillation oscillates with some frequencies determined by the detuning parameters.

When the forcing frequencies are close to the resonant frequency of the first mode, then both the amplitude and the phase of the free oscillation can eventually oscillate with a frequency that is determined by both the forcing amplitudes A_i and the detuning parameters σ_i.

If the second order approximation solution is needed, the amount and complexity of the algebraic computations required increase in a very dramatic manner. Consequently, the use of symbolic manipulation systems is strongly recommended.

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