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Letter to the Editor

Random fatigue of a higher order sandwich beam with parameter uncertainties

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1. Introduction

Sandwich constructions with cellular foam core have become more widely used in aircraft, marine vessels and other vehicles. When a sandwich structure is subjected to bending, the core material is mainly in shear. Some investigations on fatigue of sandwich panels in bending have been carried out [1–4]. Studies of the effect of material and geometric parameter uncertainties on random fatigue of sandwich structures are relatively fewer. This paper presents a study of this kind.

Fatigue models for sandwich structures can be broadly classified into three categories: the traditional S–N curve approach, phenomenological modelling, and mechanistic modelling [5]. Mechanistic fatigue models try to quantitatively account for the progression of damage such as fracture cracks in composite laminates. When sandwich structures are under random loading, there are usually several different damage mechanisms occurring at the same time such as delamination and fiber pullout. The existing mechanistic fatigue models have only been applied to simple loadings. Phenomenological fatigue models choose material properties and their degradation rates as damage indices including the stiffness or residual strength. It is observed that the stiffness of sandwich specimen in bending does not decrease below 90% of the original stiffness until just before the final failure [3]. In this study, we will adopt the S–N curve approach.

For the S–N curve model, a sufficient amount of experimental data is needed to determine the model parameters, which often involves measurements of cyclic stress magnitudes and the corresponding cycles to fatigue failure. It is well recognized that the fatigue life is not always fully characterized by the stress magnitude alone. The dependence of the mean and amplitude of the cyclic loadings may be of equal importance. For metals, there are several well-known formula available to account for the dependence of fatigue life on the mean stress. Examples include Goodman's equation, Gerber's equation and Soderbeg's equation [6]. Qiao [7] proposed an $S_a - S_m - N$ curved surface equation to consider the effect of stress amplitude and its mean

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simultaneously. Bond [8] developed a semi-empirical fatigue life prediction methodology for variable-amplitude loading of glass fibre-reinforced composites. Burman and Zenkert [3] used a two-parameter logarithmic Weibull function to fit the experimental data and a Haigh diagram to show the fatigue life dependence on the mean and amplitude of the stress. There is much less theoretical and experimental work of this kind for the sandwich foam core. This paper contributes to the accumulation of knowledge in this regard.

In this paper, a sandwich beam under white noise excitation and modelled with a high-order laminate theory [9,10] is used to demonstrate the work. In Section 2, we review the higher order model of a simply supported sandwich beam with anisotropic skins. In Section 3, the spectral density function of the stress response is calculated and later is used in the Monte Carlo simulation to generate the stress time history samples. In Section 4, a S–N curve equation incorporating the mean stress effect is proposed for the fatigue analysis of the beam under random loading. The prediction of this formula is compared with the experimental data and those of Goodman's equation and Gerber's equation. In Section 6, we present simulation results to examine the effect of material and geometric parameter uncertainties on the fatigue life of the sandwich beam.

2. Higher order sandwich beam

Let x, y and z be the co-ordinates along the length, width and thickness directions of the beam, respectively. The displacement in the x direction is given by U and in the z direction by W. Consider the mid-plane displacements expressed in terms of a finite set of orthogonal mode functions

$$U_{01}(x,t) = \sum_{i=1}^{N_{u1}} a_i(t)\phi_i(x), \quad W_{01}(x,t) = \sum_{i=1+N_{u1}}^{N_{u1}+N_{w1}} a_i(t)\phi_i(x),$$

$$U_{03}(x,t) = \sum_{i=1+N_{u1}+N_{w1}}^{N_{u1}+N_{w1}+N_{u3}} a_i(t)\phi_i(x), \quad W_{03}(x,t) = \sum_{i=1+N_{u1}+N_{w1}+N_{u3}}^{N_{u1}+N_{w1}+N_{u3}+N_{w3}} a_i(t)\phi_i(x), \tag{1}$$

where $\phi_i(x)$ are chosen to be a complete set of admissible functions of the system [11]. N_{u1} is the number of terms for the displacement $U_{01}(x,t)$. Other N's with different subscripts have a similar meaning. $N_a = N_{u1} + N_{w1} + N_{u3} + N_{w3}$ is the total number of undetermined expansion coefficients a_i. Subscript of $a_i(t)$ indicates its location in the vector $\mathbf{a} = \{a_1(t), a_2(t), \ldots, a_{N_a}(t)\}^{\mathrm{T}}$ of dimension $N_a \times 1$. Subscript of $\phi_i(x)$ corresponds to that of $a_i(t)$.

The displacement fields in the skins are expressed in terms of the mid-plane displacements according to the Euler beam theory

$$U_i(x,z,t) = U_{0i}(x,t) - z\frac{\partial W_i(x,t)}{\partial x}, \quad W_i(x,t) = W_{0i}(x,t), \quad i = 1,3. \tag{2}$$

The displacements of the core vary in the z direction as well as in the x direction and are assumed as cubic polynomials of the thickness coordinate z according to the higher order theory of sandwich panels [9]

$$U_2(x,z,t) = b_0(x,t) + b_1(x,t)z + b_2(x,t)z^2 + b_3(x,t)z^3,$$

$$W_2(x,z,t) = c_0(x,t) + c_1(x,t)z + c_2(x,t)z^2 + c_3(x,t)z^3. \tag{3}$$

Along the interfaces the above model should satisfy the displacement continuity conditions and the traction continuity conditions. A symbolic procedure can be applied to derive the functions b's and c's in terms of the mode functions in Eq. (1). The governing modal equations in matrix form for the sandwich beam can be derived by Lagrange's approach [11]

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{C}\dot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{f}, \tag{4}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix and $\mathbf{f}$ is the modal force vector.

3. Random stress response

Let $\mathbf{a} = \mathbf{U}\mathbf{y}$ where $\mathbf{U}$ is the eigenmatrix of the matrices $\mathbf{K}$ and $\mathbf{M}$. Eq. (4) can be transformed into

$$\ddot{\mathbf{y}} + \mathbf{U}^{\mathrm{T}}\mathbf{C}\mathbf{U}\dot{\mathbf{y}} + \boldsymbol{\Omega}\mathbf{y} = \mathbf{U}^{\mathrm{T}}\mathbf{f}(t), \tag{5}$$

where the superscript T denotes the transpose. The matrix $\mathbf{U}^{\mathrm{T}}\mathbf{C}\mathbf{U}$ will be diagonal if the system is classically damped [11]. Eq. (5) then becomes de-coupled. However, the matrix $\mathbf{U}^{\mathrm{T}}\mathbf{C}\mathbf{U}$ is generally not diagonal. In order to solve this non-classically damped system, an iterative approach developed in Ref. [12] can be used.

Assume now that the excitation $\mathbf{f}(t)$ is a stationary random process with a correlation matrix $\mathbf{R}_{ff}(\tau)$ and a power spectral density matrix $\Phi_{ff}(\omega)$. Let $\mathbf{H}_y(\omega)$ be the frequency response matrix of the transformed modal vector $\mathbf{y}(t)$. After the excitation acts on the system for a sufficiently long time, the random response $\mathbf{y}(t)$ becomes stationary with a power spectral density function given by

$$\Phi_{yy}(\omega) = \mathbf{H}_y\mathbf{U}^{\mathrm{T}}\Phi_{ff}\mathbf{U}\mathbf{H}_y^{\mathrm{H}}, \tag{6}$$

where the superscript H denotes Hermitian operation. The power spectral density matrix of the expansion coefficient vector $\mathbf{a}$ can be obtained as

$$\Phi_{aa}(\omega) = \mathbf{U}\mathbf{H}_y\mathbf{U}^{\mathrm{T}}\Phi_{ff}\mathbf{U}\mathbf{H}_y^{\mathrm{H}}\mathbf{U}^{\mathrm{T}}. \tag{7}$$

In general, the stress at a point on the structure can be written as a linear combination of the expansion coefficients $\sigma(t) = \mathbf{c}^{\mathrm{T}}\mathbf{a}(t)$, where the vector $\mathbf{c}$ depends on the co-ordinates of the point under consideration. Then, the power spectral density function of the stress is given by

$$\Phi_{\sigma\sigma}(\omega) = \mathbf{c}^{\mathrm{T}}\Phi_{aa}(\omega)\mathbf{c} = \mathbf{c}^{\mathrm{T}}\mathbf{U}\mathbf{H}_y\mathbf{U}^{\mathrm{T}}\Phi_{ff}\mathbf{U}\mathbf{H}_y^{\mathrm{H}}\mathbf{U}^{\mathrm{T}}\mathbf{c}. \tag{8}$$

Once $\Phi_{\sigma\sigma}(\omega)$ is obtained, we can apply the well-known simulation algorithm due to Shinozuka to simulate a large number of time history samples of the stress [13].

4. S–N curve for the foam core

The original S–N curve relates the fatigue life in cycles to the amplitude of a harmonic stress. To account for the effect of mean stress when the S–N curve is used for the fatigue analysis of metallic materials, it is common to resort to Goodman's equation, Gerber's equation and Soderbeg's equation [6]. Similar equations to Goodman's equation for sandwich foam core materials are still lacking because of limited fatigue experimental data available. In the following, we present an approach to modify the S–N curve that will account for the effect of mean stress on fatigue. This approach is also studied in Ref. [8] for glass fiber-reinforced composites.

Because the foam core of the sandwich is often failed in shear, we consider the following S–N equation:

$$\log N = c + m \log S, \tag{9}$$

where S is the maximum shear stress. N is the number of cycles to fatigue failure. c and m are determined using curve fitting methods from experiment data and are usually considered as material constants. However, c and m are generally functions of loadings as well. The limited experimental fatigue data of the foam core materials available in the literature [1] seem to suggest that c and m are functions of the stress ratio

$$R = \frac{S_{min}}{S_{max}} = \frac{S_m - S_{amp}}{S_m + S_{amp}}, \tag{10}$$

where S_m is the mean stress and S_{amp} is the amplitude. This assumption accounts for the dependence of fatigue life on the mean and amplitude of the stress. Since we have limited experimental fatigue data of the foam core, we start with the assumption that c and m are linear functions of R [8]:

$$c(R) = a + b \cdot R, \quad m(R) = d + f \cdot R, \tag{11}$$

where the coefficients a, b, d and f are determined by fitting with the experimental data.

Consider Divinycell H100 foam. The experimental data reported in Ref. [3] are used to determine the above coefficients leading to the S–N equation for the foam as

$$\log N = 2.9812 + 2.0471R - (8.3552 + 1.5886R)\log\frac{S}{\hat{S}}, \tag{12}$$

where $\hat{S}$ is the static shear strength. For Divinycell H100, $\hat{S} = 1.4$ MPa [3]. This model in Eq. (12) is valid only in a range of stress ratio: $-1 \leq R \leq 0.5$, because the experimental data fall into this range. Note also that $c(R)$ and $m(R)$ can be assumed to be non-linear functions of R provided that there are sufficient experimental data to support the assumption.

As a comparison, we present Goodman's and Gerber's equations that account for the effect of mean stress in the S–N curve model:

$$\text{Goodman's}: \quad \frac{S}{S_{eq}} + \frac{S_m}{S_f} = 1, \quad \text{Gerber's}: \quad \frac{S}{S_{eq}} + \left(\frac{S_m}{S_f}\right)^2 = 1, \tag{13}$$

where S_{eq} is the equivalent stress of fully reversed alternative loading. S_m is the mean stress. S_f is the fatigue strength of the material. Numerical examples are shown in Fig. 1 for $R = -0.5, 0.1, 0.25$ and 0.5. Gerber's predictions are conservative for all the R values considered.

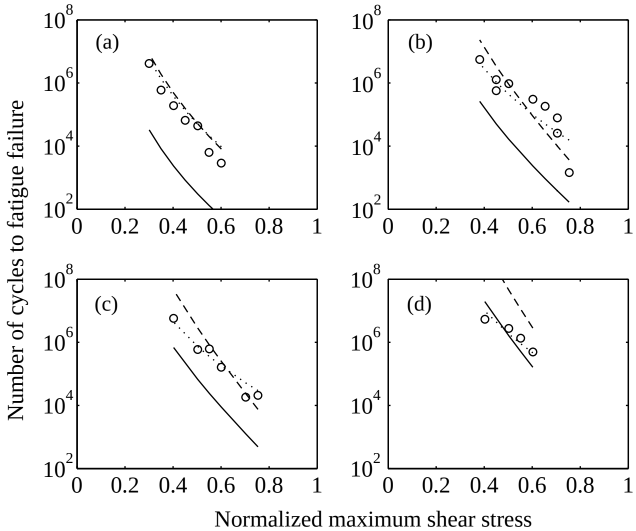

Fig. 1. Comparision between experimental data and the predictions of Goodman's equation, Gerber's equation and Eq. (12) with different stress ratios. ∘, experimental data; - - -, prediction of Goodman's equation; ——, prediction of Gerber's equation; ......, prediction of Eq. (12). (a) $R = -0.5$, (b) 0.1, (c) 0.25 and (d) 0.5.

Goodman's predictions become unconservative with the increase of the stress ratio R. The predictions based on Eq. (12) agree well with the experimental data for all the R values considered herein. The linear assumption of the dependence of c and m on R seems to be satisfactory for the given set of data for Divinycell H100. Eq. (12) is used in the following fatigue analysis of a Divinycell H100 foam core sandwich beam under random loadings.

5. Random fatigue

To estimate high cycle random fatigue life, we apply the Palmgren–Miner linear theory of damage accumulation [14]. Let S_i be an amplitude of a random stress cycle and R_i be the associated stress ratio. According to the S–N curve, an incremental fatigue damage due to this stress cycle is given by

$$\Delta D(S_i) = \frac{1}{N(S_i)}, \tag{14}$$

where $N(S_i)$ is determined from Eq. (12). Let M be the total number of random stress cycles in the entire simulation. Assume that the stress is stationary and ergodic. The accumulative damage due to M stress cycles is given by

$$\sum_{i=1}^{M} \Delta D(S_i) = \sum_{i=1}^{M} \frac{1}{N(S_i)}. \tag{15}$$

By definition, $(1/M)\sum_{i=1}^{M}(1/N(S_i))$ is a statistic estimate of $E[N(S)^{-1}]$. $E[\cdot]$ is the mathematical expectation. Therefore,

$$E[N(S)^{-1}] = \lim_{M\to\infty} \frac{1}{M}\sum_{i=1}^{M}\frac{1}{N(S_i)} \tag{16}$$

and the average damage rate per cycle is given by

$$\frac{\mathrm{d}D}{\mathrm{d}N} = \lim_{M\to\infty} \frac{1}{M}\sum_{i=1}^{M}\Delta D(S_i). \tag{17}$$

The Palmgren–Miner theory implies that the damage rate per cycle is constant. The fatigue in cycles N_f is such that

$$\frac{\mathrm{d}D}{\mathrm{d}N}\cdot N_f = 1, \tag{18}$$

i.e.,

$$N_f = E[N(S)^{-1}]^{-1}. \tag{19}$$

If we simulate the stress histories with a fixed duration in time, and the summation in Eqs. (16) and (17) is over all the stress samples, we would obtain the damage rate per unit time.

To evaluate $E[N(S)^{-1}]$, we generate many sufficiently long random samples of the stress response from Eq. (8) using the spectral method [13]. The rainflow counting scheme [15–17] is applied to obtain the values of the mean and amplitude of random stress cycles. Eq. (12) gives the fatigue life $N(S)$ for each random stress cycle. The average is taken to obtain $E[N(S)^{-1}]$ over the total number M of random stress cycles.

6. Numerical results

Extensive numerical simulations were carried out for the simply supported sandwich beam. Some of them are presented in this section. The skins of the beam are taken to be identical and anisotropic. The core material is Divinycell H100 foam. Material properties of the skins are taken from Ref. [3]: Elastic modulus: $E_1 = 20.1$ GPa, $E_2 = 5.0$ GPa, $E_3 = 9.6$ GPa; Poisson ratio: $v_{31} = 0.167$, $v_{21} = 0.075$, $v_{23} = 0.156$; Shear modulus: $G_{23} = 2.0$ GPa, $G_{13} = 2.0$ GPa, $G_{12} = 2.0$ GPa. The mass density of the skin material is $1760\,\mathrm{kg/m^3}$. The nominal dimension of the sandwich beam is length $l = 450$ mm, width $b_w = 50$ mm. The skins have the same thickness 5 mm and the core thickness is 50 mm.

Two independent white noise loadings with the same spectral density function equal to $0.06N^2$ s/rad act at the points $x_1 = 179$ mm and $x_2 = 261$ mm on the top skin. At the same time the beam is under a constant shear stress loading $\tau_m = 0.28$ MPa. This shear loading is to mimic the experimental condition reported in Ref. [3].

The fatigue life N is a random variable when the excitation is random. In all the simulations, 150 time histories of the stress with 2^{18} points in each record are used.

Recall that the objective of the paper is to study the effect of system uncertainties on the fatigue life. In the following, we present the sensitivities of the fatigue life with respect to several geometric and material constants.

6.1. Effect of core density and elastic modulus

It is well known that the density of foam core materials is quite variable from batch to batch. Within a foam core sheet, the density may even vary across the thickness. Typically, the lowest density is in the center of the sheet [1]. Here, we consider the density values with a $\pm 6\%$ variation around the nominal value $100\,\text{kg/m}^3$ and study the effect of the density uncertainty on the fatigue life. For Divinycell materials, the elastic modulus and tensile strength are often considered as linear functions of density. The following relation between the elastic modulus and the density were used [18]

$$E = -1.2902 \times 10^7 + 1.1336 \times 10^6 \rho, \tag{20}$$

where E is the elastic modulus and ρ is the density.

Fig. 2 shows that the fatigue life increases with the density and elastic modulus, and is much more sensitive to the positive deviation than the negative deviation. 6% positive deviation from the nominal value can cause 40% increase in the mean fatigue life.

6.2. Effect of geometry

We consider uncertainties associated with the thickness of the sandwich beam. The thicknesses of the core, the top bottom skins are assumed to vary $\pm 6\%$ around their nominal values. We study one uncertain parameter at a time while keeping the other two thicknesses fixed.

The results are shown in Fig. 2. It can be seen that changes in the core thickness cause larger variations in the fatigue life than the two skins. The fatigue life increases 30% when the core thickness increases 6% from the nominal value and it is quite insensitive to the variation of the skin thickness. Recall that we are studying the shear fatigue failure of the foam core due to random loadings.

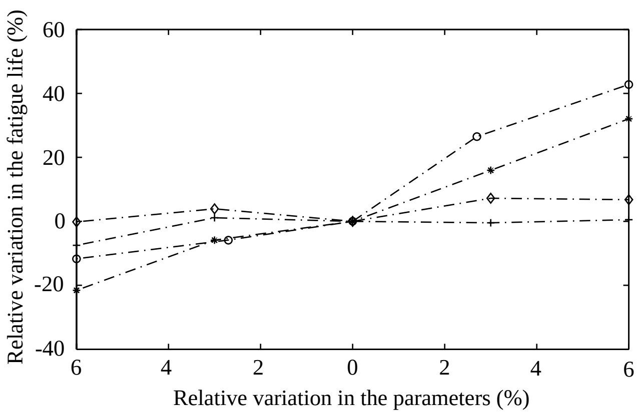

Fig. 2. Sensitivities of the fatigue life to parameter uncertainties: $\circ$, core density; $*$, core thickness; $\diamond$, bottom skin thickness; $+$, top skin thickness.

7. Concluding remarks

This paper presents a study of random fatigue analysis of a higher order sandwich beam. Sensitivities of the fatigue life to parameter uncertainties are investigated via Monte Carlo simulations. For the simply supported sandwich beam, we have found that an increase in the thickness and density of the foam core extends the fatigue life, while small variations in the skin thicknesses cause little change in the fatigue life.

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