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A DYNAMICAL BASIS FOR COMPUTING THE MODES OF EULER–BERNOULLI AND TIMOSHENKO BEAMS

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1. INTRODUCTION

In this work the modes of a uniform beam described by the Euler–Bernoulli model with axial force or the Timoshenko model are determined in a unified manner with the use of a fundamental free response. This later being referred to as the dynamical solution. Together with its first three derivatives it constitutes a dynamical basis of solutions for the underlying fourth order differential equation that governs the shape of a mode.

The flexural modes of a beam with general boundary conditions can be, in principle, determined with the use of a generic basis for the modal differential equation. However, the initial conditions satisfied by the dynamical solution allows to reduce the dimension of the algebraic modal equation that arises from satisfying the boundary conditions. This equation leads to the characteristic equation for the eigenfrequencies.

Under limit situations, the dynamical basis behaves much better than the spectral basis constructed in terms of the roots of the characteristic equation set up for seeking exponential solutions. This is the case for a static situation.

2. THE MODAL EQUATION FOR FREE VIBRATIONS

For a uniform beam with mass per unit length m, Young's modulus E, moment of inertia of the cross-section I subject to an axial force $-N$ and an external load p, we have the Euler–Bernoulli model

$$m\frac{\partial^2 v}{\partial t^2} + EI\frac{\partial^4 v}{\partial x^4} + N\frac{\partial^2 v}{\partial x^2} = p(t,\, x). \tag{1}$$

The Timoshenko beam model is described by the equation

$$M\frac{\partial^4 v}{\partial t^4} + B\frac{\partial^2 v}{\partial t^2} + Kv = p(t,\, x), \tag{2}$$

where

$$M = \alpha^2\delta^2\mathrm{I}, \quad B = \beta^2\mathrm{I} - (\alpha^2 + \delta^2)\frac{\partial^2}{\partial x^2}, \quad K = \frac{\partial^4}{\partial x^4},$$

$$\alpha^2 = \frac{m}{kAG}, \quad \delta^2 = \frac{mr^2}{EI}, \quad \beta^2 = \frac{m}{EI}.$$

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TABLE 1

Constants for the two models

Euler–Bernoulli	$g^2 = \dfrac{N}{EI}$	$a^4 = \beta^2\omega^2$
Timoshenko	$g^2 = (\alpha^2 + \delta^2)\omega^2$	$a^4 = \beta^2\omega^2 - \alpha^2\delta^2\omega^4$

Here I denotes the identity operator, r, G the rotary and shear parameters, A the transversal section area and k a geometric factor depending upon such section [1, 2].

The above models are subject to classical or non-classical spatial boundary conditions.

When no external load is acting upon the beam, free vibrations $v = X(x)e^{i\omega t}$ can be determined by solving the differential equation for the amplitude distribution $X(x)$ subject to boundary conditions. For both models, this amplitude satisfies the equation

$$X^{(iv)}(x) + g^2 X''(x) - a^4 X(x) = 0 \tag{3}$$

where the constants g and a are as shown in Table 1.

The modes are subject to the general boundary conditions

$$\begin{aligned}
A_{11}X(0) + B_{11}X'(0) + C_{11}X''(0) + D_{11}X'''(0) &= 0,\\
A_{12}X(0) + B_{12}X'(0) + C_{12}X''(0) + D_{12}X'''(0) &= 0,\\
A_{21}X(L) + B_{21}X'(L) + C_{21}X''(L) + D_{21}X'''(L) &= 0,\\
A_{22}X(L) + B_{22}X'(L) + C_{22}X''(L) + D_{22}X'''(L) &= 0.
\end{aligned} \tag{4}$$

It should be observed that for non-classical conditions, the boundary coefficients might involve the frequency ω.

The general solution of equation (3) can be written in matrix form as $X = \phi c$, where $\phi = [\phi_1\ \phi_2\ \phi_3\ \phi_4]$ denotes any basis of solutions, that is, their Wronskian is non-zero. In order to satisfy the boundary conditions, it follows that by a direct substitution and a convenient grouping, the constant vector c must be a non-zero solution of $B\Phi c = 0$, where

$$B = \begin{bmatrix} A_{11} & B_{11} & C_{11} & D_{11} & 0 & 0 & 0 & 0 \\ A_{12} & B_{12} & C_{12} & D_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{21} & B_{21} & C_{21} & D_{21} \\ 0 & 0 & 0 & 0 & A_{22} & B_{22} & C_{22} & D_{22} \end{bmatrix},$$

$$\Phi = \begin{bmatrix} \phi_1(0) & \phi_2(0) & \phi_3(0) & \phi_4(0) \\ \phi_1'(0) & \phi_2'(0) & \phi_3'(0) & \phi_4'(0) \\ \phi_1''(0) & \phi_2''(0) & \phi_3''(0) & \phi_4''(0) \\ \phi_1'''(0) & \phi_2'''(0) & \phi_3'''(0) & \phi_4'''(0) \\ \phi_1(L) & \phi_2(L) & \phi_3(L) & \phi_4(L) \\ \phi_1'(L) & \phi_2'(L) & \phi_3'(L) & \phi_4'(L) \\ \phi_1''(L) & \phi_2''(L) & \phi_3''(L) & \phi_4''(L) \\ \phi_1'''(L) & \phi_2'''(L) & \phi_3'''(L) & \phi_4'''(L) \end{bmatrix}. \tag{5}$$

Here B is the matrix of the boundary coefficients and Φ the matrix carrying the solution basis and its derivatives at both ends. We thus have the modal equation

$$Uc = 0, \quad U = B\Phi \tag{6}$$

and the characteristic equation

$$\Delta = \det U = 0.$$

3. THE DYNAMICAL BASIS

The classical or spectral basis of equation (3) comes from searching exponential type solutions. It is constructed in terms of the roots of the characteristic polynomial

$$\lambda^4 + g^2\lambda^2 - a^4 = 0. \tag{7}$$

The roots of equation (7) are given by $\lambda = \varepsilon, -\varepsilon, \mathrm{i}\delta, -\mathrm{i}\delta$, where

$$\delta = \sqrt{g^2 + \varepsilon^2}, \quad \varepsilon = \sqrt{\left(a^4 + \frac{g^4}{4}\right)^{1/2} - \frac{g^2}{2}}. \tag{8}$$

Then the spectral basis is given by

$$\phi = [\sin \delta \mathrm{x}, \ \cos \delta \mathrm{x}, \ \sinh \varepsilon \mathrm{x}, \ \cosh \varepsilon \mathrm{x}].$$

Another basis, equally or more important than the spectral basis, that will be referred to as the *dynamical basis*, is constituted by a particular solution and its derivatives [3]. The *dynamic solution* or the fundamental solution of equation (3) is defined as being the solution $h(x)$ of the equation

$$h^{(\mathrm{iv})}(x) + g^2 h''(x) - a^4 h(x) = 0 \tag{9}$$

with the initial conditions

$$h(0) = 0, \quad h'(0) = 0, \quad h''(0) = 0, \quad h'''(0) = 1.$$

The general solution is given by

$$X(x) = c_1 h(x) + c_2 h'(x) + c_3 h''(x) + c_4 h'''(x) \tag{10}$$

because the set of solutions $\{h, h', h'', h'''\}$ has a non-zero Wronskian at $t = 0$ and, consequently, it is a basis of solutions. It is of interest to observe that from equation (9) and by uniqueness, the solutions h and h'' are odd functions while h' and h''' are even functions.

The dynamical basis has the following representation with respect to the spectral basis:

$$\phi_1 = h(x) = \frac{\delta \sinh \varepsilon x - \varepsilon \sin \delta x}{\delta\varepsilon(\varepsilon^2 + \delta^2)}, \quad \phi_2 = h'(x) = \frac{\cosh \varepsilon x - \cos \delta x}{(\varepsilon^2 + \delta^2)}. \tag{11}$$

$$\phi_3 = h''(x) = \frac{\varepsilon \sinh \varepsilon x + \delta \sin \delta x}{(\varepsilon^2 + \delta^2)}, \quad \phi_4 = h'''(x) = \frac{\varepsilon^2 \cosh \varepsilon x + \delta^2 \cos \delta x}{(\varepsilon^2 + \delta^2)}. \tag{12}$$

In the vibration literature, for instance [1, 2, 4–6], the terms involving h or its derivatives might appear frequently in the final or middle of calculations but without any reference to a systematic treatment as the one given here. For the spectral basis there is not a natural preference order for the elements of the basis. This is not the case with the dynamical basis. The first element is naturally chosen as the fundamental solution h. Besides that, for a distributed load $f(x)$ its convolution with the dynamical solution gives the forced response [1, 3, 7].

3.1. THE EULER–BERNOULLI MODEL AS A LIMIT CASE

Limit situations can be obtained directly from $h(x)$ without any modification. Let us consider the basic Euler–Bernoulli model without axial force ($N = 0$). In this case, $g = 0$ and, consequently $\delta = \varepsilon = a$, Thus by taking limit as g tends to 0 in equation (12), it turns out that

$$h(x) = \frac{\sinh(ax) - \sin(ax)}{2a^3}. \tag{13}$$

The basic Euler–Bernoulli model, without axial force, reduces to the static situation when inertia effects are neglected, that is, we set $a = 0$. In this situation, we must observe that equation (7) has a single root $\lambda = 0$ of multiplicity four. This implies $\delta = \varepsilon = 0$ and, consequently, the spectral basis degenerates. *This is not the case with the dynamical basis*. For such limit situation we get without any trouble that the dynamic solution is

$$h(x) = \frac{x^3}{6}. \tag{14}$$

This later will generate the basis

$$\phi = \left[\frac{x^3}{6}, \frac{x^2}{2}, x, 1\right].$$

If we consider the Timoshenko model, we can eliminate the rotary inertia and the shear deformation effects to get the Euler–Bernoulli model without axial force. This amounts to consider $\alpha = 0$, $\delta = 0$ so that $g = 0$ and, consequently, $\delta = \varepsilon = a$. With these assumptions equation (13) is recovered.

This good behaviour of the dynamical solution h is not accidental. The reason situation being that h is defined in term of its initial values which are independent of the parameters of a given equation.

4. COMPUTING THE MODES OF A FIXED-SUPPORTED BEAM

The modes of a beam subject to diverse boundary conditions can be obtained in a systematic way by solving equation (6). We observe that due to the initial conditions of the dynamic solution h, the use of the dynamical basis will introduce a high number of zeros in the matrix basis Φ. Thus, the order of the system is automatically reduced by half.

This approach will be illustrated by considering the case of a fixed-sliding beam and fixed-supported beam. The boundary conditions for this kind of beam will imply that the matrix which registers such conditions is given by

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{15}$$

This matrix, when multiplied with Φ, activites only the elements of Φ that correspond to the derivatives present on a boundary condition, that is

$$U = \begin{bmatrix} h(0) & h'(0) & h''(0) & h'''(0) \\ h'(0) & h''(0) & h'''(0) & h^{(\mathrm{iv})}(0) \\ h(L) & h'(L) & h''(L) & h'''(L) \\ h''(L) & h'''(L) & h^{(\mathrm{iv})}(L) & h^{(\mathrm{v})}(L) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ h(L) & h'(L) & h''(L) & h'''(L) \\ h''(L) & h'''(L) & h^{(\mathrm{iv})}(L) & h^{(\mathrm{v})}(L) \end{bmatrix}. \tag{16}$$

It turns out that the characteristic equation is given by

$$\Delta = \det U = h(L)h'''(L) - h'(L)h''(L) = 0. \tag{17}$$

For each root ε of this equation, a value is obtained for δ and a that allows to fix the dynamical basis for computing the corresponding mode to an eigenfrequency ω whose value is given in terms of a. The system $Uc = 0$ is solved by elimination or with symbolic software for more complex situations.

The corresponding mode, relative to the dynamical basis, is given by

$$X_n(x) = \sigma_n h'(x, \varepsilon_n) + h(x, \varepsilon_n), \quad \sigma_n = \frac{h(L, \varepsilon_n)}{h'(L, \varepsilon_n)} \tag{18}$$

where, for the definition of $h(x)$, the dependence upon the root ε_n has been emphasized.

It should be noticed that with this methodology, the case of a supported-fixed Euler–Bernoulli beam can be handled with a simple row permutation in the matrices B and Φ or, when convenient, to employ the basis that would be generated by $h(L - x)$ instead of $h(x)$.

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