Journal of Sound and Vibration (2003) **259**(5), 1269–1275
doi:10.1006/jsvi.2002.5286, available online at http://www.idealibrary.com on IDEAL®

ON THE TRANSIENT RESPONSE OF A FLUID-LOADED STRUCTURE REPRESENTED BY A SERIES OF RESONANCE MODES

D. Habault

CNRS-Laboratoire de Mécanique et d'Acoustique, 31 *chemin Joseph Aiguier,* 13402 *Marseille cedex* 20 *France E-mail: habault@lma.cnrs-mrs.fr*

AND

P. J. T. Filippi

La Santa Severa, CNPRS-Appontement 10, *Port de la Pointe Rouge,* 13008 *Marseille, France*

(*Received* 26 *April* 2002)

1. INTRODUCTION

In two recent papers [1, 2] the authors showed how the response of a fluid-loaded structure can be represented as a series of the fluid-loaded resonance modes. Since the publication of these papers, they have found that the expression for the acoustic pressure is erroneous. The aim of this short note is to clarify their theoretical approach and to propose formulas which, to their opinion, are now correct.

For simplicity, the case of a fluid-loaded baffled plate is considered here. Use is made of notations close to those of reference [1]. A plate Σ, with boundary $\partial\Sigma$, is located in the plane $z = 0$ and extended by an infinite perfectly rigid baffle. The plate is characterized by a constant thickness h, a density μ, Young's modulus E and the Poisson ratio ν; the corresponding rigidity is $D = Eh^3/12(1-\nu^2)$. The two half-spaces $\Omega^+ = \{z > 0\}$ and $\Omega^- = \{z < 0\}$ are occupied by a perfect fluid, characterized by a density μ_0 and a sound velocity c_0. In what follows, Ω stands for $\Omega^+ \bigcup \Omega^-$. We assume that the system is excited by a force $\widetilde{F}(M,t) = \widetilde{\psi}(t) f(M)$ acting on the plate, of duration T, that is $\widetilde{\psi}(t) = 0$ for $t < 0$ and $t > T$.

Let $\widetilde{W}(M,t)$ be the plate displacement and $\widetilde{P}(M,t)$ the acoustic pressure. The governing equations are

$$\left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\widetilde{P}(M,t) = 0 \quad \text{in} \quad \Omega,$$

$$\left(D\Delta^2 + \mu h\frac{\partial^2}{\partial t^2}\right)\widetilde{W}(M,t) + \widetilde{\bar{P}}(M,t) = \widetilde{F}(M,t) \quad \text{on} \quad \Sigma,$$

$$\frac{\partial \widetilde{P}(M,t)}{\partial z} = \begin{cases} -\mu_0 \dfrac{\partial^2 \widetilde{W}(M,t)}{\partial t^2} & \text{on} \quad \Sigma, \\ 0 & \text{on} \quad \bar{\Sigma} = C\Sigma, \end{cases} \tag{1}$$

where $\widetilde{\bar{P}}(M,t) = \mathrm{Tr}^+\widetilde{P}(M,t) - \mathrm{Tr}^-\widetilde{P}(M,t)$ is the jump of the acoustic pressure across the plane $z = 0$. This system must be completed by boundary conditions on $\widetilde{W}$ along $\partial\Sigma$ and

0022-460X/02/$35.00

an outgoing-wave condition on $\widetilde{P}$. Furthermore, it is assumed that the system is at rest for $t<0$. Thus, the solution of (1) exists and is unique.

As it is usual in acoustics, the time Fourier transform $\Phi(M,\omega)$ of any function $\widetilde{\Phi}(M,t)$ is defined by

$$\Phi(M,\omega)=\int_{-\infty}^{+\infty}\widetilde{\Phi}(M,t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t.$$

The system of equations (1) becomes

$$\begin{aligned}&\left(\Delta+\frac{\omega^2}{c_0^2}\right)P(M,\omega)=0\quad\text{in}\quad\Omega,\\&\left(D\Delta^2-\mu h\omega^2\right)W(M,\omega)+\bar{P}(M,\omega)=\psi(\omega)f(M)\quad\text{on}\quad\Sigma,\\&\frac{\partial P(M,\omega)}{\partial z}\begin{cases}=\mu_0\omega^2W(M,\omega)&\text{on}\quad\Sigma,\\=0&\text{on}\quad\bar{\Sigma}=C\Sigma\end{cases}.\end{aligned}\tag{2}$$

A Sommerfeld condition on P (or any equivalent condition) must be added to these equations. The aim is to express the solutions of systems (1) and (2) in terms of the fluid-loaded resonance modes of the plate, as references [1, 2].

2. EIGENMODES AND RESONANCE MODES OF THE FLUID-LOADED PLATE

Owing to the simple geometry of the physical system, the acoustic pressure can be expressed in terms of the plate displacement by

$$P(M,\omega)=\operatorname{sgn}(z)\omega^2\mu_0\int_\Sigma W(M',\omega)\mathcal{G}_\omega(M,M')\,\mathrm{d}\sigma(M')$$

with

$$\mathcal{G}_\omega(M,M')=-\mathrm{e}^{\mathrm{i}\omega r(M,M')/c_0}/2\pi r(M,M'),\tag{3}$$

where $r(M,M')$is the distance between points M and M'. This leads to an integro-differential equation for the plate displacement:

$$\left(D\Delta^2-\mu h\omega^2\right)W(M,\omega)+2\omega^2\mu_0\int_\Sigma W(M',\omega)\mathcal{G}_\omega(M,M')\,\mathrm{d}\sigma(M')=\psi(\omega)f(M).\tag{4}$$

In the following, the variables will be omitted when there is no ambiguity.

Equation (4) can be written in its variational form which is more suitable for our purpose. Let us introduce the following notations:

$$\langle u,v^*\rangle=\int_\Sigma u(M)v^*(M)\,\mathrm{d}\sigma(M),$$

$$a(u,v)=D\int_\Sigma\left\{\Delta u(M)\Delta v^*(M)+(1-v)\left[2\frac{\partial^2u(M)}{\partial x\partial y}\frac{\partial^2v^*(M)}{\partial x\partial y}-\frac{\partial^2u(M)}{\partial x^2}\frac{\partial^2v^*(M)}{\partial y^2}-\frac{\partial^2u(M)}{\partial y^2}\frac{\partial^2v^*(M)}{\partial x^2}\right]\right\}\mathrm{d}\sigma(M),$$

$$\beta_\omega(u,v)=2\int_\Sigma\int_\Sigma u(M)\mathcal{G}_\omega(M,M')v^*(M')\,\mathrm{d}\sigma(M)\,\mathrm{d}\sigma(M').\tag{5}$$

The variational form of equation (4) is

$$a(W,U) - \mu h\omega^2\left\{\langle W,U\rangle - \frac{\mu_0}{\mu h}\beta_\omega(W,U)\right\} = \psi\langle f,U\rangle \quad \forall\ U, \tag{6}$$

where U is any function of the Hilbert space which W belongs to. The fluid-loaded plate eigenmodes $\widehat{W}_n$ and eigenvalues Λ_n, which depend on ω, are the non-zero solutions to the equation

$$a(\widehat{W}_n,U) = \Lambda_n\left\{\langle \widehat{W}_n,U\rangle - \frac{\mu_0}{\mu h}\beta_\omega(\widehat{W}_n,U)\right\} \quad \forall\, U. \tag{7}$$

They satisfy the following property:

$$a(\widehat{W}_n,\widehat{W}_m^*) = \Lambda_n\left\{\langle \widehat{W}_n,\widehat{W}_m^*\rangle - \frac{\mu_0}{\mu h}\beta_\omega(\widehat{W}_n,\widehat{W}_m^*)\right\} = 0 \quad \text{for } m\neq n,$$
$$a(\widehat{W}_n,\widehat{W}_n^*) = \Lambda_n\left\{\langle \widehat{W}_n,\widehat{W}_n^*\rangle - \frac{\mu_0}{\mu h}\beta_\omega(\widehat{W}_n,\widehat{W}_n^*)\right\} \equiv \Lambda_n N_n(\omega) \tag{8}$$

It must be noticed that $\widehat{W}_n$ is defined *modulo* any multiplicative constant. We choose here $N_n(\omega) = 1$. If, for some reason, another choice is preferred—for example $||\widehat{W}_n|| = 1$—it must be kept in mind that the eigenmode is, in fact, $\widehat{W}_n/N_n^{1/2}$.

The resonance modes W_n of the fluid-loaded plate and the resonance angular frequencies ω_n are the non-zero solutions of

$$a(W_n,U) = \mu h\omega_n^2\left\{\langle W_n,U\rangle - \frac{\mu_0}{\mu h}\beta_{\omega_n}(W_n,U)\right\} \quad \forall\, U. \tag{9}$$

As discussed in reference [1], ω_n is solution to the equation

$$\Lambda_n(\omega_n) = \mu h\omega_n^2. \tag{10}$$

This equation has two solutions, denoted by ω_n and ω_{-n}, which satisfy the following properties:

$$\omega_n = \Omega_n - \mathrm{i}\tau_n, \quad \omega_{-n} = -\Omega_n - \mathrm{i}\tau_n \ \text{ with } \quad \Omega_n > 0 \quad \text{and} \quad \tau_n > 0. \tag{11}$$

The corresponding resonance modes satisfy the relationships

$$W_n(M) = \widehat{W}_n(M,\omega_n), \quad W_{-n}(M) = \widehat{W}_n(M,-\omega_n^*) = W_n^*. \tag{12}$$

3. REPRESENTATION OF THE PLATE DISPLACEMENT AND OF THE RADIATED PRESSURE IN TERMS OF THE RESONANCE MODES

For the harmonic regime, the plate displacement W and the acoustic pressure P are given by

$$W(M,\omega) = \psi(\omega)\sum_{n=1}^{\infty}\frac{\langle f,\widehat{W}_n^*\rangle}{\Lambda_n(\omega) - \mu h\omega^2}\widehat{W}_n(M,\omega),$$
$$P(M,\omega) = \mathrm{sgn}(z)\psi(\omega)\sum_{n=1}^{\infty}\frac{\omega^2\mu_0\langle f,\widehat{W}_n^*\rangle}{\Lambda_n(\omega) - \mu h\omega^2}\int_\Sigma \widehat{W}_n(M',\omega)\mathscr{G}_\omega(M,M')\,\mathrm{d}\sigma(M'). \tag{13}$$

These expressions are equivalent to expressions (11) and (12) in reference [1].

The transient plate displacement $\widetilde{W}(M,t)$ is readily obtained by the residue method:

$$\widetilde{W}(M,t) = -\mathrm{i}\,\widetilde{\psi}(t) \underset{t}{*} Y(t) \sum_{n=1}^{\infty} \left[\frac{\langle f, W_n^* \rangle}{\Lambda_n'(\omega_n) - 2\mu h \omega_n} W_n(M) \mathrm{e}^{-\mathrm{i}\omega_n t} - \frac{\langle f, W_n^* \rangle^*}{\Lambda_n'^*(\omega_n) - 2\mu h \omega_n^*} W_n^*(M) \mathrm{e}^{+\mathrm{i}\omega_n^* t} \right]. \quad (14)$$

where $\underset{t}{*}$ stands for the time convolution product and $Y(t)$ is the Heaviside function. The corresponding expression (14) in reference [1] is erroneous. Indeed, in that paper, the inverse Fourier transform of the harmonic plate displacement (expression equivalent to the first equality in equations (13) here) does not account for the poles due to the excitation force. This is done in equation (14) through the convolution product by $\widetilde{\psi}(t)$.

Expression (15) in reference [1] for the transient sound pressure is also erroneous. This expression was obtained by applying the residue theorem, which is not possible in general (the coefficient $\omega^2/[\Lambda_n(\omega) - \mu\omega^2]$ is not a decreasing function when $|\omega| \to \infty$). Instead, the sound pressure can be expressed in terms of the plate acceleration

$$\widetilde{P}(M,t) = \mathrm{sgn}(z)\ \mu_0\ \widetilde{\mathscr{G}}(M,t) \underset{(M,t)}{*} \frac{\partial^2 \widetilde{W}(M,t)}{\partial t^2} \otimes \delta_\Sigma, \quad (15)$$

where $*_{(M,t)}$ is the space and time convolution product, and $\widetilde{\mathscr{G}}(M,t) = -\delta(t - r/c_0)/2\pi r$ is the Green function of the wave equation. When $\widetilde{W}(M,t)$ is a twice differentiable function, the convolution product in equation (15) is equal to the integral

$$\widetilde{P}(M,t) = \mathrm{sgn}(z)\ \mu_0 \int_\Sigma \frac{\partial^2 \widetilde{W}(M', t - r(M,M')/c_0)}{\partial t^2} \frac{1}{2\pi r(M,M')} \mathrm{d}\sigma(M').$$

Let us point out that this is not generally the case: due to possible discontinuities, $\partial_{t^2} \widetilde{W}(M,t)$ may contain Dirac distributions together with their first derivatives.

By introducing equation (14) into equation (15), one gets

$$\widetilde{P}(M,t) = \mathrm{sgn}(z)\ \mu_0\ \widetilde{\mathscr{G}}(M,t) \underset{(M,t)}{*} \frac{\partial^2}{\partial t^2} \Bigg\{ -\mathrm{i}\ \widetilde{\psi}(t) \underset{t}{*} Y(t) \times \sum_{n=1}^{\infty} \left[\frac{\langle f, W_n^* \rangle}{\Lambda_n'(\omega_n) - 2\mu h \omega_n} W_n(M) \mathrm{e}^{-\mathrm{i}\omega_n t} - \frac{\langle f, W_n^* \rangle^*}{\Lambda_n'^*(\omega_n) - 2\mu h \omega_n^*} W_n^*(M) \mathrm{e}^{+\mathrm{i}\omega_n^* t} \right] \Bigg\}. \quad (16)$$

Details on the numerical computation of this last formula are given in the next section.

Finally, the Fourier transform of expressions (14) and(16) gives the representations of W and P in terms of the fluid-loaded plate resonance modes:

$$W(M,\omega) = \psi(\omega) \sum_{n=1}^{\infty} \left[\frac{\langle f, W_n^* \rangle}{\Lambda_n'(\omega_n) - 2\mu h \omega_n} \frac{W_n(M)}{\omega - \omega_n} - \frac{\langle f, W_n^* \rangle^*}{\Lambda_n'^*(\omega_n) - 2\mu h \omega_n^*} \frac{W_n^*(M)}{\omega + \omega_n^*} \right],$$

$$P(M,\omega) = \mathrm{sgn}(z)\omega^2 \mu_0 \psi(\omega) \sum_{n=1}^{\infty} \left[\frac{\langle f, W_n^* \rangle}{\Lambda_n'(\omega_n) - 2\mu h \omega_n} \int_\Sigma \frac{W_n(M')}{\omega - \omega_n} \mathscr{G}_\omega(M,M')\, \mathrm{d}\sigma(M') - \frac{\langle f, W_n^* \rangle^*}{\Lambda_n'^*(\omega_n) - 2\mu h \omega_n^*} \int_\Sigma \frac{W_n^*(M')}{\omega + \omega_n^*} \mathscr{G}_\omega(M,M')\, \mathrm{d}\sigma(M') \right]. \quad (17)$$

The first equality in equations (17) is equivalent to formula (16) of reference [1], and the second equality in equations (17) is the right expression for $P(M,\omega)$.

4. COMPUTATION OF $\widetilde{P}(M, t)$

The displacement given by equations (17) is generally a function which itself (and/or its first derivative) is not continuous at $t = 0$. Furthermore, for an excitation of finite duration T, the displacement can also have discontinuities at $t = T$. The second derivative of $\widetilde{W}(M, t)$ takes the following general form:

$$\frac{\partial^2 \widetilde{W}(M,t)}{\partial t^2} = \left\{ \frac{\partial^2 \widetilde{W}(M,t)}{\partial t^2} \right\} + \delta_t \frac{\partial \widetilde{W}}{\partial t}(M, 0^+) + \frac{\mathrm{d}\delta_t}{\mathrm{d}t} \widetilde{W}(M, 0^+)$$
$$+ \delta_{t-T}\left[\frac{\partial \widetilde{W}}{\partial t}(M, T^+) - \frac{\partial \widetilde{W}}{\partial t}(M, T^-) \right] + \frac{\mathrm{d}\delta_{t-T}}{\mathrm{d}t}\left[\widetilde{W}(M, T^+) - \widetilde{W}(M, T^-) \right], \quad (18)$$

where the first term in braces is the derivative of the function $\widetilde{W}$ out of the discontinuity points and $f(\tau^{\pm}) \equiv \lim_{\varepsilon \to 0} f(\tau \pm \varepsilon)$ with $\varepsilon > 0$. For simplicity, in the series expansion of $\widetilde{\mathscr{G}}(M, t) *_{(M,t)} \partial^2 w(M, t)/\partial t^2$, let us consider one term only, say $\phi(M, t)$, given by

$$\phi(M,t) = \widetilde{\mathscr{G}}(M,t) \underset{(M,t)}{*} \frac{\partial^2}{\partial t^2}\left[\widetilde{\psi}(t) \underset{t}{*} Y(t) W_n(M) \mathrm{e}^{-\mathrm{i}\omega_n t} \right]$$
$$= \widetilde{\psi}(t) \underset{t}{*} \widetilde{\mathscr{G}}(M,t) \underset{(M,t)}{*} \frac{\partial^2}{\partial t^2}\left[Y(t) W_n(M) \mathrm{e}^{-\mathrm{i}\omega_n t} \right] \quad (19)$$

4.1. CONTRIBUTION OF THE FIRST TERM OF EQUATION (18):

This is the only term which appears if $w(M, t)$ and its first derivative are continuous functions. Its contribution is

$$\phi_1(t) = -\omega_n^2 \mathrm{e}^{-\mathrm{i}\omega_n t} \int_\Sigma \mathscr{G}_{\omega_n}(M, M') W_n(M') Y\left(t - \frac{r(M,M')}{c_0} \right) \int_0^{\theta(t,M,M')} \widetilde{\psi}(\tau) \mathrm{e}^{-\mathrm{i}\omega_n \tau} \, \mathrm{d}\tau \, \mathrm{d}\sigma(M') \quad (20)$$

where

$$\theta(t, M, M') = \begin{cases} t - \tau(M,M')/c_0 & \text{if} \quad t - \dfrac{\tau(M,M')}{c_0} < T \\ T & \text{if} \quad t - \dfrac{\tau(M,M')}{c_0} > T. \end{cases}$$

Expression (20) does not present any computational difficulty. In particular, it is equal to 0 if t is smaller than $r_{\min}/c_0$ and it is equal to

$$\phi_1(t) = -\omega_n^2 \mathrm{e}^{-\mathrm{i}\omega_n t} \int_\Sigma \mathscr{G}_{\omega_n}(M, M') W_n(M') \, \mathrm{d}\sigma(M') \int_0^T \widetilde{\psi}(\tau) \mathrm{e}^{-\mathrm{i}\omega_n \tau} \, \mathrm{d}\tau. \quad (21)$$

if t is larger than $T + r_{\max}/c_0$. $r_{\min}$ and $r_{\max}$ represent, respectively, the minimum distance and the maximum distance between M and a point on the plate.

The signal starts at $t_0 = r_{\min}/c_0$ and the radiation of the plate increases until $t_1 = T + r_{\max}/c_0$. Then it becomes a damped sine signal of the form

$$\varpi(M) \mathrm{e}^{-i\omega_n t} = \varpi(M) \mathrm{e}^{-\mathrm{i}\Omega_n t} \mathrm{e}^{-\tau_n t}.$$

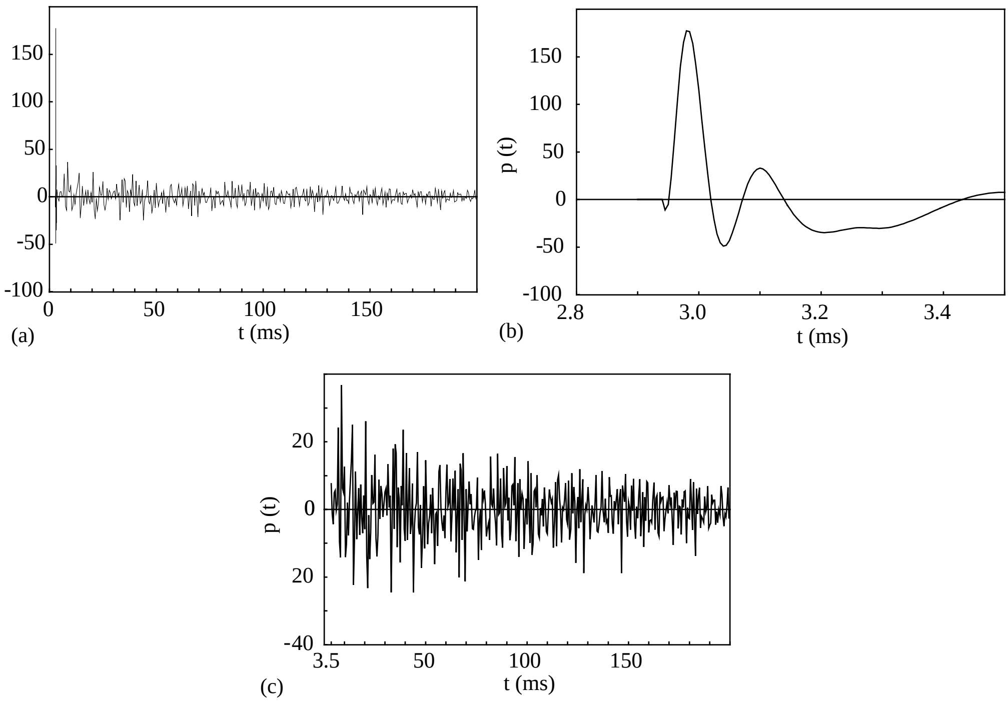

Figure 1. Acoustic pressure above the plate: (a) the first 200 ms; (b) the initial burst; (c) the reverberated sound.

The contribution of all the modes in $\widetilde{W}(M,t)$ follows the same law. The first part of the signal ($t_0 < t < t_1$) corresponds to the sound establishment, and is followed by the reverberated sound. This is illustrated by the following example.

Let us consider the following time dependence of the driving force:

$$\widetilde{\psi}(t) = \frac{1}{T}[Y(t) - Y(t-T)][1 - \cos(2\pi t/T)]. \tag{22}$$

It is well known that the limit for $T \to 0$ of this function is the Dirac distribution δ_t. The convolution product $\widetilde{\psi}(t) \underset{t}{*} Y(t) W_n(M) e^{-i\omega_n t}$ is a continuous function together with its first derivative: thus, its second derivative includes the first term of equation (18) only and there is no special numerical difficulty. Furthermore, such an excitation is certainly more realistic than a Dirac impulse. Figure 1 shows the acoustic pressure radiated by a baffled clamped plate excited by a point force with a time dependence described by equation (22): the response of the plate is computed with 20 resonance modes. More details about this example will be given in a forthcoming paper.

4.2. CONTRIBUTION OF THE SECOND AND THIRD TERMS OF EQUATION (18):

Let us point out that if $\widetilde{W}(M,0^+)$ and $\partial\widetilde{W}(M,0^+)/\partial t$ are not zero, the second and third terms of equation (18) give a non-zero contribution. They can formally be written as

$$\widetilde{\mathscr{G}} *_{(M,t)} \delta_t \frac{\partial w}{\partial t}(M,0^+) = -\int_\Sigma \frac{\delta(t - r(M,M')/c_0)}{2\pi r(M,M')} \frac{\partial w}{\partial t}(M',0^+)\, d\sigma(M'),$$

$$\widetilde{\mathcal{G}} *_{(M,t)} \frac{\mathrm{d}\delta_t}{\mathrm{d}t} w(M, 0^+) = -\int_\Sigma \frac{\partial_t \delta(t - r(M, M')/c_0)}{2\pi r(M, M')} w(M', 0^+)\, \mathrm{d}\sigma(M').$$

The first expression is the integral of $\partial w(M', 0^+)/\partial t$ along the circle arc $t = r(M, M')/c$. The second expression involves the time derivative of $\delta(t - r(M, M')/c_0)$, which can be expressed in terms of a space derivative:

$$\frac{\partial \delta(t - r/c_0)}{\partial t} = -\frac{1}{c_0} \frac{\partial \delta(t - r/c_0)}{\partial r}.$$

Using this equality implies a space derivative of $w(M')$ which is not very easy to express in a convenient form for numerical computation.

But, in practice, this is not a real difficulty. Indeed, for modelling an impact force—as, for example, the excitation of a drum or a bell—it is possible to choose a mathematical excitation as close as desired to a Dirac impulse.

5. CONCLUSION

In reference [2], formulas (28) and (29) need a few modifications as explained in section 3 of the present paper. In the same way, expressions (30) of reference [3] must be corrected.

In reference [2], the authors have presented a comparison between the computed response of an elastic shell to a transient incident acoustic wave with the experimental one. The numerical result is correct. Indeed, the solution expansion in terms of the fluid-loaded structure resonance modes has been calculated by solving directly the time-dependent equations. By doing so, the error which has been pointed out in the present note is avoided.

REFERENCES

1. D. Habault and P. J. T. Filippi 1998. *Journal of Sound and Vibration* **213**, 333–374. Light fluid approximation for sound radiation and diffraction by thin elastic plates.
2. P. J. T. Filippi, D. Habault, P.-O. Mattei and C. Maury 2001 *Journal of Sound and Vibration* **239**, 639–663. The role of the resonance modes in the response of a fluid-loaded structure.
3. C. Maury and P. J. T. Filippi 2001 *Journal of Sound and Vibration* **241**, 459–483. Transient acoustic diffraction and radiation by an axisymmetrical elastic shell: a new statement of the basic equations and a numerical method based on polynomial approximations.