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INTERACTION BETWEEN COMPRESSIVE FORCE AND VIBRATION FREQUENCY FOR A VARYING CROSS-SECTION CANTILEVER UNDER ACTION OF GENERALIZED FOLLOWER FORCE

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1. INTRODUCTION

The buckling analysis for a column subjected to a follower force was carried out in an earlier time [1, 2]. In the case of applying a follower force, the static criterion is no longer valid and it is necessary to use the dynamic criterion. In an earlier time, analysis for the problem can be referred to in references [3–7]. It is well known that the dynamic criterion for evaluating the buckling loading of a column mainly arises from the problem of the non-conservative force case. As was pointed out by Herrmann [8], the breakdown of Euler method is not a necessary consequence of non-conservativeness of the loading. Thus, it appears desirable to gain a deeper insight into the interrelation of non-conservativeness, existence or absence of adjacent equilibrium configurations and applicability of stability criteria.

Vibration and stability of a non-uniform beam subjected to a follower force was studied in references [9, 10]. The investigation was limited to the usual follower force case, i.e., $m = 1$ case in the present study. Similar study was carried out for the generalized follower force case ($m \neq 1$) [11–13].

In the knowledge of the author, there was no investigation for the force–frequency interaction of varying cross-section cantilever when the tangency coefficient m is changed gradually. In the first part of the paper, we investigate the case that a uniform cross-section cantilever is under the action of the generalized follower force P. The slope at the free end of column is denoted by $\theta = \mathrm{d}W/\mathrm{d}x$ (at $x = L$), and the force P is applied at an angle $\beta = m\theta$ ($0 \leqslant m < \infty$). In fact, $m = 0$ corresponds to the Euler buckling case and $m = 1$ corresponds to the usual follower force case. The force–frequency interaction for the cantilever with different values of (m) is investigated. The interaction can be described by a force–frequency interaction curve. It is found that the value of (m) has a significant influence to the characters of the force–frequency interaction. There is a transform value of m, which is denoted by m_{tr}. In the uniform section case, $m_{tr} = 0{\cdot}5$, which was pointed out in an earlier time [14].

In the uniform cross-section case, the force–frequency interaction can be expressed by an equation $f(\omega, p, m) = 0$ (see equation (10)), where ω is the frequency and p is the compressive force. The relevant problem for the column with varying cross-section is also studied. The problem is more complicated and has to solve numerically. The force–frequency interaction can also be expressed by an equation $f(\omega, p, m) = 0$ (see

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equation (22)). After analyzing the interaction shown by equation $f(\omega, p, m) = 0$, the buckling loading or the critical loading of the bar can be evaluated immediately.

2. ANALYSIS FOR THE UNIFORM SECTION CASE

In this paper, interaction between compressive force and vibration frequency for a cantilever under the action of the generalized follower force P is studied, and the buckling or critical loading of the bar is also investigated (Figure 1). The tangent at the right end of column is denoted by $\theta = \mathrm{d}W/\mathrm{d}x$ (at $x = L$), and the force P is applied at an angle $\beta = m\theta$ $(0 \leqslant m < \infty)$. The (m) value is called the tangency coefficient hereafter.

In the vibration analysis, after letting the displacement $w(x, t)$ in the form

$$w(x, t) = W(x)\sin(\Omega t), \tag{1}$$

the governing equation for the displacement of column with a varying cross-section takes the form [4]

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[EI(x)\frac{\mathrm{d}^2 W}{\mathrm{d}x^2}\right] + P\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} - \Omega^2 \rho A(x) W = 0 \quad (0 \leqslant x \leqslant L), \tag{2}$$

where Ω denotes the vibration frequency, ρ the mass density of materials, E Young's modulus of elasticity, $I(x)$ the moment inertia of section, and $A(x)$ the area of section.

For a cantilever shown in Figure 1, the boundary conditions will be

$$W|_{x=0} = 0, \qquad \left.\frac{\mathrm{d}W}{\mathrm{d}x}\right|_{x=0} = 0, \tag{3a, b}$$

$$\left.\frac{\mathrm{d}^2 W}{\mathrm{d}x^2}\right|_{x=L} = 0, \qquad \left.\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[EI(x)\frac{\mathrm{d}^2 W}{\mathrm{d}x^2}\right] + P(1-m)\frac{\mathrm{d}W}{\mathrm{d}x}\right\}\right|_{x=L} = 0. \tag{3c, d}$$

If the column has a circular section with the taper configuration, the two functions $I(x)$ and $A(x)$, have the following expression:

$$I(x) = I_0 g(x), \quad \text{where } I_0 = \pi a^4/4, \;\; g(x) = \left(1 + \frac{ex}{L}\right)^4, \tag{4}$$

$$A(x) = A_0 h(x), \quad \text{where } A_0 = \pi a^2, \;\; h(x) = \left(1 + \frac{ex}{L}\right)^2, \tag{5}$$

where "a" denotes the radius at the left section (at $x = 0$), and "e" represents the degree of the taper configuration. In the following study, the force P and the vibration frequency Ω may be normalized as

$$P = pEI_0\left(\frac{\pi}{L}\right)^2 \quad (p = k^2), \qquad \Omega = \omega\left(\frac{EI_0}{\rho A_0}\right)^{1/2}\left(\frac{\pi}{L}\right)^2. \tag{6a, b}$$

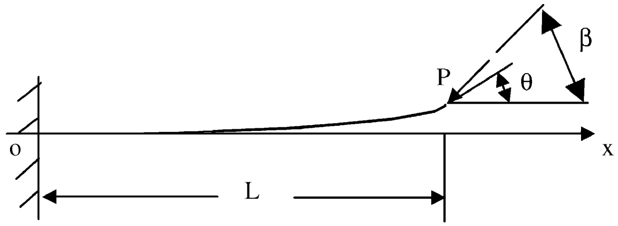

Figure 1. A cantilever under the action of the generalized follower force.

The constant section case will be considered first. In this case, there are $e = 0, h(x) = 1$ and $g(x) = 1$ in equations (4) and (5), and equation (2) has a general solution [4]

$$W(x) = A\cosh(\delta_1 x) + B\sinh(\delta_1 x) + C\cos(\delta_2 x) + D\sin(\delta_2 x), \tag{7}$$

where

$$\delta_1 = \left[\sqrt{\omega^2 + \frac{p^2}{4}} - \frac{p}{2}\right]^{1/2}\left(\frac{\pi}{L}\right), \qquad \delta_2 = \left[\sqrt{\omega^2 + \frac{p^2}{4}} + \frac{p}{2}\right]^{1/2}\left(\frac{\pi}{L}\right). \tag{8}$$

Substituting equation (7) into conditions (3a–d) yields

$$A + C = 0, \qquad \delta_1 B + \delta_2 D = 0, \tag{9a, b}$$

$$[A\cosh(\delta_1 L) + B\sinh(\delta_1 L)]\delta_1^2 - [C\cos(\delta_2 L) + D\sin(\delta_2 L)]\delta_2^2 = 0, \tag{9c}$$

$$[A\sinh(\delta_1 L) + B\cosh(\delta_1 L)]\left[\delta_1^2 + p(1-m)\left(\frac{\pi}{L}\right)^2\right]\delta_1 + [C\sin(\delta_2 L) - D\cos(\delta_2 L)]\left[\delta_2^2 - p(1-m)\left(\frac{\pi}{L}\right)^2\right]\delta_2 = 0. \tag{9d}$$

To obtain a non-trivial solution for A, B, C and D, the relevant determinant composed of the coefficients of equations (9a–d) should vanish. By the use of this condition, we have the governing equation for the parameters ω and p as follows:

$$f(\omega, p, m) = 2\omega^2 + mp^2 + [2\omega^2 + (1-m)p^2]\cosh(\delta_1 L)\cos(\delta_2 L) - (1-2m)\omega p\sinh(\delta_1 L)\sin(\delta_2 L) = 0. \tag{10}$$

It is seen that the equation depends on the tangency coefficient (m). Equation (10) may be written in an alternative form:

$$p = g(\omega, m). \tag{11}$$

Clearly, it is not easy to obtain the function $g(\omega, m)$ in an explicit form.

Before analyzing the interaction between (ω) and (p) in the general case, a particular case of $\omega = 0$ is studied as follows. From equation (8), it is found that $\delta_1 = 0, \delta_2 = \sqrt{p}\pi/L$. Substituting these results into equation (10) yields

$$\cos(k\pi) = -\frac{m}{1-m} \quad \text{or} \quad k = \frac{1}{\pi}\arccos\left(-\frac{m}{1-m}\right) \quad (p = k^2). \tag{12}$$

From equation (12) we see that if and only if $m \leqslant 0{\cdot}5$ the relevant solution for k exists. In order to know the relation of (k) versus (m), the first six $k_i\,(i = 1, 2, \ldots, 6)$ in the three cases (m = 0·49, 0·499, 0·5) are listed in Table 1.

TABLE 1

The first six normalized $k_i\,(i = 1, 2, \ldots, 6)$ for the cantilever under the action of generalized follower force ($p = k^2$)

m	k_1	k_2	k_3	k_4	k_5	k_6
0·49	0·972	1·028	2·972	3·028	4·972	5·028
0·499	0·991	1·009	2·991	3·009	4·991	5·009
0·50	1·000	1·000	3·000	3·000	5·000	5·000

From equation (12) and Table 1 we see the following points: (1) if $m<0{\cdot}5$, the eigenvalues $k_i\,(i=1,2,\ldots,6)$ are located on the isolating position, (2) if $m<0{\cdot}5$ and $m\to 0{\cdot}5$, the difference between two successful eigenvalues becomes smaller, for example, $k_{2i}-k_{2i-1}=0{\cdot}056$ at $m=0{\cdot}49$, and $k_{2i}-k_{2i-1}=0{\cdot}018$ at $m=0{\cdot}499$, (3) if $m=0{\cdot}5$ the two successive eingenvalues merge into one value, for example, $k_{2i}-k_{2i-1}=0$, and $k_1=k_2=1{\cdot}000$. In fact, this is the double root case, (4) if $m>0{\cdot}5$, there is no solution for (k) from equation (12). From the above analysis, the following definition is introduced:

If the first two eigenvalues merge into one value, i.e., $k_2-k_1=0$, at some particular value of (m), this particular value of (m) is called the mode transform value m_{tr}.

Obviously, in the present case $m_{tr}=0{\cdot}5$ [14]. Later, it will be seen that the interaction between force and frequency is quite different for two cases $m\leqslant m_{tr}$ and $m>m_{tr}$.

Similarly, in the $p=0$ case, the successive zeros of the function $f(\omega,0,m)$ are denoted by $\omega_j\,(j=1,2,\ldots)$ respectively. The first two zeros $\omega_1=0{\cdot}356$ and $\omega_2=2{\cdot}233$ are indicated in Figure 2.

The function $p=g(\omega,m)$ shown by equation (11) is evaluated by using the half-division technique [15]. This function, in turn, is called the force–frequency curve hereafter. Clearly, the mentioned function generally has a complicated character.

The obtained force–frequency curves $p=g(\omega,m)$ can be separated on two cases. For the $m\leqslant m_{tr}$ cases ($m_{tr}=0{\cdot}5$), the relevant curves for $m=0{\cdot}0, 0{\cdot}20, 0{\cdot}49$, and $0{\cdot}50$ are plotted in Figure 2. In fact, there are many curves for a given value of $m\,(m\leqslant m_{tr})$. However, the most interested curves are first two of them. For example, if $m=0{\cdot}00$, the two curves are the AC and BH shown in Figure 2. Similarly, there are AD and BG curves for the case of $m=0{\cdot}2$, and AE and BF curves for the case of $m=0{\cdot}49$. Finally, for the case of $m=m_{tr}=0{\cdot}5$, the left ends of the two force–frequency curves are merged into one point Q, and the force–frequency curves become AQ and BQ. In fact, the case $m=m_{tr}=0{\cdot}5$ is the limitation for which the static criterion of the buckling analysis is still valid. In this case, it is natural to define the buckling loading by the following definition:

In the case of $m\leqslant m_{tr}=0{\cdot}5$, the lowest root of (p) from the equation $f(0,p,m)=0$ is the buckling loading of the cantilever.

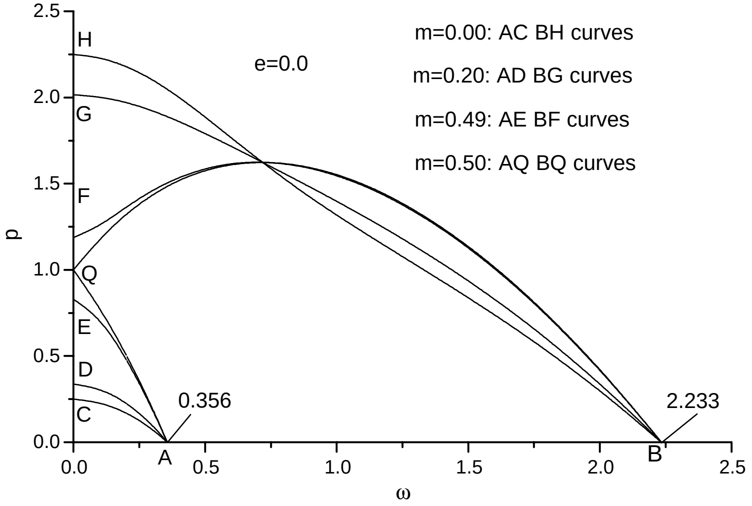

Figure 2. The force–frequency interaction curves for the cases $m\leqslant m_{tr}$ ($m_{tr}=0{\cdot}5$).

For the cases of $m = 0, 0{\cdot}2, 0{\cdot}49$ and $0{\cdot}50$, the normalized buckling forces $p = h_1(m)$ are listed in Table 2.

For the $m > m_{tr}$ ($m_{tr} = 0{\cdot}5$) cases, the relevant curves for $m = 0{\cdot}51, 0{\cdot}55, 0{\cdot}80$ and $1{\cdot}00$ are plotted in Figure 3. From Figure 3 we see that the force–frequency curve is significantly changed, if one compares to the case of $m < m_{tr}$. For example, in the case of $m = 0{\cdot}51$ case, from Figure 3 we see that there is no solution for (p) from the equation $f(0, p, m) = 0$. That is to say, the static formulation of the buckling problem cannot give a solution of (p). Secondly, the relevant force–frequency curve ARB (for $m = 0{\cdot}51$ case) is actually deformed from two curves AQ and BQ (for $m = 0{\cdot}50$ case). Furthermore, the curves are gradually changed from ARB (for $m = 0{\cdot}51$) to ASB ($m = 0{\cdot}55$), to ATB ($m = 0{\cdot}80$), to AUB ($m = 1{\cdot}00$).

Following the idea proposed by Timoshenko [4], when the p value is increasing a complex eigenvalue ω will be encountered. Also, the complex eigenvalue of ω will make the motion divergent. Therefore, it is natural to define the critical force by the following dynamic criterion.

In the case of $m > m_{tr} = 0{\cdot}5$, the highest value of p for which the relevant real eigenvalue ω can be obtained from $f(\omega, p, m) = 0$ is the critical force of the cantilever.

For both cases $m \leqslant m_{tr} = 0{\cdot}5$ and $m > m_{tr} = 0{\cdot}5$, the buckling or critical force and relevant frequency are expressed by

$$p = h_1(m) \quad \text{(for both cases)}, \tag{13}$$

TABLE 2

Dependence of the normalized critical force (p) on the parameter (m), in the uniform section case

m	0·0	0·2	0·49	0·50	0·51	0·55	0·80	1·00
$h_1(m)$	0·240	0·337	0·829	1·000	1·627	1·632	1·782	2·032
$h_2(m)$					0·732	0·788	1·009	1·118

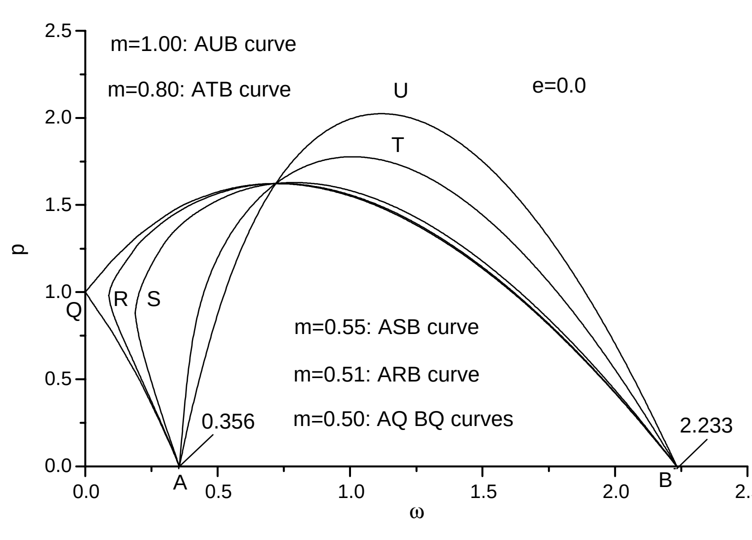

Figure 3. The force–frequency interaction curves for the cases $m > m_{tr}$ ($m_{tr} = 0{\cdot}5$).

$$\omega = h_2(m) \quad (\text{for case } m > m_{tr} = 0{\cdot}5). \tag{14}$$

Note that the calculated values for $h_1(m)$ in Table 2 are obtained from different criteria, the static criterion (for $m \leqslant m_{tr} = 0{\cdot}5$) and the dynamic criterion (for $m > m_{tr} = 0{\cdot}5$). From Table 2, it is seen that the calculated values for the function $h_1(m)$ are discontinuous at the point $m = m_{tr} = 0{\cdot}5$.

3. ANALYSIS AND NUMERICAL SOLUTION FOR THE VARYING SECTION CASE

In this section, it is assumed that the varying cross-section has a taper configuration. After substituting equations (4) and (5) into equations (2), the governing equation of motion becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[g(x)\frac{\mathrm{d}^2 W}{\mathrm{d}x^2}\right] + p\left(\frac{\pi}{L}\right)^2\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} - \omega^2\left(\frac{\pi}{L}\right)^4 h(x)W = 0 \quad (0 \leqslant x \leqslant L). \tag{15}$$

The boundary conditions (3a–d) are rewritten as

$$W|_{x=0} = 0, \qquad \left.\frac{\mathrm{d}W}{\mathrm{d}x}\right|_{x=0} = 0, \tag{16a, b}$$

$$\left.\frac{\mathrm{d}^2 W}{\mathrm{d}x^2}\right|_{x=L} = 0, \qquad \left.\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[g(x)\frac{\mathrm{d}^2 W}{\mathrm{d}x^2}\right] + p(1-m)\left(\frac{\pi}{L}\right)^2\frac{\mathrm{d}W}{\mathrm{d}x}\right\}\right|_{x=L} = 0. \tag{16c, d}$$

In fact, for any given ω and p, for equation (15) we can solve the following initial-boundary value problems:

$$W|_{x=0} = 0, \quad \left.\frac{\mathrm{d}W}{\mathrm{d}x}\right|_{x=0} = 0, \quad \left.\frac{\mathrm{d}^2 W}{\mathrm{d}x^2}\right|_{x=0} = 1, \quad \left.\frac{\mathrm{d}^3 W}{\mathrm{d}x^3}\right|_{x=0} = 0 \ \ (\text{the fundamental problem } Q), \tag{17}$$

$$W|_{x=0} = 0, \quad \left.\frac{\mathrm{d}W}{\mathrm{d}x}\right|_{x=0} = 0, \quad \left.\frac{\mathrm{d}^2 W}{\mathrm{d}x^2}\right|_{x=0} = 0, \quad \left.\frac{\mathrm{d}^3 W}{\mathrm{d}x^3}\right|_{x=0} = 1 \ \ (\text{the fundamental problem } S). \tag{18}$$

The relevant solution is called the fundamental solution $Q(S)$ respectively. The obtained solutions are denoted by

$$W = q(x, \omega, p) \quad (0 \leqslant x \leqslant L) \ \ (\text{for the fundamental problem } Q), \tag{19}$$

$$W = s(x, \omega, p) \quad (0 \leqslant x \leqslant L) \ \ (\text{for the fundamental problem } S). \tag{20}$$

The mentioned numerical solution for the functions $q(x, \omega, p)$ and $s(x, \omega, p) (0 \leqslant a \leqslant L)$ can be easily obtained by using the well-known Runge–Kutta integration rule [16].

Clearly, we can seek the general solution in the form

$$W(x, \omega, p) = c_1 q(x, \omega, p) + c_2 s(x, \omega, p). \tag{21}$$

After substituting equation (21) into condition (16c,d), in order that the non-trivial solution for the coefficients c_1, c_2 exists, the relevant determinant should vanish. Therefore, we have

$$f(\omega, p, m) = 0, \tag{22}$$

where

$$f(\omega, p, m) = f_{11}f_{22} - f_{12}f_{21}, \tag{23}$$

$$f_{11} = q''(L,\omega,p) \qquad f_{12} = s''(L,\omega,p),$$

$$f_{21} = g(L)q'''(L,\omega,p) + p(1-m)\left(\frac{\pi}{L}\right)^2 q'(L,\omega,p),$$

$$f_{22} = g(L)s'''(L,\omega,p) + p(1-m)\left(\frac{\pi}{L}\right)^2 s'(L,\omega,p), \tag{24}$$

where, for example, $s'''(L,\omega,p)$ denotes the value of $\mathrm{d}s^3(x,\omega,p)/\mathrm{d}x^3$ at $x = L$. Equation (22), or $f(\omega,p,m) = 0$, represents the force–frequency interaction mentioned previously.

In the following numerical analysis it is assumed $e = 0{\cdot}5$ in equations (4) and (5). As before, in the $\omega = 0$ case, the successive zeros of (p) for the function $f(0,p,m)$ are denoted by p_j $(j = 1,2\ldots)$ respectively. A particular value of m is called m_{tr}, which is defined such that the condition $p_1 = p_2$ is satisfied. In the studied case $(e = 0{\cdot}5)$, it is found $m_{tr} = 0{\cdot}6014$, and the corresponding buckling loading is $p_1 = p_2 = p_r = 2{\cdot}090$.

Similarly, in the $p = 0$ case, the successive zeros of the function $f(\omega,0,m)$ are denoted by ω_j $(j = 1,2,\ldots)$ respectively. The first two zeros are $\omega_1 = 0{\cdot}299$ and $\omega_2 = 2{\cdot}463$.

As mentioned above the force–frequency interaction curves $p = g(\omega,m)$ can be evaluated numerically [15]. For the $m = 0{\cdot}00$, $0{\cdot}30$, $0{\cdot}6014$ and $0{\cdot}61$ cases, the relevant curves are plotted in Figure 4. From Figure 4 we see that, for $m = 0{\cdot}00$ and $0{\cdot}30$ cases, there are two separated curves AC and BH (for the $m = 0{\cdot}00$ case), AD and BG curves (for the $m = 0{\cdot}30$ case). However, in the case of $m = m_{tr} = 0{\cdot}6014$, the left ends of curves are merged into one point Q. Also, in the case of $m = 0{\cdot}61 > m_{tr}$, the curve becomes ARB in Figure 4. Similarly, for the $m = 0{\cdot}80$, $1{\cdot}00$, $1{\cdot}20$, $1{\cdot}40$ and $1{\cdot}60$ cases, the relevant curves are plotted in Figure 5.

Similarly, the buckling or critical force can be obtained from the static criterion (for the $m \leqslant m_{tr}$ case) or from the dynamic criterion (for the $m > m_{tr}$ case). For both cases $m \leqslant m_{tr}$ and $m > m_{tr}$, $(m_{tr} = 0{\cdot}6014)$, the buckling or critical force and the relevant frequency are expressed by

$$p = h_1(m) \quad \text{(for both cases)}, \tag{25}$$

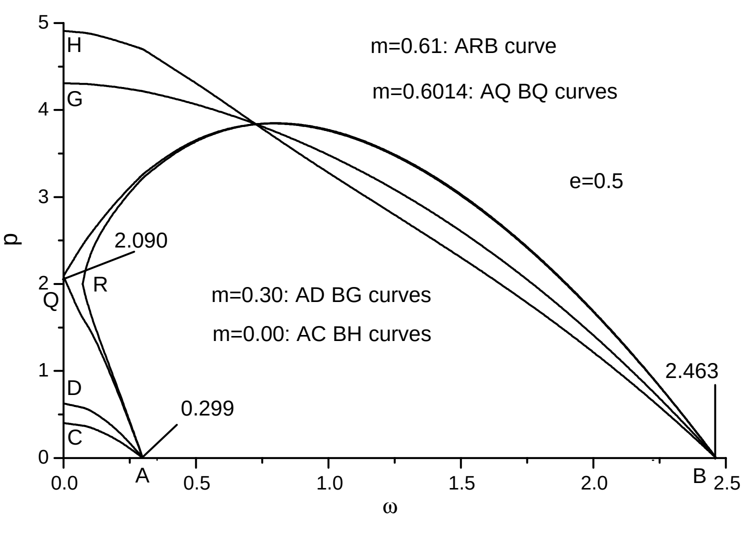

Figure 4. The force–frequency interaction curves for the cases $m = 0{\cdot}00$, $0{\cdot}30$, $0{\cdot}6014$ and $0{\cdot}61$ $(m_{tr} = 0{\cdot}6014)$ in the varying cross-section case.

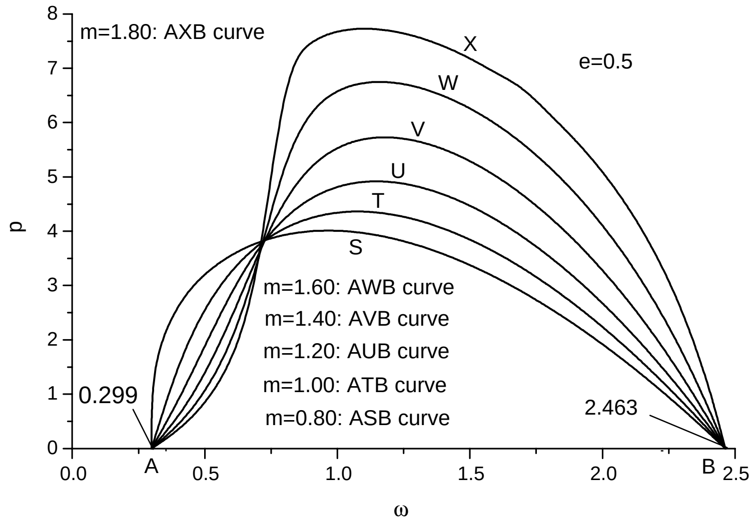

Figure 5. The force–frequency interaction curves for the cases from $m = 0{\cdot}80$ to $1{\cdot}80$ in the varying cross-section case.

Table 3

Dependence of the normalized critical force (p) on the parameter (m), in the tape section case

m	0·0	0·3	0·59	0·6014	0·61	0·80	1·20	1·60
$h_1(m)$	0·400	0·625	1·673	2·090	3·854	4·020	4·932	6·765
$h_2(m)$					0·799	0·964	1·145	1·155

$$\omega = h_2(m) \quad \text{(for the case } m > m_{tr} = 0{\cdot}6014). \tag{26}$$

The calculated values are plotted in Table 3. It is seen from Table 3 that the calculated values for the function $h_1(m)$ are discontinuous at the point $m = m_{tr}$.

4. CONCLUSIONS

The analysis of interaction between the compressive force and the vibration frequency plays an important role in the present study. The interaction can be expressed by $f(\omega, p, m) = 0$, or in an alternative form $p = g(\omega, m)$. The function $g(\omega, m)$ generally depends on the tangency coefficient (m). Generally, the function $g(\omega, m)$ is a complicated function, particularly in the varying cross-section case. In this paper, an effective numerical method is suggested to obtain the function $g(\omega, m)$.

Secondly, on the basis of many interaction curves $p = g(\omega, m)$ from different values of (m), for example, shown by Figures 4 and 5, one can easily determine whether the static or dynamic criterion will be used. In addition, the critical force can be evaluated immediately. It is found that if the tangency coefficient (m) is larger then the critical force is higher. Physically, the lateral component of the compressive force can elevate the critical loading.

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REFERENCES

1. M. Beck 1952 *Zeitschrift für angewandte Mathematik und Physik* **3**, 225–228. Die Knicklast des einseitig eingespannten, tangential gedrückten Stabes.
2. K. S. Deineko and M. J. Leonov 1955 *Applied Mathematics and Mechanics* **19,** 738–744. Dynamic method of investigating the stability of a compressed bar. (In Russian)
3. H. Ziegler 1956 in *Advances in Applied Mechanics*, Vol. IV, 3 (H. L. Dryden and Th. Von Karman, editors) 351–403. New York: Academic Press. On the concept of elastic stability.
4. S. P. Timoshenko and J. M. Gere 1961 *Theory of Elastic Stability*. New York: McGraw-Hill.
5. V. V. Bolotin 1963 *Nonconservative Problems of the Theory of Elastic Stability*. New York: Pergamon.
6. H. Leipholz 1980 *Stability of Elastic Systems*. Netherlands: Noordhoff.
7. G. Herrmann 1967 *Applied Mechanics Review* **20**, 103–108. Stability of equilibrium of elastic systems subjected to nonconservative forces.
8. G. Herrmann and R. W. Bungay 1964 *American Society of Mechanical Engineers Journal of Applied Mechanics* **31**, 435–440. On the stability of elastic systems subjected to nonconserative forces.
9. T. Irie, G. Yamada and I. Takahashi 1980 *Journal of Sound and Vibration* **70**, 503–512. Vibration and stability of a non-uniform Timoshenko beam subjected to a follower force.
10. S. Y. Lee and Y. H. Kuo 1992 *Journal of Sound and Vibration* **155**, 291–301. Exact solutions for the non-conservative elastic stability of non-uniform columns.
11. B. N. Rao and G. V. Rao 1988 *Journal of Sound and Vibration* **122**, 188–191. Applicability of static or dynamic criterion for the stability of a non-uniform cantilever column subjected to a tip-concentrated subtangential follower force.
12. S. Y. Lee and K. C. Hsu 1995 *Journal of Sound and Vibration* **186**, 111–123. Elastic instability of beams subjected to a partially tangential force.
13. S. Y. Lee, T. Y. Chen and W. R. Wang 1995 *Journal of Sound and Vibration* **188**, 25–38. Non-conservative instability of a Timoshenko beam subjected to a partially tangential force.
14. G. Dzhanelidze 1958 *Trudy Leningradskogo Politekhnicheskogo Instituta* **192**, 21–27. On the stability of rods due to the action of follower forces. (In Russian)
15. Y. Z. Chen 2000 *Communications in Numerical Methods in Engineering* **16**, 343–355. Target function method for evaluating natural vibration frequency of bending beam with adhesive mass.
16. F. B. Hildebland 1974 *Introduction to Numerical Analysis.* New York: McGraw-Hill.