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Letter to the Editor

On the “modes” of non-homogeneously damped rods consisting of two parts

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1. Introduction

Recently, an interesting study [1] was published in this journal in which the eigencharacteristics of a continuous beam model with damping, are determined using the separation of variables approach. The beam considered has different stiffness, damping and mass properties in each of its two parts. Motivated by this publication, the present paper deals with an axially vibrating rod consisting of two parts as a counterpart of that publication. Unlike the cited publication where overdamped and underdamped “modes” are investigated separately, here, both of them will be handled simultaneously, again via separation of variables approach. Probable applications of these systems include rods composed of two different cross-sections with different damping subject to impulsive axial forces in civil engineering applications. Such systems can also be encountered in oil well drilling practices.

2. Theory

Let it be assumed that an axially vibrating rod consists of two parts, one of length L_1 with stiffness k_1, viscous damping coefficient c_1, and mass per unit length m_1, and the second of length L_2, with corresponding parameters k_2, c_2 and m_2. These parameters are assumed to be constant along each rod segment, and contain contributions from the rod and any surrounding medium. Fig. 1 shows the rod diagrammatically.

Due to the presence of external viscous damping it is more appropriate to work with complex variables. It will be assumed that the axial displacements $w_i(x, t)$ $(i = 1, 2)$ of both parts of the rod are the real parts of some complex quantities denoted as $z_i(x,t)$. Keeping in mind that actually, one is interested only in the real parts of the expressions below, the equations of motion of the rod can

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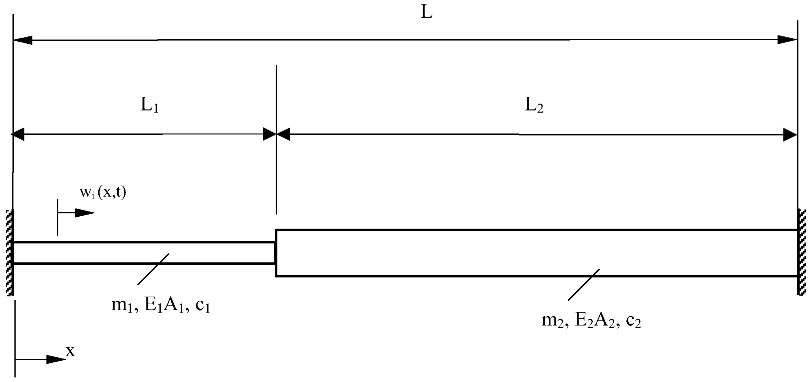

Fig. 1. Axially vibrating elastic rod consisting of two parts.

be written as

$$k_i z_i''(x,t) - m_i \ddot{z}_i(x,t) - c_i \dot{z}_i(x,t) = 0, \quad i = 1,2 \tag{1}$$

with $k_i = (EA)_i$, where $(EA)_i$ denotes the axial rigidity of the ith portion and x is the axial position along the rod. Dots and primes denote partial derivatives with respect to time t and position co-ordinate x.

The corresponding boundary conditions are

$$z_1(0,t) = 0, \quad z_2(L,t) = 0,$$

$$z_1(L_1,t) = z_2(L_1,t), \quad k_1 z_1'(L_1,t) = k_2 z_2'(L_1,t). \tag{2}$$

Let it be assumed that

$$z_i(x,t) = Z_i(x) D_i(t), \quad i = 1,2 \tag{3}$$

according to the separation of the variables approach, where both functions $Z_i(x)$ and $D_i(t)$ are complex functions in general. Substituting Eq. (3) into Eq. (1) gives

$$\frac{k_i}{m_i}\frac{Z_i''}{Z_i} = \frac{\ddot{D}_i + (c_i/m_i)\dot{D}_i}{D_i} := \kappa_i, \tag{4}$$

where the κ_i are complex constants to be determined. Here, primes and dots denote derivatives with respect to position x and time t. To satisfy the last two of the boundary conditions (2) these time functions must be equal, so that $D_1(t) = D_2(t) = D(t)$. Thus, the differential equations for $Z_i(x)$ may be written using Eq. (4) as follows:

$$Z_i''(x) - \frac{m_i}{k_i}\kappa_i Z_i(x) = 0, \quad i = 1,2. \tag{5}$$

The time function is assumed now as an exponential function

$$D(t) = e^{\lambda t}, \tag{6}$$

where λ represents an eigenvalue of the system which is complex in general. With this $D(t)$, the second equality in Eq. (4) gives

$$\kappa_i = \frac{c_i}{m_i}\lambda + \lambda^2, \quad i = 1,2. \tag{7}$$

With the abbreviation

$$\nu_i^2 = \frac{m_i}{k_i}\kappa_i, \quad i = 1, 2 \tag{8}$$

the first equation in Eq. (4) can be written as

$$Z_i''(x) - \nu_i^2 Z_i(x) = 0, \quad i = 1, 2. \tag{9}$$

The general solutions of the differential equations (9) can be expressed as

$$Z_i(x) = \bar{A}_i e^{\nu_i x} + \bar{B}_i e^{-\nu_i x}, \quad i = 1, 2, \tag{10}$$

where $\bar{A}_i$ and $\bar{B}_i$ denote complex constants to be determined. In terms of the $Z_i(x)$, the boundary conditions in Eq. (2) can be formulated as

$$Z_1(0) = Z_2(L) = 0, \quad Z_1(L_1) = Z_2(L_1), \quad k_1 Z_1'(L_1) = k_2 Z_2'(L_1). \tag{11}$$

The substitution of expressions (10) into Eq. (11) yields the following set of four homogeneous equations for the determination of $\bar{A}_i$ and $\bar{B}_i$:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & e^{\nu_2 L} & e^{-\nu_2 L} \\ e^{\nu_1 L_1} & e^{-\nu_1 L_1} & -e^{\nu_2 L_1} & -e^{-\nu_2 L_1} \\ k_1\nu_1 e^{\nu_1 L_1} & -k_1\nu_1 e^{-\nu_1 L_1} & -k_2\nu_2 e^{\nu_2 L_1} & k_2\nu_2 e^{-\nu_2 L_1} \end{bmatrix} \begin{bmatrix} \bar{A}_1 \\ \bar{B}_1 \\ \bar{A}_2 \\ \bar{B}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{12}$$

Let the matrix of the coefficients be denoted by $\mathbf{A}$. For a non-trivial solution, the determinant of the matrix $\mathbf{A}$ should be zero

$$\det \mathbf{A}(\nu_1, \nu_2) = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & e^{\nu_2 L} & e^{-\nu_2 L} \\ e^{\nu_1 L_1} & e^{-\nu_1 L_1} & -e^{\nu_2 L_1} & -e^{-\nu_2 L_1} \\ k_1\nu_1 e^{\nu_1 L_1} & -k_1\nu_1 e^{-\nu_1 L_1} & -k_2\nu_2 e^{\nu_2 L_1} & k_2\nu_2 e^{-\nu_2 L_1} \end{vmatrix} = 0. \tag{13}$$

Though not necessary, after some rearrangements, the evaluation of the determinant above yields

$$\begin{aligned} \det \mathbf{A}(\nu_1, \nu_2) = {} & (k_2\nu_2 - k_1\nu_1)\,[e^{\nu_2 L} e^{-(\nu_1+\nu_2)L_1} - e^{-\nu_2 L} e^{(\nu_1+\nu_2)L_1}] \\ & - (k_2\nu_2 - k_1\nu_1)[e^{\nu_2 L} e^{(\nu_1-\nu_2)L_1} - e^{-\nu_2 L} e^{-(\nu_1-\nu_2)L_1}] = 0. \end{aligned} \tag{14}$$

Using the definitions given by Eqs. (7) and (8) ν_1 and ν_2 can be expressed as functions of the eigenvalue λ:

$$\nu_1(\lambda) = \pm\sqrt{\frac{m_1}{k_1}\left[\left(\frac{c_1}{m_1}\right)\lambda + \lambda^2\right]}, \quad \nu_2(\lambda) = \pm\sqrt{\frac{m_2}{k_2}\left[\left(\frac{c_2}{m_2}\right)\lambda + \lambda^2\right]}. \tag{15}$$

Hence Eq. (14) becomes

$$\det \mathbf{A}(\nu_1(\lambda), \nu_2(\lambda)) = \det \mathbf{A}(\lambda) = 0 \tag{16}$$

from which λ can be obtained, which is a complex number in general. Now via Eq. (15) ν_1 and ν_2 can be obtained. If these ν_1 and ν_2 are substituted into the coefficients matrix $\mathbf{A}$ in Eq. (12) the unknowns $\bar{A}_i$ and $\bar{B}_i$ $(i = 1, 2)$ can be determined up to an arbitrary constant. Hence, $Z_i(x)$ in Eq. (10) are obtained.

Table 1
Physical parameters for the numerical example

	Rod 1	Rod 2	
		Case 1	Case 2
L_i (m)	1	2	2
m_i (kg/m)	10	20	40
c_i (kg/ms)	0	100	600
E_iA_i (N)	100	100	100

Table 2
Lower eigenvalues of the numerical example

Continuous model	Finite element model
Case 1	
$-2.34674 \pm 0.82665i$	$-2.34674 \pm 0.82665i$
$-2.06672 \pm 4.90241i$	$-2.06672 \pm 4.90242i$
$-1.61100 \pm 7.83051i$	$-1.61102 \pm 7.83053i$
$-1.63223 \pm 9.92247i$	$-1.63234 \pm 9.92276i$
$-2.02490 \pm 12.60870i$	$-2.02506 \pm 12.60940i$
$-1.99895 \pm 15.55919i$	$-1.99931 \pm 15.56080i$
$-1.67162 \pm 18.27967i$	$-1.67314 \pm 18.28370i$
$-1.70472 \pm 20.42707i$	$-1.70587 \pm 20.43730i$
Case 2	
-0.22170	-0.22170
-1.17668	-1.17668
-3.71253	-3.71254
$-7.17204 \pm 4.99519i$	$-7.17164 \pm 4.99531i$
$-7.12849 \pm 8.66093i$	$-7.13036 \pm 8.66532i$
$-0.99800 \pm 9.26632i$	$-0.99842 \pm 9.26683i$
-10.78322	-10.78240
-13.49824	-13.49860
$-7.12529 \pm 11.71717i$	$-7.11971 \pm 11.68780i$
-14.65507	-14.65480
$-7.11832 \pm 14.53285i$	$-7.21323 \pm 14.92130i$

Returning to Eq. (3) considering Eq. (10) and introducing

$$\lambda = \lambda_{re} + j\lambda_{im}, \quad v_i = v_{i_{re}} + jv_{i_{im}},$$

$$\bar{A}_i = \bar{A}_{i_{re}} + j\bar{A}_{i_{im}}, \quad \bar{B}_i = \bar{B}_{i_{re}} + j\bar{B}_{i_{im}} \quad (j = \sqrt{-1}) \tag{17}$$

the axial displacements of the two rod portions $w_i(x, t)$ are determined, after lengthy calculations as

$$w_i(x, t) = \mathrm{Re}[z_i(x, t)] = e^{\lambda_{re}t} S_i(x) \cos \lambda_{im} t - e^{\lambda_{re}t} Q_i(x) \sin \lambda_{im} t, \tag{18}$$

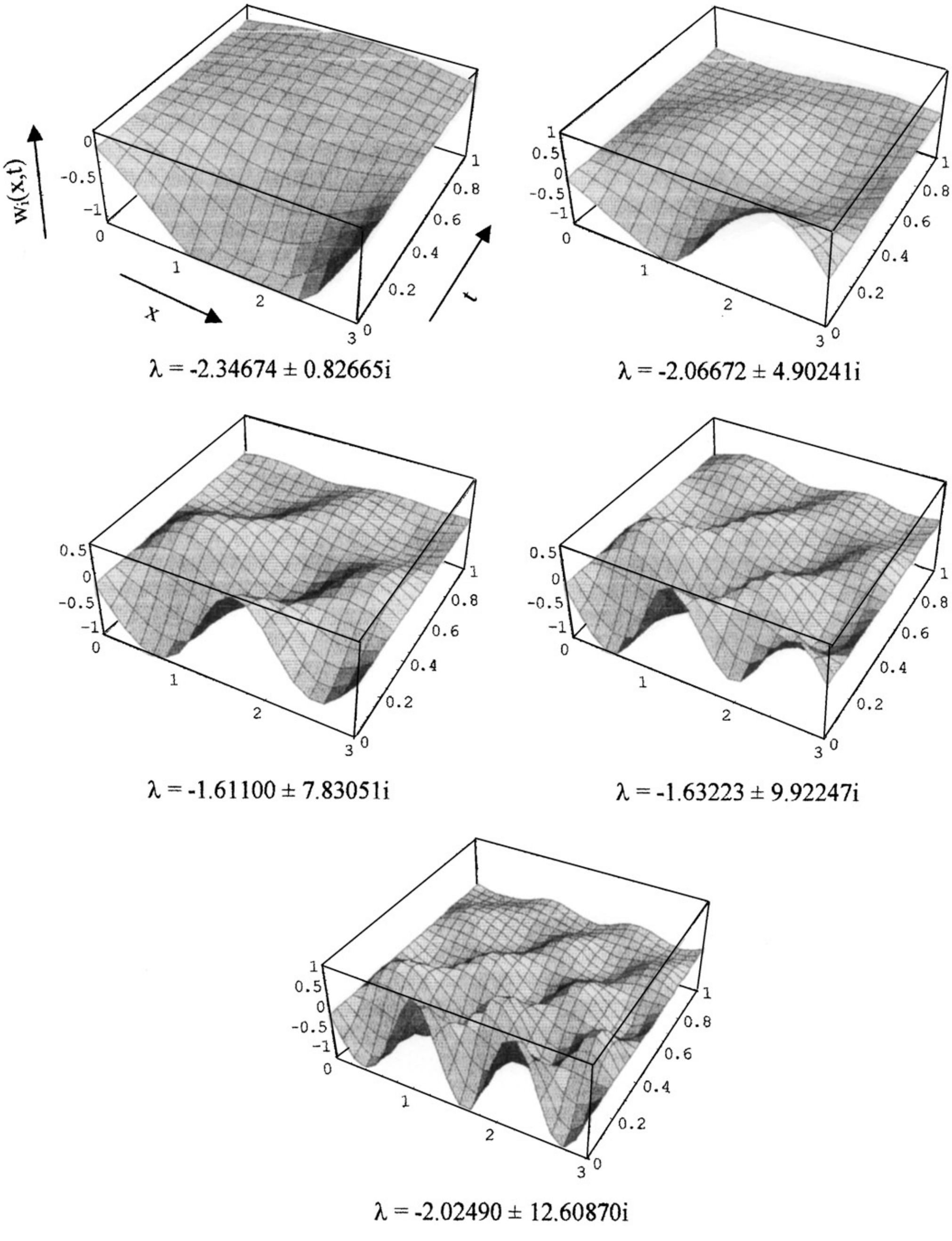

Fig. 2. Three-dimensional plots of $w_i(x, t)$—surfaces corresponding to the first five underdamped eigenvalues for case 1.

where the following abbreviations are introduced:

$$S_i(x) = \mathrm{e}^{v_{i_{re}}x}(\bar{A}_{i_{re}} \cos v_{i_{im}}x - \bar{A}_{i_{im}} \sin v_{i_{im}}x) + \mathrm{e}^{-v_{i_{re}}x}(\bar{B}_{i_{re}} \cos v_{i_{im}}x + \bar{B}_{i_{im}} \sin v_{i_{im}}x),$$

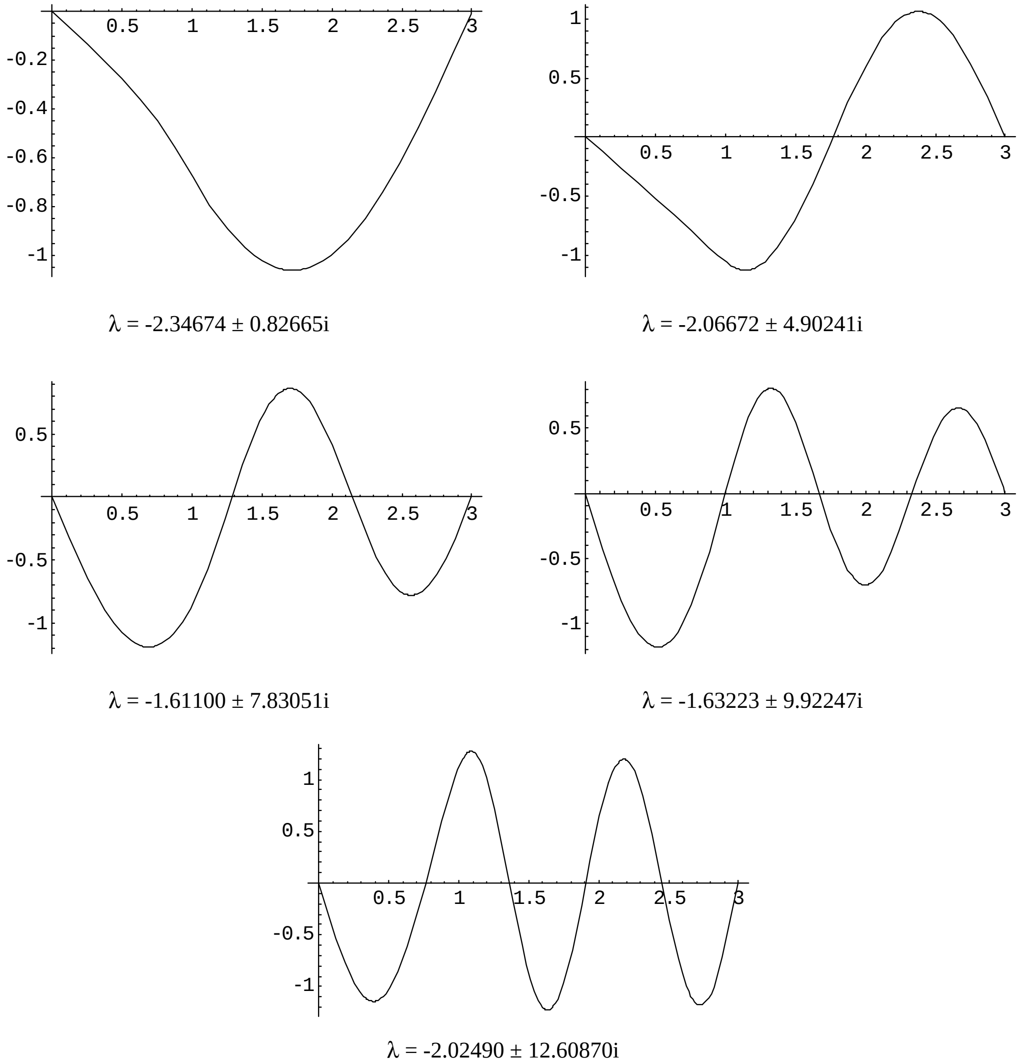

Fig. 3. Intersection curves of the surfaces in Fig. 2 with $t = 0$ plane.

$$
\begin{aligned}
Q_i(x) = {} & \mathrm{e}^{v_{ire}x}(\bar{A}_{i_{re}} \sin v_{i_{im}}x + \bar{A}_{i_{im}} \cos v_{i_{im}}x) \\
& + \mathrm{e}^{-v_{ire}x}(\bar{B}_{i_{im}} \cos v_{i_{im}}x - \bar{B}_{i_{re}} \sin v_{i_{im}}x).
\end{aligned} \tag{19}
$$

The expressions of the axial displacements can be put in a more compact form, as

$$
w_i(x, t) = \mathrm{e}^{\lambda_{re}t} C_i(x) \cos(\lambda_{im}t - \varepsilon_i(x)) \tag{20}
$$

with

$$
\begin{aligned}
C_i(x) &= \sqrt{S_i^2(x) + Q_i^2(x)}, \\
\tan \varepsilon_i(x) &= -\frac{Q_i(x)}{S_i(x)}.
\end{aligned} \tag{21}
$$

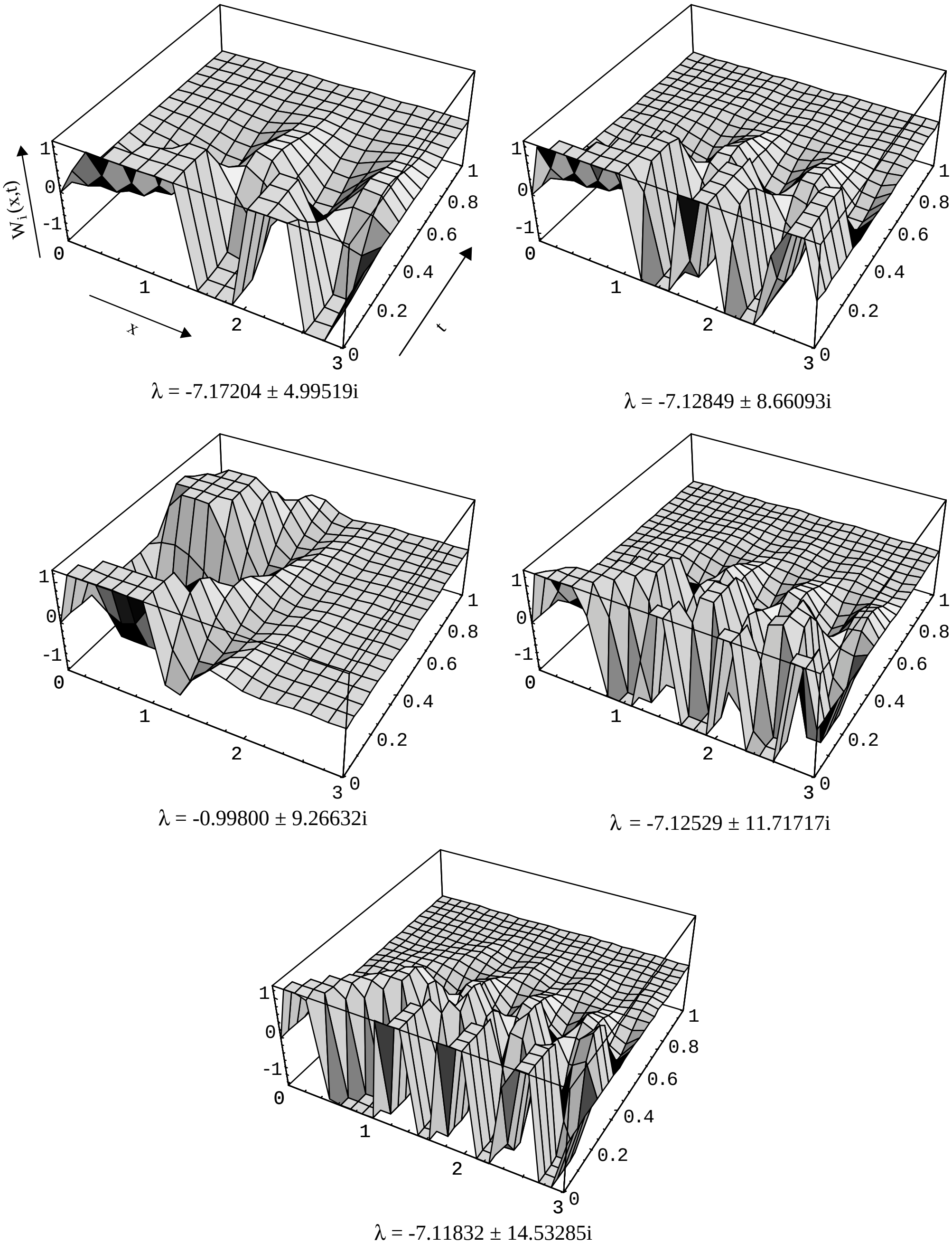

Fig. 4. Three-dimensional plots of $w_i(x, t)$—surfaces corresponding to the first five underdamped eigenvalues for case 2.

$w_i(x, t)$, $i = 1, 2$ determine the axial displacement distribution over the length of the viscously damped rod when it vibrates at an eigenvalue λ. Due to the apparent presence of a phase, which is a function of the position co-ordinate x, the authors preferred to use the expression "mode" or "eigenfunction" as seldom as possible. Whenever necessary, those words were used in quotation marks.

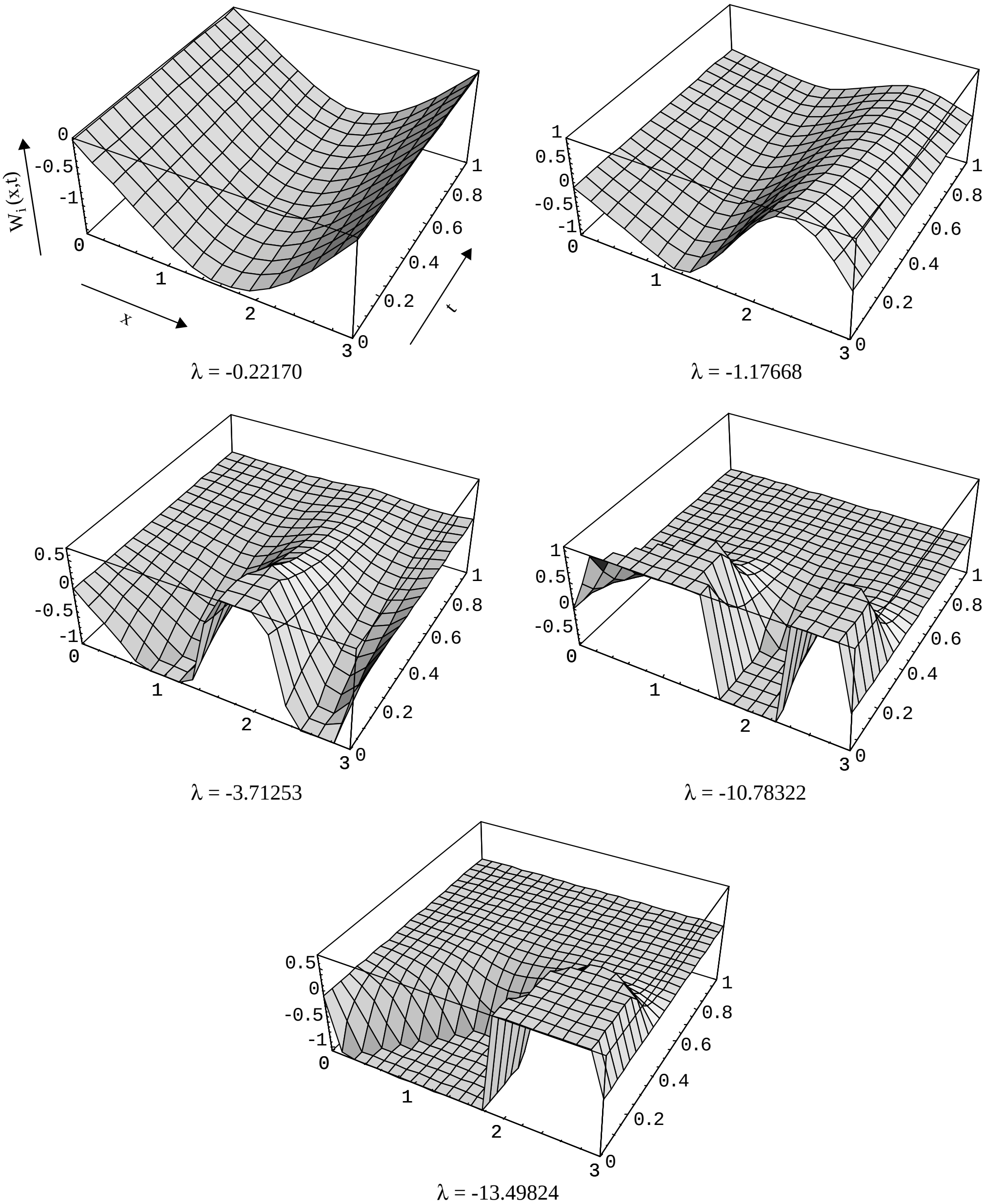

Fig. 5. Three-dimensional plots of $w_i(x, t)$—surfaces corresponding to the first five overdamped eigenvalues for case 2.

3. Numerical evaluations

This section is devoted to the numerical evaluation of the expressions obtained. The computation will be demonstrated using a rod with the parameters given in Table 1. The eigenvalues and "eigenfunctions" are computed using the procedure outlined above, and also

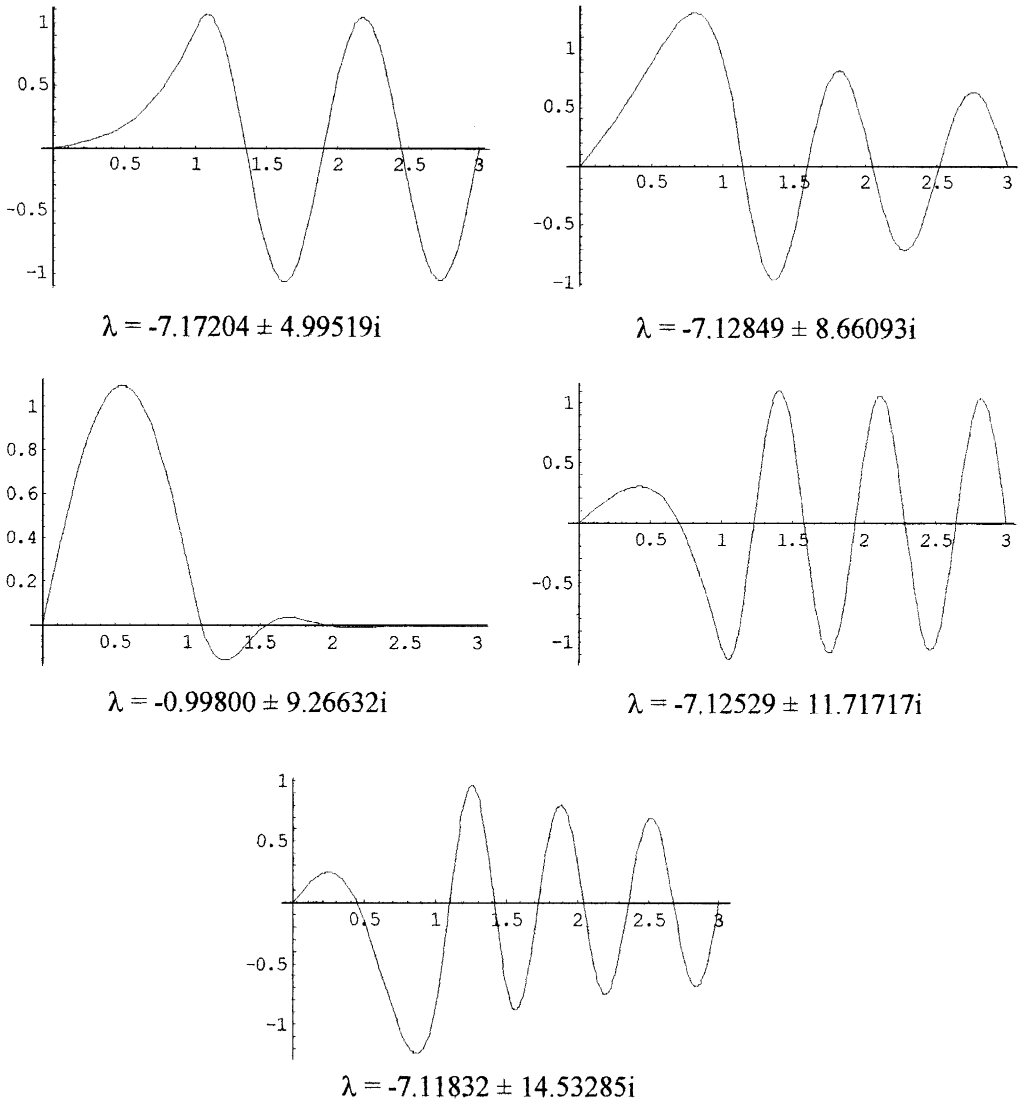

Fig. 6. Intersection curves of the surfaces in Fig. 4 with $t = 0$ plane.

using a finite element model for comparison. The finite element model has 45 elements of equal length, giving a total of 90 degrees of freedom.

As in Ref. [1], two cases are considered: the first represents a lightly damped case, and the second the situation when a more viscous medium surrounds the second part of the rod. Table 2 reflects the results for the lower "modes" for both cases. Case 2 has a large number of overdamped "modes", whereas all the "modes" in case 1 are underdamped.

Fig. 2 shows the three dimensional plots of $w_i(x, t)$ for the first five eigenvalues λ_1 to λ_5. In Fig. 3, as representatives of the damped "eigenbehavior", those curves are plotted which result from the intersection of the surfaces above with the $t = 0$ plane. These curves could

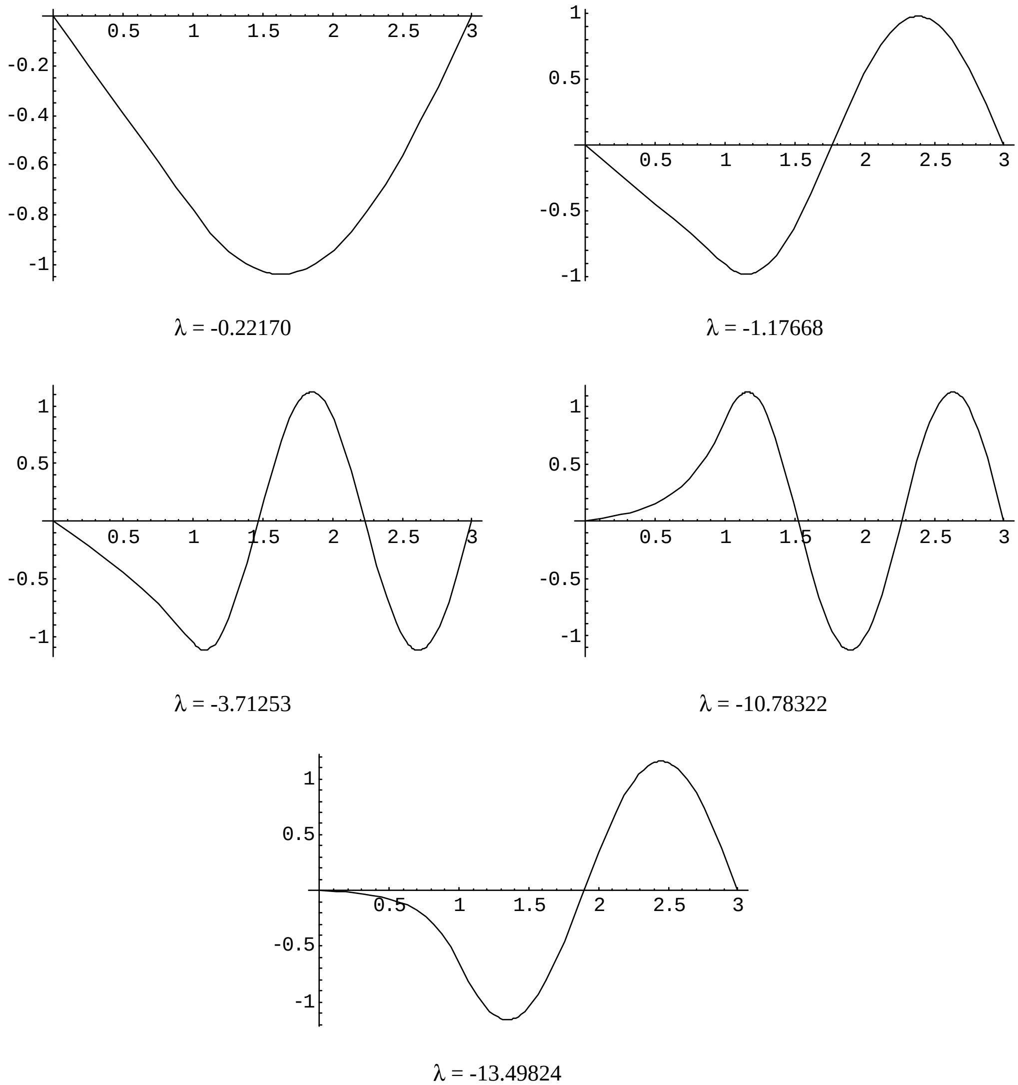

Fig. 7. Intersection curves of the surfaces in Fig. 6 with $t = 0$ plane.

be viewed as eigenfunctions. It is seen from Fig. 2 that the amplitudes of the intersection curves corresponding to increasing t values decrease in accordance with the damped character of the system.

Figs. 4 and 5 show the first five $w_i(x, t)$—surfaces corresponding to the first five underdamped and overdamped eigenvalues λ for case 2, respectively. Figs. 6 and 7 represent the corresponding curves resulting from the intersection with $t = 0$ plane.

The curves in Figs. 3, 6 and 7 are similar to those given in Ref. [1]. The observation in Ref. [1] can also be made here, namely that the majority of the displacements in the lower "modes" is local to the underdamped part of the rod.

4. Conclusions

This study is concerned with the establishment of a method to compute the eigenvalues and eigenfunctions of a continuous, viscously damped rod, consisting of two parts having different physical parameters.

Both the overdamped and the underdamped eigenvalues and corresponding eigenfunctions have been computed for two different sets of parameters. For high damping the lower underdamped modes seem to be local to the undamped part of the rod.

References

[1] M.I. Friswell, A.W. Lees, The modes of non-homogeneous damped beams, Journal of Sound and Vibration 242 (2001) 355–361.