Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 260 (2003) 191–194
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

# An almost semicentennial formula for a simple approximation of the natural frequencies of Bernoulli–Euler beams

M.J. Maurizi[a,*], P.M. Bellés[a], H.D. Martín[b]

[a] Department of Engineering, Universidad Nacional del Sur, Avda. Alem 1253, 8000 Bahia Blanca, Argentina
[b] FRVT, Universidad Tecnológica Nacional, 2600 Venado Tuerto, Argentina

Received 17 May 2002; accepted 4 June 2002

Even though its applications were rarely discussed, the brief note which is referred to in the title should be cited in the present investigations on the subject. The reason is that the mentioned study is probably the earliest investigation developed on the adoption of approximate formulae for the determination of frequencies of free transverse vibrations of particularly restrained beams.

As a part of a program initiated in 1952 by the Structural Research Laboratory, Department of Civil Engineering, of the University of Illinois (USA), the professors Newmark and Veletsos [1] pioneered the employment of the basic formulae that would be of use to structural engineers to estimate, at the preliminary design stage, a simple approximation of the natural frequencies in the case of beams elastically restrained against rotation at the ends.

Consider a uniform beam to be oscillating in one of its natural modes of free vibration. $EI$ is the modulus of flexural rigidity, $m$ is its mass per unit length, and $L$ denotes the span length between supports which are non-deflecting but both of them offer a linear resistance to end rotations of the beam (see Fig. 1).

The restraining moments at the corresponding ends are related to the end rotations by the equations

$$M_1 = K_{r1}\theta_1 \quad \text{and} \quad M_2 = K_{r2}\theta_2. \tag{1,2}$$

Here $K_{r1}$ and $K_{r2}$ are the rotational spring constants and they are in connection to the characteristics of the beam by the dimensionless coefficients as follows:

$$\beta_1 = \frac{K_{r1}L}{EI} \quad \text{and} \quad \beta_2 = \frac{K_{r2}L}{EI}. \tag{3,4}$$

It can be pointed out that for a hinged end $\beta = 0$ and for a clamped end $\beta \to \infty$.

The frequencies $f_n$ of the elastically restrained beam have been stated as the product of a dimensionless coefficient $C_n$ multiplied by the fundamental frequency of same beam hinged at the

*Corresponding author. Fax: 54-291-459-5157.
*E-mail address:* dtoinge@criba.edu.ar (M.J. Maurizi).


PII: S0022-460X(02)01129-X

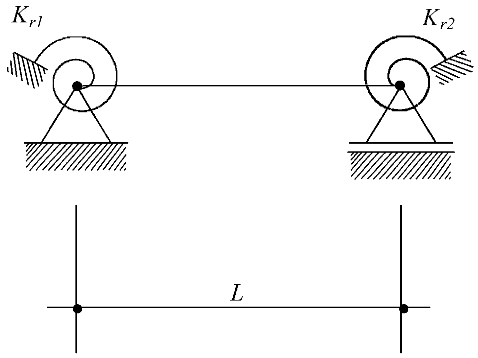


Fig 1. Beam elastically restrained at the ends.

ends

$$f_n = C_n \frac{\pi}{2L^2}\sqrt{\frac{EI}{m}}. \tag{5}$$

Furthermore, the following simple approximation for the coefficient $C_n$, for the fundamental frequency ($n = 1$) and the higher natural frequencies, was also reported in Ref. [1]:

$$C_n = \left[n + \frac{1}{2}\left(\frac{\beta_1}{5n + \beta_1}\right)\right]\left[n + \frac{1}{2}\left(\frac{\beta_2}{5n + \beta_2}\right)\right]. \tag{6}$$

Additionally, when the end restraints are very small, i.e., for values of the coefficients $\beta$ approaching zero, the value of $C_n$ is proposed under the following form:

$$C_n \cong n^2 + 0.1(\beta_1 + \beta_2). \tag{7}$$

These approximate equations are similar to those developed by the same authors in a previous paper [2] in relation to buckling loads for partially restrained beams.

On the other hand, the general equation of frequencies for generally restrained Bernoulli–Euler beams presented by Maurizi et al. [3], and also quoted in Refs. [4–7], can be used for a wide range of combinations of boundary conditions. In the present study the transcendental equation is as follows [6]:

$$2R_1R_2\varphi_1(y_n)y_n^2 + (R_1 + R_2)\varphi_6(y_n)y_n - \varphi_4(y_n) = 0, \tag{8}$$

where

$$(y_n/L)^4 = k_n^4 = \omega_n^2(\rho A/EI), \tag{9}$$

$$\varphi_1(y_n) = \sin y_n \sinh y_n, \tag{10a}$$

$$\varphi_4(y) = \cos y_n \cosh y_n - 1, \tag{10b}$$

$$\varphi_6(y_n) = \sin y_n \cosh y_n - \sinh y_n \cos y_n \tag{10c}$$

Table 1
Comparison of the frequency parameters between the exact values and the N–V method

| $\beta_1$ | $\beta_2$ | Mode number | | | | | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | |
| 0 | 0 | 3.14159265 | 6.28318531 | 9.42477796 | 12.56637061 | 15.70796327 | Exact |
| | | 3.14159265 | 6.28318531 | 9.42477796 | 12.56637061 | 15.70796327 | N–V |
| 0.01 | 10 | 3.66600954 | 6.68815630 | 9.75207435 | 12.84001779 | 15.94263705 | Exact |
| | | 3.62940846 | 6.66515656 | 9.73440922 | 12.82589889 | 15.93110087 | N–V |
| 0 | 100 | 3.88918500 | 7.00322722 | 10.11854568 | 13.23541270 | 16.35372435 | Exact |
| | | 3.81699044 | 6.96065983 | 10.08463369 | 13.20465869 | 16.32419428 | N–V |
| 1 | 10,000 | 4.04143802 | 7.13313310 | 10.24653159 | 13.37459282 | 16.50651642 | Exact |
| | | 4.00442698 | 7.10348394 | 10.23171197 | 13.36677608 | 16.50443133 | N–V |
| 10,000 | 10,000 | 4.72909537 | 7.85163553 | 10.99341157 | 14.13434261 | 17.27531056 | Exact |
| | | 4.71160397 | 7.85241241 | 10.99322162 | 14.13403162 | 17.27484240 | N–V |
| $\infty$ | $\infty$ | 4.73004074 | 7.85320462 | 10.99560784 | 14.13716549 | 17.27875966 | Exact |
| | | 4.73004074 | 7.85320462 | 10.99560784 | 14.13716549 | 17.27875966 | N–V |

and

$$R_1 = EI/K_{r1}L, \quad R_2 = EI/K_{r2}L. \tag{11a, b}$$

Obviously, $R_1 \equiv 1/\beta_1$ and $R_2 \equiv 1/\beta_2$.

In Table 1 the first five natural frequencies of a beam elastically restrained against rotation at the ends are presented. Numerical results were determined by means of expression (6) (henceforth N–V) and subsequent application of Eq. (5) and are tabulated for various dimensionless coefficients $\beta$. Additionally, the free vibration frequencies were computed according to the Bernoulli–Euler theory by employing Eq. (8) (henceforth "exact").

It appears that, the maximum difference between the exact solution and the approximations does not exceed 4% for the fundamental frequency and a little over 1% for the higher frequencies [1]. It was found that the absolute maximum error of Eq. (6) is obtained for a hinged–clamped beam [1] and the results show that errors greater than 2% in the fundamental frequency occur only for beams having one end either hinged or practically unrestrained and the other end clamped or approximately fixed.

On the other hand, the results obtained by using relation (7) are in error by less than 1.5% when the sum $(\beta_1 + \beta_2)$ is less than 1.0.

Clearly, the published work of Newmark and Veletsos [1] reveals that the calculations of the parameters, which significantly affect the dynamic performance of beams, are simplified greatly with the use of these approximate formulations.

## Acknowledgements

The present study has been sponsored by the Secretary of Science and Technology (Universidad Nacional del Sur), Research and Development Program 2001–2002, while the second author has been supported by the Comisión de Investigaciones Científicas de la Provincia de Buenos Aires (CIC).

## References

[1] N.M. Newmark, A.S. Veletsos, A simple approximation for the natural frequencies of partially restrained bars, Journal of Applied Mechanics 19 (4) (1952) 563.
[2] N.M. Newmark, A simple approximate formula for effective end-fixity of columns, Journal of the Aeronautical Sciences 16 (1949) 116.
[3] M.J. Maurizi, R.E. Rossi, J.A. Reyes, General equation of frequencies for vibrating uniform one-span beams, Department of Engineering, Universidad Nacional del Sur, Bahía Blanca, Argentina, 1976.
[4] M.S. Hess, Vibration frequencies for a uniform beam with central mass and elastic supports, Journal of Applied Mechanics 31 (1964) 556–557.
[5] C. Kameswara Rao, S. Mirza, A note on vibrations of generally restrained beams, Journal of Sound and Vibration 130 (1989) 453–465.
[6] M.J. Maurizi, R.E. Rossi, J.A. Reyes, Comments on "A note on vibrations of generally restrained beams", Journal of Sound and Vibration 147 (1991) 167–171.
[7] T. Nànàsi, Relation between frequency equations of single-span beams, Journal of Sound and Vibration 196 (1996) 529–533.