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A BETTI–MAXWELL RECIPROCAL THEOREM FOR A ROTORDYNAMIC SYSTEM WITH GYROSCOPIC TERMS

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INTRODUCTION

Conservation laws have been under consideration for a long time. The classical method in constructing conservation laws, based on Noether's theorem, can only be applied, if a Lagrangian function is available for the system of interest. By using the recently developed neutral action (NA) method, this requirement can be dropped, since a given set of governing partial differential equations is sufficient to construct conservation laws. But even if a Lagrangian function is available, the NA method delivers the same results as Noether's method, if, in addition, the Bessel-Hagen extension would be applied.

1. DEFINITION OF CONSERVATION LAWS

A mechanical system is considered that can be described by a system of q differential equations

$$\Delta_\beta(x_i, v_\alpha, v_{\alpha,k}) = 0, \quad \beta = 1, 2, \ldots, q, \tag{1}$$

with

$$x_i, \quad i, k = 1, 2, \ldots, m, \qquad v_\alpha, v_{\alpha,k}, \quad \alpha = 1, 2, \ldots, \mu,$$

in which x_i denote the independent variables and v_α, $v_{\alpha,k}$ denote the dependent variables. The abbreviation $v_{\alpha,k}$ stands for $\mathrm{d}v_\alpha/\mathrm{d}x_k$. If any set of m associated functions

$$P_i, \quad i = 1, 2, \ldots, m \tag{2}$$

satisfies

$$P_{i,i} = 0 \quad \text{(local formulation)} \tag{3}$$

along all solutions of equation (1), then equation (3) is denoted as a conservation law.

A conservation law may also be written in an integral form. Let B be a body with an infinitesimal volume element $\mathrm{d}V$, which is enclosed by a surface S with area element $\mathrm{d}A$ and unit outward normal vektor n_i (Figure 1). By using the divergence theorem, one can write

$$\int_B P_{i,i}\,\mathrm{d}V = \int_S P_i n_i\,\mathrm{d}A = 0 \quad \text{(global formulation)} \tag{4}$$

leading to a conservation law in a global formulation.

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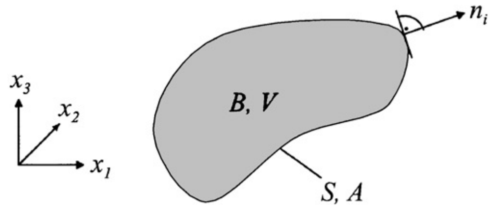

Figure 1. Body B with volume V, surrounding surface S with area A and unit outward normal vektor n_i.

2. CONSTRUCTING CONSERVATION LAWS

The classical way of constructing conservation laws has been established by the mathematician Noether in 1918 [1] and was extended by Bessel-Hagen [2] in 1921. For systems without a Lagrangian, no procedure existed for a systematic construction of conservation laws, until the NA method was advanced [3]. This method can be used on the subject of material or configurational mechanics [4] as well as on dynamics. All that is required is that the system can be described by a set of differential equations

$$\Delta_\beta(x_i, v_\alpha, v_{\alpha,k}) = 0. \tag{5}$$

And even if the governing equations can be calculated from a Lagrange function, the application of the NA method is usefull, since it leads to the same conservation laws in a much simpler way in comparison to the formalism above.

First, the concept of a "Null Lagrangian" will be introduced. If a Lagrange function can be written as $\tilde{L} = \mathrm{d}g_i/\mathrm{d}x_i = g_{i,i}$ with $g_i = g_i(x_k, v_\alpha, v_{\alpha,j})$, it can be shown that

$$\tilde{L} = g_{i,i} \;\Leftrightarrow\; E_\alpha(\tilde{L}) = 0, \tag{6}$$

i.e., it satisfies the Euler–Lagrange equation identically. $\tilde{L}$ is then called a "Null Lagrangian". Setting the variation of the action integral

$$A = \int_B \tilde{L}\,\mathrm{d}V \tag{7}$$

of such a Null Lagrangian to zero, one obtains

$$\delta A = 0 \;\Leftrightarrow\; E_\alpha(\tilde{L}) = 0, \tag{8}$$

where δA denotes the variation of the dependent variables. This means that the action integral A does not depend on the explicit functional dependence of $g_i(x_k)$ inside the domain of integration, but only on the values at the boundary S. So the idea is to seek for characteristic functions f_α such that

$$f_\alpha \Delta_\alpha = P_{i,i}. \tag{9}$$

From equations (6) and (8), it follows

$$E_\beta(f_\alpha \Delta_\alpha) = E_\beta(P_{i,i}) = E_\beta(\tilde{L}) = 0 \;\Leftrightarrow\; \delta A = 0 \tag{10}$$

with

$$A = \int_B \tilde{L}\,\mathrm{d}V = \int_B f_\alpha \Delta_\alpha\,\mathrm{d}V. \tag{11}$$

The characteristics f_α have to be determined from equation (10). The action integral behaves neutrally under its variation, so the formalism is called the "NA method".

3. SYSTEMS WITH GYROSCOPIC FORCES

From classical mechanics, it is well known that for elastostatic systems, the Betti–Maxwell reciprocal theorem is valid [5]. With the help of the NA method, it is possible to calculate an analogous theorem for rotordynamic systems with gyroscopic forces. Figure 2 shows a system with gyroscopic forces. Gyroscopic terms frequently occur if the co-ordinate system is not fixed to an inertial system. The system in Figure 2 consists of a turntable (moment of inertia I) rotating around the vertical axis. A masspoint m is located in a radial groove and is connected to the center by a spring (initial length r_0, spring constant c). The table itself is connected to a drive via an elastic clutch (torsional spring constant c_T). The angular velocity Ω of the shaft is taken to be constant, the angle φ denotes the torsion between the upper and lower part of the elastic clutch. The system can be described by a set of non-linear differential equations [6]

$$(I + mr^2)\ddot{\varphi} + (\Omega + \dot{\varphi})2mr\dot{r} + c_T\varphi = 0, \qquad m\ddot{r} - rm(\Omega + \dot{\varphi})^2 + c(r - r_0) = 0. \quad (12)$$

They can be linearized around a stationary solution φ_s, r_s by

$$\varphi = \varphi_s + \bar{\varphi}, \qquad r = r_s + \bar{r}, \quad (13)$$

in which $\varphi_s = 0$ and $r_s = cr_0/(c - m\Omega^2)$. Considering only small disturbances $\bar{\varphi}$, $\bar{r}$ the linearization leads to a set of linear differential equations

$$(I + mr_s^2)\ddot{\bar{\varphi}} + 2mr_s\Omega\dot{\bar{r}} + c_T\bar{\varphi} = 0, \qquad m\ddot{\bar{r}} - 2mr_s\Omega\dot{\bar{\varphi}} + (c - m\Omega^2)\bar{r} = 0. \quad (14)$$

To apply the NA method formulate

$$f_\varphi\Delta_\varphi = f_\varphi[(I + mr_s^2)\ddot{\bar{\varphi}} + 2mr_s\Omega\dot{\bar{r}} + c_T\bar{\varphi}], \qquad f_r\Delta_r = f_r[m\ddot{\bar{r}} - 2mr_s\Omega\dot{\bar{\varphi}} + (c - m\Omega^2)\bar{r}]. \quad (15)$$

Using the concept of the "Null Lagrangian" (10) one sets

$$E(f_\varphi\Delta_\varphi + f_r\Delta_r) = 0, \quad (16)$$

which is satisfied by the special formulation

$$f_\varphi = \dot{\bar{\varphi}}, \qquad f_r = \dot{\bar{r}}. \quad (17)$$

One then obtains the divergence

$$P_{t,t} = \dot{\bar{\varphi}}\Delta_\varphi + \dot{\bar{r}}\Delta_r. \quad (18)$$

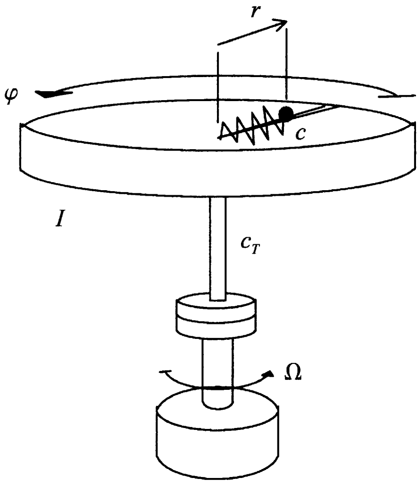

Figure 2. Mechanical system with gyroscopic forces: masspoint on a turntable.

After integration the conserved current is

$$P_t = \tfrac{1}{2} I \dot{\bar{\varphi}}^2 + \tfrac{1}{2} c_T \bar{\varphi}^2 + \tfrac{1}{2} m \dot{\bar{r}}^2 + \tfrac{1}{2} c \bar{r}^2 - \tfrac{1}{2} m \Omega^2 \bar{r}^2. \tag{19}$$

Equation (19) is a well-known energy-related quantity. The total energy of the system is not a conserved quantity [6]. Beside this classical conservation law, other conserved currents can be calculated with the use of the NA method. The special formulation

$$f_\varphi = \bar{\varphi}, \qquad f_r = \bar{r} \tag{20}$$

can be used as a characteristic and leads to the divergence

$$P_{t,t} = \bar{\varphi} \Delta_\varphi + \bar{r} \Delta_r. \tag{21}$$

This means that any solution $\bar{r}$, $\bar{\varphi}$ obeying equation (14), but belonging, for instance, to a different initial value problem, may serve as a characteristic for a conservation law. By using the energy-related equation (19) and the identity

$$\bar{\varphi} \ddot{\bar{\varphi}} = (\bar{\varphi} \dot{\bar{\varphi}})^{\bullet} - \dot{\bar{\varphi}}^2 \tag{22}$$

one obtains after some calculation, the Betti–Maxwell reciprocal theorem

$$p_\varphi^{(1)} \bar{\varphi}^{(2)} + p_r^{(1)} \bar{r}^{(2)} = p_\varphi^{(2)} \bar{\varphi}^{(1)} + p_r^{(2)} \bar{r}^{(1)}. \tag{23}$$

In equation (23), the abbreviation for the canonical momenta

$$p_\varphi^{(i)} = (I + m r_s^2) \dot{\bar{\varphi}}^{(i)} + 2 m r_s \Omega \bar{r}^{(i)}, \qquad p_r^{(i)} = m \dot{\bar{r}}^{(i)} \tag{24}$$

have been used. This theorem leads to the conclusion that we have an infinity of conserved currents parametrized by solutions $\bar{r}$, $\bar{\varphi}$ of equation (14).

4. CONCLUSIONS

The classical procedure of constructing conservation laws is to apply Noether's theorem. It requires the existence of a Lagrangian for the system under consideration. Furthermore, this method demands the knowledge of infinitesimal transformations, which have to be calculated in a separate step. Some further conservation laws can be calculated using the Bessel-Hagen extension, since the equations of motion are left unchanged when a so-called "gauge function" is added to the Lagrangian. In this case, another seperate calculation has to be performed. The same conservation laws as above can be obtained by using the NA method, having to calculate only one unknown set of functions f_α. It was shown that with the NA method even a Betti–Maxwell reciprocal theorem can be derived for a system with gyroscopic forces. Moreover, the NA method can also be applied in the absence of a Lagrangian (e.g., dissipative systems), since only the governing differential equations are required for this procedure. It seems that with a systematic treatment even more conservation laws can be obtained for this problem. Studies along this line are in progress and will be dealt with in a forthcoming paper.

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