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Quantification of non-viscous damping in discrete linear systems

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Abstract

The damping forces in a multiple-degree-of-freedom engineering dynamic system may not be accurately described by the familiar 'viscous damping model'. The purpose of this paper is to develop indices to quantify the extent of any departures from this model, in other words the amount of 'non-viscosity' of damping in discrete linear systems. Four indices are proposed. Two of these indices are based on the non-viscous damping matrix of the system. A third index is based on the residue matrices of the system transfer functions and the fourth is based on the (measured) complex modes of the system. The performance of the proposed indices is examined by considering numerical examples.

1. Introduction

The true nature of the damping forces in a dynamic system is often not known with any great accuracy. The most common approach in vibration modelling is to assume a model in which it is supposed that the instantaneous generalized velocities are the only relevant state variables that determine damping. This approach was first introduced by Rayleigh [1] via his famous 'dissipation function', a quadratic expression for the energy dissipation rate with a symmetric matrix of coefficients, the 'damping matrix'. The equations of motion describing the free vibration of such a linear system with N degrees of freedom can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}. \tag{1}$$

Here $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the $N \times N$ mass, damping and stiffness matrices respectively, and $\mathbf{q}(t)$ is the vector of the generalized co-ordinates (A list of nomenclature is given). The dynamics of such

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systems have been extensively studied, and are quite well understood. This damping model, which we will call 'viscous damping', is generally the only damping model allowed in commercial finite element (FE) codes [2], and it is also the only damping model usually taken into account in experimental modal analysis (EMA) [3]. In fact, in most EMA and FE methods a further idealization of viscous damping is used, known as 'proportional damping' or 'classical damping'. This simplification, also pointed out by Rayleigh, allows the damping matrix to be diagonalized simultaneously with the mass and stiffness matrices, preserving the simplicity of real normal modes as in the undamped case.

It is well recognized that proportional damping is very rarely a physically realistic model because practical experience in modal testing shows that most real-life structures possess complex modes instead of real normal modes. Complex modes can arise from viscous damping, provided it is non-proportional. However, the physical justification for viscous damping is hardly more convincing than that for proportional damping. Any causal model which makes the energy dissipation functional non-negative is a possible candidate for a damping model. Such damping models, in which the damping force depends on anything other than the *instantaneous* generalized velocities, will be called in this paper 'non-viscous' damping models. Recently in a series of papers Adhikari and Woodhouse [4–7] have considered the problem of identification of viscous and non-viscous damping from vibration measurements. In these studies, attention was focused on the following questions:

(1) From experimentally determined complex modes can one identify the underlying damping mechanism? Is it viscous or non-viscous? Can the correct model parameters be found experimentally?
(2) Is it possible to establish experimentally the *spatial distribution* of damping?
(3) Is it possible that more than one damping model with corresponding 'correct' sets of parameters may represent the system response equally well, so that the identified model becomes non-unique?
(4) Does the selection of damping model matter from an engineering point of view? Which aspects of behaviour are wrongly predicted by an incorrect damping model?

These questions highlight the distinction between viscous and non-viscous damping models, simply because most vibration analysis textbooks and computer packages only allow viscous damping, and the aim here is to address the question of whether this restriction matters in practice.

2. Linear models of non-viscous damping

Damping models in which the dissipative forces depend on any quantity other than the instantaneous generalized velocities, then will be called non-viscous damping models. Clearly a wide range of choice is possible. The discussion in this paper is confined to linear systems only, and the most general way to model damping within the linear range is through the class of damping models which depend on the past history of motion via convolution integrals over suitable kernel functions or Green functions. The equations of motion of a discrete system with such a damping model can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \int_0^t \mathcal{G}(t-\tau)\dot{\mathbf{q}}(\tau)\mathrm{d}\tau + \mathbf{K}\mathbf{q}(t) = \mathbf{0}. \tag{2}$$

Here $\mathcal{G}(t) \in \mathbb{R}^{N\times N}$ is a symmetric matrix of the damping kernel functions, $\mathcal{G}_{jk}(t)$. The kernel functions, or others closely related to them, are described under many different names in the literature of different subjects: for example, retardation functions, heredity functions, after-effect functions and relaxation functions. Golla and Hughes [8], and McTavish and Hughes [9] have used damping models of form (2) in the context of viscoelastic structures. The damping kernel functions are commonly defined in the frequency/Laplace domain. Taking the Laplace transform of Eq. (2) and assuming zero initial conditions one obtains

$$\mathbf{D}(s)\mathbf{q} = \mathbf{0}, \tag{3}$$

where

$$\mathbf{D}(s) = s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K}, \tag{4}$$

is the dynamic stiffness matrix and $\mathbf{G}(s)$ is the Laplace transform of $\mathcal{G}(t)$. The elements of $\mathbf{G}(s)$ could in principle have any mathematical form as long as they represent a causal dissipative function.

By choosing specific forms of $\mathbf{G}(s)$, a wide variety of particular linear damping models can be obtained as special cases of this general non-viscous model. Some examples are as follows:

(1) *Viscous damping model:* by choosing $\mathbf{G}(s) = \mathbf{C}$, $\forall s$, Eq. (2) reduces to the case of viscous damping as in Eq. (1).

(2) *Exponential damping model:* this model was introduced by Biot [10] and can be obtained by choosing

$$\mathbf{G}(s) = \sum_{j=1}^{n} \frac{\mu_j}{s+\mu_j}\mathbf{C}_j. \tag{5}$$

Here μ_j are known as relaxation parameters and $\mathbf{C}_j$ are associated damping coefficient matrices. It is often argued that this is the physically most realistic non-viscous damping model [11]. When $\mu_j \to \infty$, $\forall j$ this model reduces to the case of viscous damping.

(3) *Fractional derivative damping model:* Bagley and Torvik [12], Torvik and Bagley [13], Gaul et al. [14] and Maia et al. [15] have considered damping modelling in terms of fractional derivatives of the displacements. By choosing

$$s\mathbf{G}(s) = \sum_j s^{\nu_j}\mathbf{g}_j, \tag{6}$$

where $\mathbf{g}_j$ are complex constant matrices and ν_j are fractional powers, Eq. (2) gives this fractional derivative model. The familiar viscous damping appears as a special case when $\nu_j = 1$. The review papers by Slater et al. [16], Rossikhin and Shitikova [17] and Gaul [18] give further discussions on this topic.

It is clear that a wide variety of linear non-viscous damping models can be represented by the convolution integral approach. For this reason in this paper systems of the form (2) are considered as a basis to examine the distinction between viscous and non-viscous damping models.

The specific purpose of the present study is to quantify the amount of 'non-viscosity' of damping present in a system. The amount of non-viscosity of damping, in other words the extent of departure from the usual viscous model, is important because, as already explained, most vibration analysis and simulation methods assume viscous damping, and one may want to know how accurate such a method can be expected to be. In Section 3, a brief analytical background on non-viscously damped systems is given. Non-viscosity indices are discussed in Section 4. In Section 4.1 two indices of non-viscosity, based on the first moment and the Laplace transform of the non-viscous damping matrix $\mathcal{G}(t)$, are proposed. An index of non-viscosity based on the residues of the transfer function matrix is developed in Section 4.2. Section 4.3 develops an index based on only the measured complex modes. The relative advantages and disadvantages of the four proposed indices and the situations when each of them is likely to be applicable are discussed in Section 4.4. In Section 5, application of the proposed indices is illustrated through numerical examples. The behaviour of the proposed indices in the context of errors that arise by making a viscous damping assumption for a non-viscously damped system is discussed in Section 6 and some conclusions are drawn in Section 7.

3. Analytical background

Woodhouse [19] and Adhikari [20,21] have shown that conventional modal analysis can be extended to non-viscously damped systems of form (2). The eigenvalue problem associated with this equation can be defined by taking the Laplace transform

$$\lambda_k^2 \mathbf{M}\mathbf{z}_k + \lambda_k \mathbf{G}(\lambda_k)\mathbf{z}_k + \mathbf{K}\mathbf{z}_k = \mathbf{0}, \tag{7}$$

where $\mathbf{G}(s)$ is the Laplace transform of $\mathcal{G}(t)$. The eigenvalue problem of form (7) has been discussed by Adhikari [20,21]. The eigenvalues, λ_k, associated with Eq. (7) are roots of the characteristic equation

$$\det[\mathbf{D}(s)] = 0. \tag{8}$$

Because $\mathcal{G}(t)$ is real, $\mathbf{G}^*(s) = \mathbf{G}(s^*) \forall s$. Upon using this and taking the complex conjugate of (8) it is clear that

$$\det\left[{s^*}^2\mathbf{M} + s^*\mathbf{G}(s^*) + \mathbf{K}\right] = 0. \tag{9}$$

This implies that if s satisfies Eq. (8) then so does s^*. Thus, the eigenvalues of non-viscously damped systems either appear in complex conjugate pairs or become purely real. From Eq. (7) it is easy to observe that, when λ_k appear in complex conjugate pairs $\mathbf{z}_k$ also appear in complex conjugate pairs, and when λ_k is real $\mathbf{z}_k$ can also be taken to be real. Suppose the order of the characteristic Eq. (8) is m. In general m is more than $2N$, that is $m = 2N + p$; $p \geqslant 0$. Thus, although the system has N degrees of freedom (d.o.f.), the number of eigenvalues is more than $2N$. This is a major difference between non-viscously damped systems and viscously damped systems where the number of eigenvalues is exactly $2N$, including any multiplicities.

It may be noted that the phenomenon of 'proportional damping', much discussed for viscously damped systems, can also appear in non-viscously damped systems. Adhikari [22] has shown that system (2) can be diagonalized by undamped modes if and only if any one of the following conditions is satisfied:

$$\text{(a)}\ \mathbf{K}\mathbf{M}^{-1}\mathcal{G}(t) = \mathcal{G}(t)\mathbf{M}^{-1}\mathbf{K},$$

$$\text{(b)}\ \mathbf{M}\mathbf{K}^{-1}\mathcal{G}(t) = \mathcal{G}(t)\mathbf{K}^{-1}\mathbf{M},$$

$$\text{(c)}\ \mathbf{M}\mathcal{G}^{-1}(t)\mathbf{K} = \mathbf{K}\mathcal{G}^{-1}(t)\mathbf{M}. \tag{10}$$

In this paper the damping is assumed to be non-proportional, that is, the system matrices in general do not satisfy any of the above relationships.

4. Non-viscosity indices

4.1. Indices based on the non-viscous damping matrix

In this section two indices of non-viscosity will be developed. It is assumed that the non-viscous damping matrix $\mathcal{G}(t)$ is available beforehand. Thus, the indices to be developed here are best suited for analytical applications.

4.1.1. Index based on the first moment of the non-viscous damping matrix

It was mentioned that when

$$\mathcal{G}(t) = \mathbf{C}\delta(t), \tag{11}$$

the non-viscously damped system (2) reduces to the viscously damped system (1). The first two moments of $\mathcal{G}(t)$ given by Eq. (11) are

$$\mathcal{M}_0 = \int_0^\infty \mathcal{G}(t)\,\mathrm{d}t = \mathbf{C}, \tag{12}$$

$$\text{and} \quad \mathcal{M}_1 = \int_0^\infty t\mathcal{G}(t)\,\mathrm{d}t = \mathbf{O}, \tag{13}$$

where $\mathbf{O}$ is a $N \times N$ null matrix. It is clear that the first moment $\mathcal{M}_1$ will not be a null matrix if $\mathcal{G}(t)$ in Eq. (11) is not expressed in terms of the delta function. Thus $\mathcal{M}_1$ can be used to quantify the amount of non-viscosity of the damping. This idea was first introduced by Adhikari and Woodhouse [4] for the special case when

$$\mathcal{G}(t) = \mathbf{C}g(t), \tag{14}$$

where $g(t)$ is some (scalar) non-viscous damping function. The degree of non-viscosity was quantified by means of a characteristic time constant defined via the first moment of $g(t)$. Here this idea is extended to a more general case when $\mathcal{G}(t)$ is not necessarily restricted in the form of Eq. (14).

Define a matrix

$$\boldsymbol{\Theta}_1 = \mathcal{M}_0^{-1}\mathcal{M}_1. \tag{15}$$

Clearly, for viscously damped systems $\boldsymbol{\Theta}_1$ will be a null matrix. Thus, the value of $\boldsymbol{\Theta}_1$ can be used to quantify the amount of non-viscosity of damping, by considering a suitable norm of $\boldsymbol{\Theta}_1$. A good choice seems to be the so-called l_2 matrix norm, denoted by $\|\bullet\|$. Further note that $\boldsymbol{\Theta}_1$ has the dimension of time. To express the index in a non-dimensional form, it can be normalized by the minimum time period of the undamped system, denoted by T_{min}. Thus the first index of non-viscosity, γ_1, is defined as

$$\gamma_1 = \frac{\|\boldsymbol{\Theta}_1\|}{T_{min}} = \frac{\|\mathcal{M}_0^{-1}\mathcal{M}_1\|}{T_{min}}. \tag{16}$$

4.1.2. Index based on the Laplace transform of the non-viscous damping matrix

The Laplace transform of $\mathcal{G}(t)$, denoted by $\mathbf{G}(s)$, is defined as

$$\mathbf{G}(s) = \int_0^\infty \mathrm{e}^{-st}\mathcal{G}(t)\,\mathrm{d}t \tag{17}$$

for any $s \in \mathbb{C}$. Expanding e^{-st}, Eq. (17) gives

$$\mathbf{G}(s) = \int_0^\infty \left[1 - st + \frac{s^2t^2}{2!} - \frac{s^3t^3}{3!} + \cdots\right]\mathcal{G}(t)\,\mathrm{d}t = \mathcal{M}_0 - s\mathcal{M}_1 + \frac{s^2\mathcal{M}_2}{2!} - \frac{s^3\mathcal{M}_3}{3!} + \cdots, \tag{18}$$

where

$$\mathcal{M}_r = \int_0^\infty t^r\mathcal{G}(t)\,\mathrm{d}t, \tag{19}$$

is the rth moment of the non-viscous damping matrix $\mathcal{G}(t)$. Now, premultiplying Eq. (18) by $\mathcal{M}_0^{-1}$ and subtracting the result from an $N \times N$ identity matrix gives

$$\mathbf{I} - \mathcal{M}_0^{-1}\mathbf{G}(s) = s\mathcal{M}_0^{-1}\mathcal{M}_1 - \mathcal{M}_0^{-1}\left(\frac{s^2\mathcal{M}_2}{2!} - \frac{s^3\mathcal{M}_3}{3!} + \cdots\right). \tag{20}$$

For $s = 1$, the above expression reduces to

$$\mathbf{I} - \mathcal{M}_0^{-1}\mathbf{G}(1) = \mathcal{M}_0^{-1}\mathcal{M}_1 - \mathcal{M}_0^{-1}\mathcal{M}_2/2! + \mathcal{M}_0^{-1}\mathcal{M}_3/3! - \cdots. \tag{21}$$

The first term on the right-hand is the same as $\boldsymbol{\Theta}_1$ given by Eq. (15). The higher order terms appearing on the right-hand of Eq. (21) include the effect of the higher order moments of $\mathcal{G}(t)$. From Eq. (11), it is easy to observe that for viscous damping all the higher order moments of $\mathcal{G}(t)$ would be null matrices. Thus, the expression given in Eq. (21) can be used to quantify the amount of non-viscosity of damping. Upon taking the l_2 matrix norm of Eq. (21) and from Eq. (18) noting that

$$\mathbf{G}(0) = \mathcal{M}_0, \tag{22}$$

the second index of non-viscosity of damping is defined as

$$\gamma_2 = \frac{\|\mathbf{I} - \mathbf{G}(0)^{-1}\mathbf{G}(1)\|}{T_{min}}. \tag{23}$$

To use this equation it is required to evaluate $\mathbf{G}(s)$ at $s = 0$. Recently, Adhikari [23] has proposed a method from which $\mathbf{G}(0)$ can be evaluated from experimentally measured transfer functions. However, currently there is no method to experimentally obtain $\mathbf{G}(1)$. Thus, the index of non-viscosity given by Eq. (23) is best suited for analytical work only. An index which is more suitable for experimental analysis is proposed in the next section.

4.2. Index based on transfer function residues

So far, no assumption regarding the functional form of $\mathbf{G}(s)$ has been made. The next proposed index depends for its strict application on a particular assumption about the mathematical behaviour, namely that $G_{jk}(s)$ is analytic except at isolated poles, and also that the elements of $\mathbf{G}(s)$ remain finite as $s \to \infty$. This assumption in turn implies that the elements of $\mathbf{G}(s)$ are at most of order $1/s$ in s or constant, as in the case of viscous damping. Table 1 shows some non-viscous damping functions which have been used in the literature. Observe that the first five damping functions shown in the table satisfy the condition just described. All these models are basically variants of the general exponential model (relaxation function) proposed by Biot [10]. In the literature it has been argued, by for example Cremer and Heckl [11], that among the possible damping functions, the relaxation (exponential) function is the only one likely to be physically justified. However, from a mathematical point view this will not necessarily be true. For example, the damping function 8 in Table 1 (Gaussian model) has its only singularity when $s \to \infty$, but it

Table 1
A representative selection of non-viscous damping functions in the Laplace domain

Model Number	Damping function	Author and year of publication
1	$G(s) = \sum_{k=1}^{n} \dfrac{a_k s}{s + b_k}$	Biot [10]—1955
2	$G(s) = as \int_0^\infty \dfrac{\gamma(\rho)}{s+\rho} \mathrm{d}\rho$ $\gamma(\rho) = \begin{cases} \dfrac{1}{\beta-\alpha}, & \alpha \leqslant \gamma \leqslant \beta \\ 0, & \text{otherwise} \end{cases}$	Buhariwala [24]—1982
3	$G(s) = \dfrac{E_1 s^\alpha - E_0 b s^\beta}{1 + b s^\beta}$ $0<\alpha<1, \quad 0<\beta<1$	Bagley and Torvik [12]—1983
4	$sG(s) = G^\infty \left[1 + \sum_k \alpha_k \dfrac{s^2 + 2\xi_k \omega_k s}{s^2 + 2\xi_k \omega_k s + \omega_k^2} \right]$	Golla and Hughes [8]—1985 and McTavish and Hughes [9]—1993
5	$G(s) = 1 + \sum_{k=1}^{n} \dfrac{\Delta_k s}{s + \beta_k}$	Lesieutre and Mingori [25]—1990
6	$G(s) = c \dfrac{1 - e^{-st_0}}{st_0}$	Adhikari [26]—1998
7	$G(s) = \dfrac{c}{st_0} \dfrac{1 + 2(st_0/\pi)^2 - e^{-st_0}}{1 + 2(st_0/\pi)^2}$	Adhikari [26]—1998
8	$G(s) = c e^{s^2/4\mu} \left[1 - \operatorname{erf}\left(\dfrac{s}{2\sqrt{\mu}} \right) \right]$	Adhikari and Woodhouse [4]—2001

satisfies the necessary causality and energetic conditions to be a valid damping model. The fractional-derivative damping models mentioned earlier also violate the assumption to be made here, for a different reason. The associated functions $G_{jk}(s)$ are multi-valued, and have a branch point at the origin of the complex plane and would require a more sophisticated analysis. Such damping models will not be covered by the following discussion.

In a modal testing procedure a set of transfer functions is measured, typically by exciting a structure at some chosen grid of points and observing response at a fixed point. From the measured transfer functions, the *poles* and the *residues* can be extracted. Under the restriction on $\mathbf{G}(s)$ given in the previous paragraph, in Ref. [21] it was shown that the transfer function matrix of non-viscously damped systems (2) can be expressed as

$$\mathbf{H}(s) = \sum_{k=1}^{m} \frac{\mathbf{R}_k}{s - \lambda_k}, \tag{24}$$

where $\mathbf{R}_k$ is the residue matrix corresponding to the kth pole λ_k and $s = \mathrm{i}\omega$, where ω denotes frequency. The poles λ_k can be related to the natural frequencies, ω_k and the damping factors, ζ_k, as

$$\lambda_k, \lambda_k^* \approx -\zeta_k \omega_k \pm \mathrm{i}\omega_k. \tag{25}$$

The residue matrix $\mathbf{R}_k$ is related to the corresponding mode shape by

$$\mathbf{R}_k = \frac{\mathbf{z}_k \mathbf{z}_k^{\mathrm{T}}}{\theta_k}, \tag{26}$$

where θ_k, the normalization constant, is given by

$$\theta_k = \mathbf{z}_k^{\mathrm{T}} \left. \frac{\partial \mathbf{D}(s)}{s} \right|_{s=\lambda_k} \mathbf{z}_k. \tag{27}$$

Eqs. (24)–(27) also hold for viscously damped systems except that $m = 2N$ because the order of the characteristic polynomial is $2N$ for viscously damped systems.

Upon noting that $\mathbf{H}(s) = \mathbf{D}^{-1}(s)$, it can be proved that (see Appendix A for details)

$$\sum_{k=1}^{m} \mathbf{R}_k = \mathbf{O}. \tag{28}$$

If the damping is not too high one would expect that among the m eigenvalues, $2N$ will appear in complex conjugate pairs corresponding to perturbed versions of the eigenvalues of the undamped system. The remaining eigenvalues will be associated with the internal behaviour of the damping model and might be expected to be purely real or else far from the imaginary axis. For convenience, arrange the eigenvalues in the sequence

$$\lambda_1, \lambda_2, \ldots, \lambda_N, \lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*, \lambda_{2N+1}, \ldots, \lambda_m. \tag{29}$$

Corresponding to the N complex conjugate pairs of eigenvalues, the N eigenvectors together with their complex conjugates are called *elastic modes* [20,21]. These modes are related to the N modes of vibration of the structural system. The modes corresponding to the 'additional' $p = 2N - m$ eigenvalues are called *non-viscous modes*. These modes are induced by the non-viscous effect of the damping mechanism.

Now, separate the sum on the left-hand of Eq. (30) to obtain

$$\sum_{k=1}^{2N} \mathbf{R}_k + \sum_{k=2N+1}^{m} \mathbf{R}_k = \mathbf{O}. \tag{30}$$

Recalling the arrangement of the eigenvalues in Eq. (29), one has

$$\mathbf{R}_{N+k} = \mathbf{R}_k^* \quad \text{for } 1 \leqslant k \leqslant N,$$

$$\mathbf{R}_{2N+k} = \mathbf{R}_{nv_k} \quad \text{for } 1 \leqslant k \leqslant p, \tag{31}$$

where $(\bullet)_{nv}$ denotes the non-viscous terms of $(\bullet)$. In view of Eqs. (31), (30) can be rewritten as

$$2\sum_{k=1}^{N} \Re(\mathbf{R}_k) = -\sum_{k=1}^{p} \mathbf{R}_{nv_k}. \tag{32}$$

The left-hand of the above equation corresponds to only the elastic modes, while the right-hand corresponds to only the non-viscous modes. Usually, the damping of a structure is sufficiently light so that all elastic modes are sub-critically damped, i.e., all of them are oscillatory in nature. In this case, the transfer functions of a system have 'peaks' corresponding to all the elastic modes (although these peaks may overlap if natural frequencies are closely spaced). The natural frequencies and the damping factors can be obtained by examining each peak separately, for example, by using the circle fitting method [3]. Estimation of ω_k and ζ_k is likely to be good if the peaks are well separated. Once the poles are known, the residues can be obtained straightforwardly; see Refs. [27,28] for example.

As was mentioned earlier, for passive systems encountered in practice the non-viscous modes are likely to be over-critically damped. Thus, in contrast to the elastic modes, they do not produce any peaks in the transfer functions. As a consequence, the poles and the residues corresponding to non-viscous modes cannot be obtained by the usual techniques of experimental modal analysis. However, due to Eq. (32), the sum of the residues corresponding to the non-viscous modes can be obtained because the left-hand side of this equation can be measured experimentally. Thus, from Eq. (32) it is clear that the left-hand side, $2\sum_{k=1}^{N} \Re(\mathbf{R}_k)$, can be used as a measure of non-viscosity of damping. For viscously damped systems, the quantity $2\sum_{k=1}^{N} \Re(\mathbf{R}_k)$ will exactly be a null matrix. In view of this discussion, the third index of non-viscosity is defined as

$$\gamma_3 = 2\left|\sum_{k=1}^{N} \Re(\mathbf{R}_k)\right|. \tag{33}$$

The above quantity provides a useful measure of non-viscosity of damping on the assumption that the residues corresponding to *all* the modes are known. Because modal truncation is inevitable in experimental work, the index given in Eq. (33) may not in practice quantify the amount of non-viscosity exactly. An index is proposed next which does not suffer from this drawback.

4.3. Index based on complex modes

Classical normal modes exist only if the damping is proportional, that is, if any one of the conditions given in Eq. (10) is satisfied. Real-life structures in general do not satisfy any of these conditions and experimental modal analysis normally gives complex modes. In this section, an index to quantify non-viscosity of damping is proposed, which utilizes measured complex modes.

To derive a single index of non-viscosity, it has been assumed that (a) the damping is small so that first-order perturbation theory is applicable, (b) the non-viscous damping matrix can be expressed as in Eq. (14), and (c) the mass matrix of the system is known. Suppose $G(\omega)$ denotes the Fourier transform of the (scalar) damping function $g(t)$ shown in Eq. (14). Separating the real and imaginary parts of $G(\omega)$ write

$$G(\omega) = G_R(\omega) + \mathrm{i}G_I(\omega). \tag{34}$$

Under such assumptions, Adhikari and Woodhouse [5] have shown that the ratio of the imaginary and real parts of $G(\omega)$, evaluated at the undamped natural frequency ω_j, can be expressed as

$$\frac{G_I(\omega_j)}{G_R(\omega_j)} = -\frac{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j}{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{v}_j} \quad \text{for } j = 1, 2, \ldots, N, \tag{35}$$

where $\mathbf{u}_j$ and $\mathbf{v}_j$ are, respectively, the real and imaginary parts of complex mode $\mathbf{z}_j$. Note that only the elastic modes have to be used in Eq. (35). For viscously damped systems $g(t) = \delta(t)$ (see Eq. (11)), and taking the Fourier transform of $g(t)$ one obtains

$$G(\omega) = 1. \tag{36}$$

This implies that for viscously damped systems $G_I(\omega) = 0$. For this reason, the ratio given by Eq. (35) is zero for viscously damped systems and differs from zero for non-viscously damped systems. This fact may be utilized to quantify the amount of non-viscosity of damping. To obtain a numerical index one may simply take the average of the ratio given by Eq. (35) over all j. The fourth index of non-viscosity is then defined as

$$\gamma_4 = \left| \frac{1}{N} \sum_{j=1}^{N} -\frac{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j}{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{v}_j} \right|. \tag{37}$$

The above quantity might be expected to provide an accurate measure of non-viscosity of damping if the damping is non-proportional and the non-viscous damping matrix has the special form given by Eq. (14), for example, if the physical damping in the system satisfies an exponential model with only one relaxation time.

4.4. Discussion

Due to their inherent differences in origin and nature, it is not possible to normalize the four proposed indices so that their absolute values are directly comparable to each other. The choice of a particular index depends upon what information is available. If the non-viscous damping matrix is available, either in the time domain or in the frequency domain, one can readily use the first or the second index. Since one cannot hope to know the non-viscous damping matrix of a structure in advance, these two indices are therefore useful for analytical studies only. However, note that the mass and the stiffness matrices are not required to obtain these indices. When the non-viscous damping matrix is not known, the third and the fourth indices may be used to quantify the non-viscosity of damping. However, both the indices have their own limitations. The third index relies on having all the modes, which is not possible for most experimental analysis. Truncation of the set of modes will degrade the usefulness of this index. The fourth index, although it does not suffer from this drawback, is strictly valid only for systems with non-proportional damping of the particular form given by Eq. (14). In the subsequent sections, the behaviour of all four indices and the consequences of the limitations just described are explored by numerical examples.

5. Numerical examples

5.1. Example 1: A four-d.o.f. system

A four-d.o.f. system with non-viscous damping is considered to illustrate the use of the four non-viscosity indices suggested above. The mass and stiffness matrices of the system are taken to be

$$\mathbf{M} = \mathrm{diag}[1, 2, 2, 1] \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 5 & -3 & 0 & 0 \\ -3 & 7 & -4 & 0 \\ 0 & -4 & 7 & -3 \\ 0 & 0 & -3 & 5 \end{bmatrix}. \tag{38}$$

The matrix of damping functions is assumed to be of the form

$$\mathcal{G}(t) = \mathrm{diag}\left[\delta(t) + \mu_1 \mathrm{e}^{-\mu_1 t}, \frac{\delta(t) + \mu_2 \mathrm{e}^{-\mu_2 t}}{5}, \frac{\delta(t) + \mu_2 \mathrm{e}^{-\mu_2 t}}{5}, \frac{\delta(t) + \mu_3 \mathrm{e}^{-\mu_3 t}}{10}\right]. \tag{39}$$

This implies that the damping mechanism is a linear combination of viscous and exponential damping models. It may be verified that none of the conditions for proportionality of damping given in Eq. (10) is satisfied. The eigenvalues and the eigenvectors of the system were obtained by following the procedure in Refs. [20,21]. The system has four elastic modes (appearing in complex conjugate pairs) and four non-viscous modes.

To calculate the first index of non-viscosity, γ_1, it is required to obtain the first two moments of $\mathcal{G}(t)$. Using the expression for $\mathcal{G}(t)$ in Eq. (39) one obtains

$$\mathcal{M}_0 = \int_0^\infty \mathcal{G}(t)\,\mathrm{d}t = 2 \times \mathrm{diag}\left[1, \frac{1}{5}, \frac{1}{5}, \frac{1}{10}\right], \tag{40}$$

and

$$\mathcal{M}_1 = \int_0^\infty t\mathcal{G}(t)\,\mathrm{d}t = \mathrm{diag}\left[\frac{1}{\mu_1}, \frac{1}{5\mu_2}, \frac{1}{5\mu_2}, \frac{1}{10\mu_3}\right]. \tag{41}$$

From $\mathcal{M}_0$ and $\mathcal{M}_1$, the matrix $\mathbf{\Theta}_1$ in Eq. (15) is obtained as

$$\mathbf{\Theta}_1 = 1/2 \times \mathrm{diag}\left[\frac{1}{\mu_1}, \frac{1}{\mu_2}, \frac{1}{\mu_2}, \frac{1}{\mu_3}\right] \tag{42}$$

By using this, γ_1 can be easily calculated from Eq. (16). Fig. 1 shows the values of γ_1 for values of μ_1 ranging from 0.5 to 15, while using fixed values $\mu_2 = 7.5$ and $\mu_3 = 5.0$. Observe that γ_1 is high for small values of μ_1. This fact is intuitively appealing because for small values of μ_1 the damping

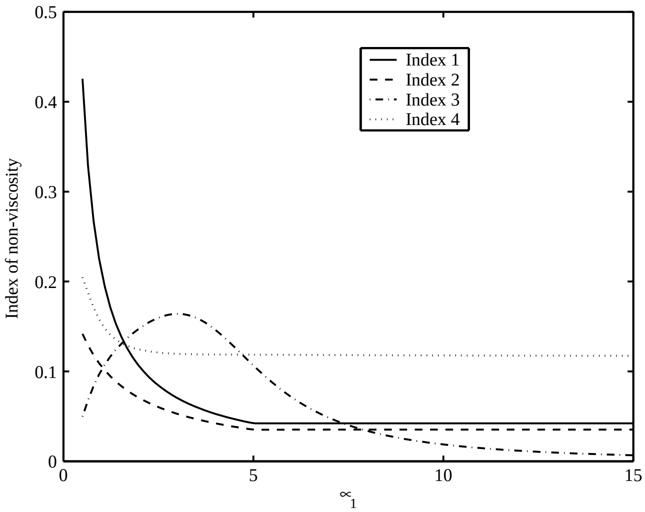

Fig. 1. Indices of non-viscosity for different values of μ_1, $\mu_2 = 7.5$ and $\mu_3 = 5.0$ for the four-d.o.f. system defined in Eqs. (38) and (39).

function has a long 'tail' and departs further from the viscous damping case, where the equivalent function would be a delta function having no tail.

Taking the Laplace transform of $\mathscr{G}(t)$ one obtains

$$\mathbf{G}(s) = \mathrm{diag}\left[1 + \frac{\mu_1}{s+\mu_1}, \frac{1}{5} + \frac{\mu_1}{5(s+\mu_1)}, \frac{1}{5} + \frac{\mu_1}{5(s+\mu_1)}, \frac{1}{10} + \frac{\mu_1}{10(s+\mu_1)}\right]. \tag{43}$$

From $\mathbf{G}(s)$ one can easily obtain $\mathbf{G}(0)$ and $\mathbf{G}(1)$, and consequently calculate the second index of non-viscosity, γ_2, given by Eq. (23). The residue matrices are obtained from Eq. (26) by using the four elastic modes of the system. The values of γ_3 are again plotted in the same figure. Note that in the region where μ_1 is less than 3 or so, this index behaves in the opposite manner to γ_1 and γ_2. However, for μ_1 greater than 3 or so, the behaviour of γ_3 is recognizably similar to those of γ_1 and γ_2. Although γ_3 shows a discrepancy from the two previous indices, we emphasize that unlike them, γ_3 can be calculated without knowing the non-viscous damping matrix. For this reason, γ_3 might be expected to have more applicability in practice.

Finally, consider the fourth index of non-viscosity, γ_4, given by Eq. (37). Only the real and imaginary parts of the measured complex modes (that is, only the elastic modes) are required to obtain this index. As before, the values of γ_4 are shown in Fig. 1. It is clear that the trend of γ_4 is similar to those of γ_1 and γ_2. From this particular example it may be concluded that the first, second and the fourth indices behave in a similar way, while the third index shows some discrepancy for lower values of μ_1.

5.2. Example 2: A 30-d.o.f. system

To see if the results of the previous example are typical, it is necessary to look at a wide range of systems with different damping mechanisms. In this section, a larger system consisting of a linear array of 30 spring–mass oscillators and non-viscous dampers has been considered. The assumed model of damping, expressed by a linear combination of two exponential models, is a step further than the previous example.

The mass and stiffness of all units are assumed to be the same, so that the mass and the stiffness matrices are given by

$$\mathbf{M} = m_u, \tag{44}$$

and

$$\mathbf{K} = k_u \bar{\mathbf{I}}, \tag{45}$$

where the tri-diagonal matrix is

$$\bar{\mathbf{I}} = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{bmatrix}. \tag{46}$$

The non-viscous damping matrix is assumed to be of the form

$$\mathcal{G}(t) = \mu_1 \mathrm{e}^{-\mu_1 t} \mathbf{C}_1 + \mu_2 \mathrm{e}^{-\mu_2 t} \mathbf{C}_2, \tag{47}$$

where

$$\mathbf{C}_1 = c_1 \mathbf{I} \tag{48}$$

and

$$\mathbf{C}_2 = c_2 \bar{\mathbf{I}}_{p,l}, \tag{49}$$

where $\bar{\mathbf{I}}_{p,l}$ is a tri-diagonal matrix similar to that in Eq. (46) except that it is non-zero only between the pth and lth entries along the diagonal and the super-diagonal. The first damping term in Eq. (47) corresponds to a set of non-viscous dampers connecting each mass to the ground. The second damping term in Eq. (47) corresponds to dampers connected between adjacent masses, but only between the pth and the lth masses.

For the numerical simulation it is assumed that $m_u = 1\,\mathrm{kg}$, $k_u = 4.0 \times 10^5\,\mathrm{N/m}$, $c_1 = 25\,\mathrm{Nm/s}$, $c_2 = 200\,\mathrm{Nm/s}$, $\mu_2 = \mu_1/4$, $p = 8$ and $l = 17$. The indices of non-viscosity are shown in Fig. 2 for values of μ_1 varying between 250 and $1500\,\mathrm{s}^{-1}$. For higher values of the relaxation parameter μ_1 damping is close to viscous. Since in this problem μ_2 is expressed in terms of μ_1 it would be expected that all indices should show comparatively low values for higher μ_1. Indices 1 and 2 are very close to each other and their values gradually decrease with increasing μ_1 as expected. The value of index 4, although oscillatory, also shows a decreasing trend with increasing μ_1. Index 3, however, shows an opposite trend, with the index value increasing with increasing μ_1. Here, this index clearly gives a wrong indication of the actual damping behaviour.

Recall that index 3 is obtained by using a 'modal sum' from Eq. (33). The effect of modal truncation can also be investigated by using this example. Fig. 3 shows the index values obtained by retaining 5, 10, 20 and 30 (all) modes. Note that modal truncation does not affect the index

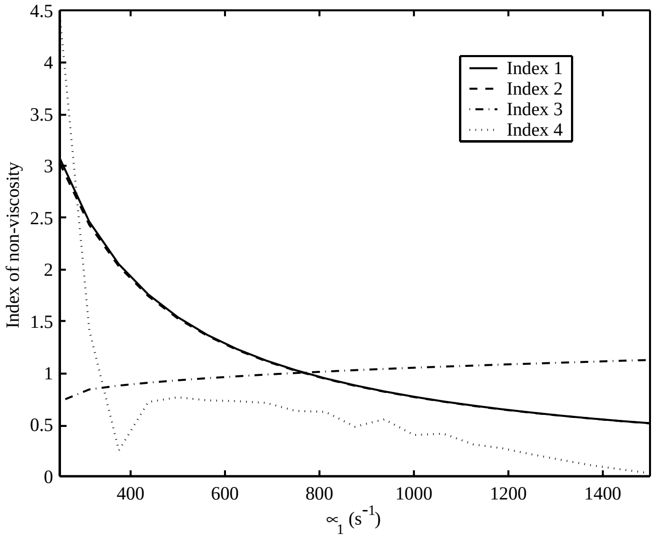

Fig. 2. Indices of non-viscosity for different values of μ_1 for a 30-d.o.f. system defined in the text.

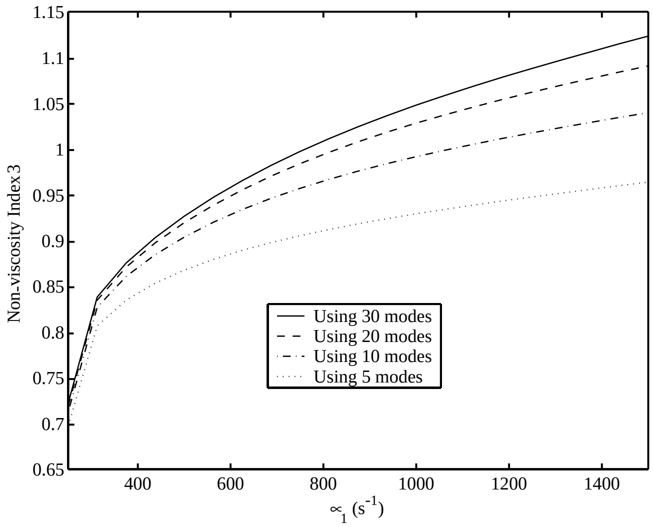

Fig. 3. Non-viscosity index 3 obtained by retaining different number of modes in Eq. (33), for the same system as Fig. 2.

values significantly, and they still indicate the damping behaviour wrongly as all of them increase with increasing μ_1.

From Eq. (37) observe that index 4 is obtained by averaging the ratio over all modes. The effect of using a reduced number of modes for this index in shown in Fig. 4. All the index values, obtained by retaining different number of modes in the calculation, decrease with increasing μ_1. This indicates that, even with as few as 10 or 5 modes, index 4 can be used with some success. However, using more modes improves the result because for any value of μ_1 the index obtained by

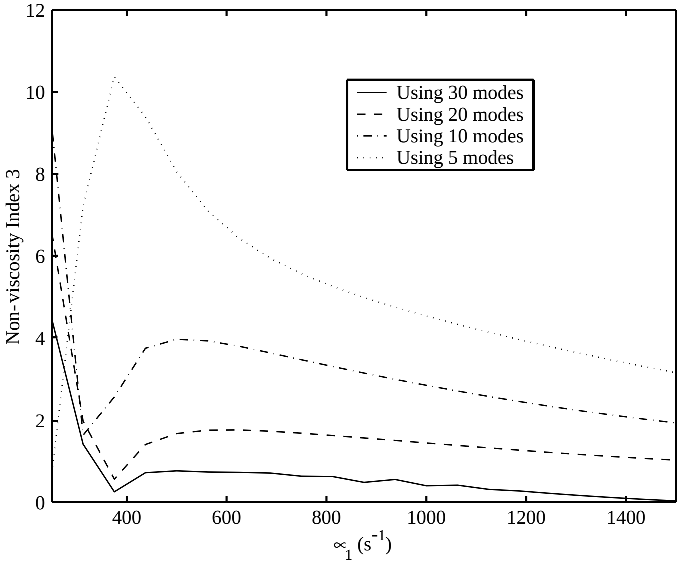

Fig. 4. Non-viscosity index 4 obtained by using different number of modes in Eq. (37), for the same system as Fig. 2.

using a higher number of modes has a lower value, and use of the full set of modes produces the most satisfactory result.

From this example, and also the previous one, it may be concluded that indices 1, 2 and 4 can give a good indication of non-viscosity of damping. The performance of index 3 has not been very convincing and in some cases it shows the opposite of the expected damping behaviour. Therefore, it is suggested that for practical purposes indices 1, 2 and 4 should be used and index 3 should be avoided.

6. Error analysis

The numerical values of the non-viscosity indices proposed here are unbounded except that $\gamma_i \geqslant 0, \ \forall i$. The lack of an upper bound may be regarded as a possible drawback because from a *single* value of γ_i it is not in general possible to comprehend the degree of non-viscosity of damping. One useful way to interpret the non-viscosity indices is to analyze the errors that arise if one makes the assumption of viscous damping for a non-viscously damped system. There are various possible choices of a quantity to measure the error: for example, the difference between the time response or the frequency response at some d.o.f. of a structure. Here, we consider the l_2 norm of the transfer function matrix. If a system is perfectly viscously damped, Eq. (12) would give the viscous damping matrix. By using this viscous damping matrix, the transfer function matrix of the equivalent viscously damped system can be obtained from Eq. (24) with $m = 2N$, so that

$$\mathbf{H}^{(v)}(s) = \sum_{k=1}^{2N} \frac{\mathbf{R}_k^{(v)}}{s - \lambda_k^{(v)}} \tag{50}$$

where $(\bullet)^{(v)}$ symbolizes 'for viscous damping'. The frequency-dependent error is now given by

$$\varepsilon(\mathrm{i}\omega) = \frac{\left|\mathbf{H}(\mathrm{i}\omega) - \mathbf{H}^{(v)}(\mathrm{i}\omega)\right|}{|\mathbf{H}(\mathrm{i}\omega)|}. \tag{51}$$

The aim here is to understand the behaviour of this quantity in the light of the non-viscosity indices proposed here.

For numerical illustration, we consider the four-d.o.f. system of Section 5.1. The equivalent viscous damping matrix for the example considered may be obtained from Eq. (40). In Fig. 5, the quantity $\varepsilon(\mathrm{i}\omega)$ is plotted for values of μ_1 ranging from 0.5 to 15 as considered before. The error decreases as μ_1 increases. In view of the non-viscosity indices shown in Fig. 1, it appears that the error due to making the viscous damping assumption is greater when the values of the non-viscosity indices are greater and vice versa. This shows that the indices of non-viscosity proposed here do indeed give a good qualitative indication of the error which would be incurred by making a viscous damping assumption. For this example, it appears that a viscous damping model should not be used for a non-viscously damped system if the non-viscosity index (any one of them) has a value 0.4 or higher.

Another interesting fact to emerge from Fig. 5 is the frequency dependence of the error. Note that the error has peaks around the system natural frequencies. This implies that a viscous damping assumption for a non-viscously damped system is likely to produce more error if the driving frequency is near to a system's natural frequency. This is as one might have guessed: it is well known that the effect of damping is most significant near the natural frequencies. Since the indices developed here are not frequency dependent, they cannot indicate directly the presence or level of this variation of the error, and this could be regarded as a shortcoming of all these indices for quantitative purposes. Further work is needed to determine how important this might be.

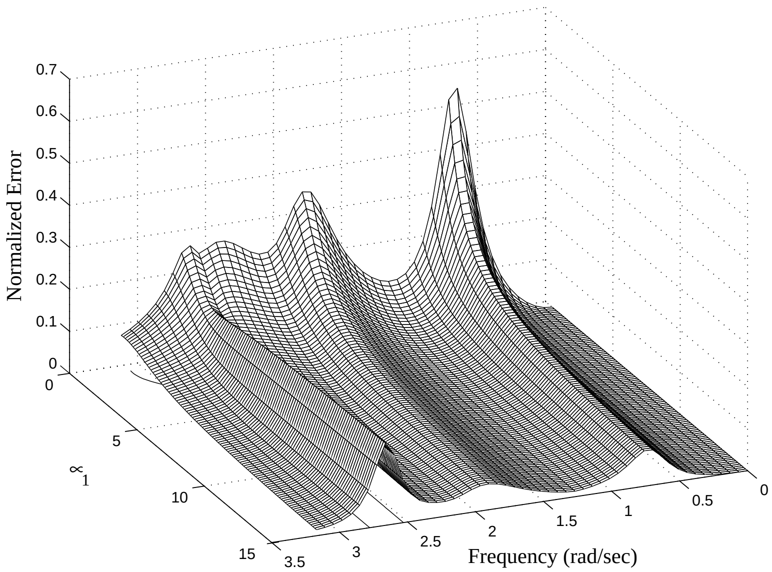

Fig. 5. Error in the norm of the transfer function matrix due to the viscous damping assumption, $\mu_2 = 7.5$ and $\mu_3 = 5.0$, for the system used to calculate Fig. 1.

7. Conclusions

Quantification of the amount of non-viscosity of damping in linear multiple-degree-of-freedom dynamic systems has been considered. Four indices, based on (1) moments of the non-viscous damping matrix, (2) the Laplace transform of the non-viscous damping matrix, (3) transfer function residues and (4) complex modes, have been proposed. The first and the second indices are suitable for analytical studies, while the other two are aimed at using experimental data. The relative merits and demerits of these indices have been discussed. Indices 1, 2 and 4 were shown to behave in a similar way, but index 3 was found to have counter-intuitive behaviour and thus to be less useful.

The indices of non-viscosity proposed here are useful to understand the justification of the viscous damping assumption commonly used in practice. If the values of the non-viscosity indices are high then the viscous damping assumption may not be suitable. Through a numerical study, it was shown that the error in the frequency response function incurred due to the viscous damping assumption indeed increases for higher levels of non-viscosity. It was observed that the error also depends on the forcing frequency—if the forcing frequency is near to a system's natural frequency, the error is higher. Further research is worth pursuing in this direction. It has been assumed that all the information required to obtain these indices is known exactly. Further work is also needed to understand their sensitivity and robustness to errors in measured data.

Appendix A. Summation of the residues of the transfer function matrix

For non-viscously damped systems the transfer function matrix can be expressed by Eq. (24) From the definition of the transfer function matrix,

$$\mathbf{H}(s) = \mathbf{D}^{-1}(s), \tag{A.1}$$

where the dynamic stiffness matrix $\mathbf{D}(s)$ is given by Eq. (4). Rewrite the expression of the dynamic stiffness matrix as

$$\mathbf{D}(s) = s^2\mathbf{M}\left[\mathbf{I}_N + \frac{\mathbf{M}^{-1}}{s}\left(\mathbf{G}(s) + \frac{\mathbf{K}}{s}\right)\right]. \tag{A.2}$$

Taking the inverse of this equation and expanding the right-hand side one obtains

$$\mathbf{H}(s) = \frac{\mathbf{M}^{-1}}{s^2} + \frac{1}{s^3}\left(-\mathbf{M}^{-1}\mathbf{G}(s)\mathbf{M}^{-1}\right) + \frac{1}{s^4}\left(\mathbf{M}^{-1}\left[\mathbf{G}(s)\mathbf{M}^{-1}\mathbf{G}(s) - \mathbf{K}\right]\mathbf{M}^{-1}\right) + \cdots. \tag{A.3}$$

Now, express a general term of the expression of the transfer function matrix given by Eq. (24) as

$$\begin{aligned}\frac{\mathbf{R}_k}{s-\lambda_k} &= \left[s\left(1-\frac{\lambda_k}{s}\right)\right]^{-1}\mathbf{R}_k \\ &= \frac{1}{s}\mathbf{R}_k + \frac{1}{s^2}\left[\lambda_k\mathbf{R}_k\right] + \frac{1}{s^3}\left[\lambda_k^2\mathbf{R}_k\right] + \frac{1}{s^4}\left[\lambda_k^3\mathbf{R}_k\right] + \cdots.\end{aligned} \tag{A.4}$$

By using the above expression, the transfer function matrix in Eq. (24) can be expressed as

$$\mathbf{H}(s) = \frac{1}{s}\left[\sum_{k=1}^{m} \mathbf{R}_k\right] + \frac{1}{s^2}\left[\sum_{k=1}^{m} \lambda_k \mathbf{R}_k\right] + \frac{1}{s^3}\left[\sum_{k=1}^{m} \lambda_k^2 \mathbf{R}_k\right] + \frac{1}{s^4}\left[\sum_{k=1}^{m} \lambda_k^3 \mathbf{R}_k\right] + \cdots. \tag{A.5}$$

Comparing Eqs. (A.3) and (A.5) makes it clear that their right sides are equal. Multiplying these equations by s and taking the limit as $s \to \infty$ (and recalling that $\lim_{s\to\infty} \|\mathbf{G}(s)\|$ is assumed to be bounded) one obtains

$$\sum_{k=1}^{m} \mathbf{R}_k = \mathbf{O}. \tag{A.6}$$

This implies that the sum of all the residues of the transfer function matrix of non-viscously damped systems is a null matrix. This result also holds for a viscously damped system provided $m = 2N$ is used.

Appendix B. Nomenclature

$\mathbf{C}$	viscous damping matrix
$\mathbf{D}(s)$	dynamic stiffness matrix
$\mathcal{G}(t)$	non-viscous damping matrix in the time domain
$\mathbf{G}(s)$	Laplace transform of $\mathcal{G}(t)$
$g(t)$	non-viscous damping function
$G(\omega)$	Fourier transform of the (scalar) damping function $g(t)$
$G_R(\omega)$	real part of $G(\omega)$
$G_I(\omega)$	imaginary part of $G(\omega)$
$\mathbf{H}(\mathrm{i}\omega)$	transfer function matrix
$\mathbf{I}$	identity matrix
i	unit imaginary number, $\mathrm{i} = \sqrt{-1}$
$\mathbf{K}$	stiffness matrix
$\mathbf{M}$	mass matrix
$\mathcal{M}_r$	rth moment of $\mathcal{G}(t)$
m	order of the characteristic polynomial
N	degrees-of-freedom of the system
$\mathbf{O}$	null matrix
p	number of non-viscous modes
$\mathbf{q}(t)$	vector of generalized co-ordinates
$\mathbf{R}_k$	residue matrix corresponding to kth mode
s	Laplace domain parameter
t	time
T_{min}	minimum time period for the system
$\mathbf{u}_k$	real part of $\mathbf{z}_k$
$\mathbf{v}_k$	imaginary part of $\mathbf{z}_k$
$\mathbf{z}_k$	kth mode of the system
λ_k	kth complex natural frequency of the system

ω_k	kth undamped natural frequency
ζ_k	kth modal damping factor
θ_k	normalization constant associated with the kth mode
$\varepsilon(\mathrm{i}\omega)$	error in the norm of transfer function matrix
μ_j	constants associated with exponential damping function ($j = 1, 2, 3$)
γ_j	index of non-viscosity ($j = 1, 2, 3, 4$)
$\delta(t)$	Dirac delta function
diag	diagonal matrix
$\det(\bullet)$	determinant of $(\bullet)$
$\mathbb{C}$	space of complex numbers
$\mathbb{R}$	space of real numbers
$\Re(\bullet)$	real part of $(\bullet)$
$(\bullet)^{\mathrm{T}}$	matrix transpose of $(\bullet)$
$(\bullet)^{-1}$	matrix inverse of $(\bullet)$
$(\dot{\bullet})$	derivative of $(\bullet)$ with respect to t
$(\bullet)^*$	complex conjugate of $(\bullet)$
$(\bullet)^{(v)}$	$(\bullet)$ for viscously damped systems
$(\bullet)_{nv}$	non-viscous part of $(\bullet)$
$\Vert\bullet\Vert$	l_2 matrix norm

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