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A simplified method for the free vibration and flutter analysis of bridge decks

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Abstract

A simplified method for the free vibration and flutter analysis of bridge decks is presented. Bending–torsion coupled beam theory with warping stiffness included is used in the structural idealization of bridge decks in order to derive explicit formulae for natural frequencies and mode shapes. These are used to perform the flutter analysis. The time-dependent aerodynamic forces are modelled using Theodorsen type flat plate theory. Expressions for generalized mass, generalized stiffness and generalized aerodynamic force terms are derived in compact explicit form. The flutter problem is then formulated by summing algebraically the analytical expressions for generalized mass, generalized stiffness and generalized aerodynamic forces, and the associated flutter determinant is expanded in analytical form. Finally, the flutter speed and flutter frequency are thereby determined by using a standard root finding procedure. The method is demonstrated by numerical results. This is followed by some concluding remarks.

1. Introduction

Free vibration and flutter analysis of bridge decks is an important area of investigation. After the collapse of the Tacoma suspension bridge in 1940, the importance of such research has been recognized widely by scientists and engineers. The serious nature of the flutter instability of the Tacoma bridge is well documented in films, reports and research papers. Since this disaster research in the area of free vibration and flutter analysis of bridges has expanded with many novel ideas. A small, but carefully selected, sample of the literature [1–22] shows that a wide range of procedures and methodologies for tackling the problem is now available. One striking feature of this published literature is the overlap between the problems of free vibration and flutter analysis

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of high aspect ratio aircraft wings and those of bridges. In particular, the structural model of a high aspect ratio aircraft wing based on beam theory has often been used to idealize bridge decks, especially those with long spans [2,6,7,11,21] for which the length to thickness ratio is usually very high. In the idealization of the unsteady aerodynamic forces on bridges arising from wind, gust and turbulence, Theodorsen's theory [23] of harmonically oscillating flat plate, which is well established in aeroelastic studies of aircraft wings, has been used [8,13,22] to carry out the flutter analysis. However, one important difference between the two applications is in the boundary conditions. For aircraft wings, cantilever end (boundary) conditions are often used whereas simple-support end conditions are naturally more appropriate for bridges. Although the origin of the flutter problem lies historically in the aeronautical engineering field, many of its fundamental principles can generally be applied to civil engineering structures such as bridges. Of course, the numerical values of the input data for the two types of problems can be markedly different (for examples, the values of bending and torsional rigidities, mass per unit length, mass moment of inertia per unit length, length, etc.). The author's own experience in the field of aircraft vibration and flutter analysis suggests that a technology transfer of this nature from aeronautical to civil engineering is of great value.

In the preliminary stage of this research, a literature survey was carried out. Refs. [1–22] furnish a chronological development in the field, which provide the reader with a useful introduction to the subject. These papers are broadly classified under (i) free vibration and (ii) flutter analysis, bearing in mind that free vibration analysis is a fundamental pre-requisite before carrying out a flutter analysis when the normal mode method is used. The key publications are briefly reviewed as follows.

1.1. Free vibration analysis

Steinman [1] is one of the earliest investigators to derive a set of simple formulae for natural frequencies and mode shapes of suspension bridges using a representation of the deflected shape by sine waves, and the overall structure by equivalent rigidity and mass properties. Later Vellozzi [2] gave an analytical formulation for the dynamic behaviour of suspension bridges by accounting for a moving load. He used beam theory and derived the governing differential equations of motion by applying Newton's second law. Both free and force vibration problems were solved for a single-span suspension bridge and the method was illustrated by numerical results. In the late 197055 Abdel-Ghaffar [3–5] used the finite element method to analyze the vertical, torsional and lateral free oscillations of suspension bridges. Van der Woude [6] investigated the natural modes and frequencies of a simple (single) span suspension bridge with straight backstays by using Lagrange's equation and Fourier series representation of cable displacements. He conducted model tests in the laboratory and his experimental results agreed quite well with his theoretical predictions. Hayashikawa and Watanabe [7] studied the vertical vibration of suspension bridges using Timoshenko's beam theory and obtained a general solution of the resulting fourth order differential equation. They illustrated their method by numerical results obtained for the famous Innoshima bridge built in Japan. Other contributors to the free vibration analysis of suspension bridges include Brownjohn [17], Hayashikawa [18], Rossikhin and Shitikova [20] and Holubova-Tajcova [21].

1.2. Flutter analysis

Agar [8] and Miyata and Yamada [9] solved the flutter problem of suspension bridges using numerical methods based on normal modes and Theodorsen-type unsteady aerodynamics [23]. Later Scanlan and Jones [10] made a noteworthy contribution when they presented an empirically based formulation for flutter analysis in the frequency domain using flutter derivatives. Their investigation is focussed around experimentally determined flutter derivatives, and a full three-dimensional modal analysis of the structure. Tinh [11] developed a theoretical formula to determine the critical flutter speed of a suspension bridge and tabulated results for a wide range of parameters so that estimates of flutter speed for different suspension bridges can be obtained. Agar [12] extended his earlier work [8] and addressed the issue of how the degree of refinement of the basic structural model and the number of natural modes included in the analysis affect the flutter prediction. Kobayashi and Nagaoka [13] and Preidikman and Mook [16] showed the effect of an active control technology in suppressing the flutter of suspension bridges. The identification of flutter derivatives by experimental means can be found in the work of Zasso et al. [14] and Singh et al. [15]. Beith [19] used a practical engineering method by de-coupling the equation of motion to response in each mode while investigating the flutter characteristics of long span bridges. Katsuchi et al. [22] presented an analytical investigation on multimode coupled flutter and buffeting of the Akashi-Kaikyo bridge and compared their results with related wind tunnel tests. Their theoretical results compared favourably with experimental ones.

1.3. Object of this paper

The object of this paper is to give a simplified method for the free vibration and flutter analysis of bridge decks using the normal mode method and generalized co-ordinates. Particular emphasis is placed in generating all explicit algebraic expressions, which are needed for the free vibration and flutter analysis. These expressions are short, compact and concise and their derivations involve detailed algebraic simplifications. In the structural idealization of the bridge it is assumed that the overall structure can be modelled by its *effective* bending, torsional and warping rigidities together with representative values of its mass and inertia properties. It is assumed that the shear centre and centroid of the bridge cross-section are non-coincident so that coupled bending–torsion beam theory with warping included is needed to describe the free vibratory motion. (This is particularly relevant for lateral vibration of suspension bridges as opposed to vertical (transverse) vibration.) First an explicit analytical formula is derived, which gives the natural frequencies of a suspension bridge for simply supported (pinned–pinned) end conditions. Mode shapes (which are coupled in bending and torsion as a result of non-coincident mass and shear centre) are expressed in explicit analytical form. Next the expressions for generalized mass and stiffness in each mode are derived in explicit algebraic form. In the aerodynamic idealization of the bridge, Theodorsen's theory [23] is used and the unsteady aerodynamic forces are expressed in modal co-ordinates. Explicit expressions for the elements of the generalized aerodynamic matrix are derived in algebraic form by simplifying the algebra considerably. These are complex expressions whose real and imaginary parts are identified. Once the analytical expressions for generalized mass, generalized stiffness and generalized aerodynamic force in each mode are obtained individually in explicit form, they are summed algebraically to formulate the complex flutter function, which is primarily a function of two unknown variables, namely the air-speed and the frequency. The zeros of this function, which give the flutter speed and flutter frequency, are obtained by a standard root finding procedure.

2. Theory

2.1. Free vibration analysis

A single span suspension bridge is shown in Fig. 1 in a schematic diagram. In this paper, attention is confined to the central part of the bridge (Section 2 of the Figure) which is generally more susceptible to free vibration and flutter phenomena than the rest of the structure. This portion of the bridge is idealized as a uniform bending–torsion coupled beam, possessing an equivalent representative, but effective, values of bending (EI), torsional (GJ) and warping ($E\Gamma$) rigidities, mass per unit length (m) and mass moment of inertia per unit length (I_α), respectively.

The co-ordinate system and notation for the central part of the bridge (which is represented by a uniform bending–torsion coupled beam) is shown in Fig. 2. The length and width (semi-width) are taken to be L and $2b$ (b), respectively. The elastic axis is assumed to be at a distance ba_h from the mid-chord (mid-width) position whereas the mass axis is assumed to be at a distance bx_α from the elastic axis as shown. (Note that a_h and x_α are both non-dimensional quantities expressed as fractions of semi-chord and they are positive in the positive direction of X as shown.) The elastic axis, which is coincident with the Y-axis, is allowed to deflect out of plane by $h(y,t)$, while the cross–section is allowed to rotate or twist about OY by $\psi(y,t)$, where y and t denote distance form the origin and time, respectively.

Using bending-torsion coupled theory with warping stiffness included, the governing differential equations of motion in free vibration can be written as [24]

$$EI\frac{\partial^4 h}{\partial y^4} + m\frac{\partial^2 h}{\partial t^2} - mbx_\alpha\frac{\partial^2 \psi}{\partial t^2} = 0, \tag{1}$$

$$E\Gamma\frac{\partial^4 \psi}{\partial y^4} - GJ\frac{\partial^2 \psi}{\partial y^2} + I_\alpha\frac{\partial^2 \psi}{\partial t^2} - mbx_\alpha\frac{\partial^2 h}{\partial t^2} = 0. \tag{2}$$

Fig. 1. A single span suspension bridge.

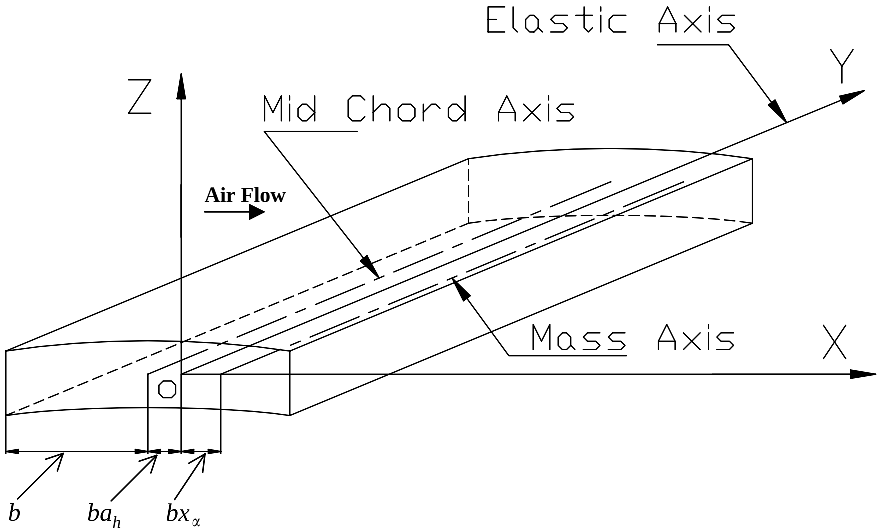

Fig. 2. Co-ordinate system and notation for a suspension bridge idealized as a bending–torsion coupled beam.

Assuming harmonic oscillation with circular frequency ω, then

$$h(y,t) = H(y)e^{i\omega t},$$
$$\psi(y,t) = \Psi(y)e^{i\omega t}. \tag{3}$$

Substituting Eq. (3) into Eqs. (1) and (2) gives

$$EI\frac{d^4H}{dy^4} - m\omega^2 H + mbx_\alpha\omega^2\Psi = 0. \tag{4}$$

and

$$E\Gamma\frac{d^4\Psi}{dy^4} - GJ\frac{d^2\Psi}{dy^2} - I_\alpha\omega^2\Psi + m\omega^2 bx_\alpha H = 0. \tag{5}$$

If simple support end conditions (pinned–pinned) are assumed then the mode shapes for bending displacement (H) and torsional rotation (Ψ) in the nth mode can be assumed as

$$H_n(y) = C_n \sin\frac{n\pi y}{L},$$
$$\Psi_n(y) = D_n \sin\frac{n\pi y}{L}, \tag{6}$$

where C_n and D_n are the amplitudes of bending displacement and torsional rotation in the nth mode, respectively. (Note that the simple supports prevent local torsional rotations.)

Substituting Eqs. (6) into Eqs. (4) and (5) gives

$$\left[EI\left(\frac{n\pi}{L}\right)^4 - m\omega^2\right]C_n + m\omega^2 bx_\alpha D_n = 0 \tag{7}$$

and

$$m\omega^2 b x_\alpha C_n + \left[GJ\left(\frac{n\pi}{L}\right)^2 + E\Gamma\left(\frac{n\pi}{L}\right)^4 - I_\alpha\omega^2\right]D_n = 0. \tag{8}$$

Elimination of C_n and D_n from Eqs. (7) and (8) yields a quadratic equation in ω^2 whose two roots which give the natural frequencies of the beam (bridge) for any integer value of $n(n = 1, 2, 3, 4, \ldots)$ are given by

$$\omega_n^2 = \frac{B_n^2 + T_n^2 \pm \sqrt{(B_n^2 - T_n^2)^2 + 4(1-\mu)B_n^2 T_n^2}}{2\mu}, \tag{9}$$

where

$$B_n^2 = \frac{EIn^4\pi^4}{mL^4}, \tag{10}$$

$$T_n^2 = \frac{GJn^2\pi^2L^2 + E\Gamma n^4\pi^4}{I_\alpha L^4}, \tag{11}$$

and

$$\mu = 1 - \frac{m(bx_\alpha)^2}{I_\alpha}. \tag{12}$$

Eq. (9) can be used to determine any number of natural frequencies of the bending–torsion coupled beam (bridge). Note that for the uncoupled (degenerate) case (when bending and torsional motions are decoupled) $x_\alpha = 0$ and hence $\mu = 1$ from Eq. (12). The substitution of $\mu = 1$ in Eq. (9) gives the natural frequencies as B_n and T_n which are natural frequencies of a simple Bernoulli-Euler beam in bending and torsional vibration, respectively.

From Eq. (7) the ratio of the bending and torsional amplitudes follows as

$$\frac{C_n}{D_n} = \frac{bx_\alpha}{(1 - n^4\pi^4 EI/m\omega_n^2 L^4)} = \frac{bx_\alpha}{(1 - B_n^2/\omega_n^2)}. \tag{13}$$

2.2. Generalized mass and generalized stiffness

The generalized mass M_n and generalized stiffness K_n in the nth mode of the bending–torsion coupled beam can be derived using the procedure put forward by Bishop and Price [25]. These are, respectively given by

$$M_n = \int_0^1 [(mH_n^2 + I_\alpha \Psi_n^2) - 2mx_\alpha H_n \Psi_n]\, d\xi, \tag{14}$$

and

$$K_n = \int_0^1 [EI(H_n'')^2 + GJ(\Psi_n')^2]\, d\xi, \tag{15}$$

where ξ is the non-dimensional length given by

$$\xi = y/L, \tag{16}$$

and

H_n and Ψ_n are the mode shapes given by Eq. (6)

However, a simpler way to calculate the generalized stiffness K_n would be to use the following equation:

$$K_n = \omega_n^2 M_n, \tag{17}$$

where ω_n, the nth natural frequency has already been calculated from Eq. (9) prior to the calculation of the mode shapes H_n and Ψ_n.

Using the (sine function) expressions for H_n and Ψ_n given by Eq. (6), the three ξ dependent integrals, which appear in Eq. (14), can be replaced by $C_n^2/2$, $D_n^2/2$ and $C_n D_n/2$ to give the generalized mass as

$$M_n = \tfrac{1}{2}\{mC_n^2 + I_\alpha D_n^2 - 2mbx_\alpha C_n D_n\} \tag{18}$$

and the generalized stiffness K_n follows from Eq. (17).

2.3. Generalised aerodynamic coefficients

The generalized aerodynamic coefficients are derived by applying the principle of virtual work. The aerodynamic strip theory of Theodorsen for unsteady lift and moment [23] and the normal modes obtained above, are used when applying the principle of virtual work. Thus if the bending displacement and torsional rotation in the ith mode are $H_i(\xi)$ and $\Psi_i(\xi)$, and the corresponding generalized co-ordinate is $q_i(t)$, then for a number of n modes, the time-dependent bending displacement $h(\xi, t)$ and torsional rotation $\psi(\xi, t)$ can be expressed, respectively, as

$$h(\xi, t) = \sum_{i=1}^{n} H_i(\xi) q_i(t), \tag{19}$$

and

$$\psi(\xi, t) = \sum_{i=1}^{n} \Psi_i(\xi) q_i(t). \tag{20}$$

Eqs. (19) and (20) can be written in matrix form as

$$\begin{bmatrix} h(\xi, t) \\ \psi(\xi, t) \end{bmatrix} = \begin{bmatrix} H_1(\xi) & H_2(\xi) & \cdots & \cdots & H_n(\xi) \\ \Psi_1(\xi) & \Psi_2(\xi) & \cdots & \cdots & \Psi_n(\xi) \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ \vdots \\ q_n(t) \end{bmatrix}. \tag{21}$$

If $L(\xi)$ and $M(\xi)$ are, respectively, the unsteady lift and moment at a spanwise distance $\xi = y/L$ from the root, the virtual work (δW) done by the aerodynamic forces is given by

$$\delta W = \sum_{i=1}^{n} \delta q_i \int_0^1 [L(\xi)H_i(\xi) + M(\xi)\Psi_i(\xi)]\,\mathrm{d}\xi. \tag{22}$$

Eq. (22) can be written as

$$\begin{bmatrix} \dfrac{\delta W_1}{\delta q_1} \\ \dfrac{\delta W_2}{\delta q_2} \\ \vdots \\ \vdots \\ \dfrac{\delta W_n}{\delta q_n} \end{bmatrix} = \int_0^1 \begin{bmatrix} H_1 & \Psi_1 \\ H_2 & \Psi_2 \\ \vdots & \vdots \\ \vdots & \vdots \\ H_n & \Psi_n \end{bmatrix} \begin{bmatrix} L(\xi) \\ M(\xi) \end{bmatrix} \mathrm{d}\xi. \tag{23}$$

The unsteady lift $L(\xi)$ and moment $M(\xi)$ in two-dimensional flow given by Theodorsen can be expressed as [26–28]

$$\begin{bmatrix} L(\xi) \\ M(\xi) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} h(\xi,t) \\ \psi(\xi,t) \end{bmatrix}, \tag{24}$$

where

$$\begin{aligned}
A_{11} &= -\pi\rho U^2\{-k^2 + 2C(k)\mathrm{i}k\},\\
A_{12} &= \pi\rho U^2 b[(a_h k^2 + \mathrm{i}k) + 2C(k)\{1 + \mathrm{i}k(0.5 - a_h)\}],\\
A_{21} &= -\pi\rho U^2 b\{2C(k)\mathrm{i}k(0.5 + a_h) - k^2 a_h\},\\
A_{22} &= \pi\rho U^2 b^2[2(0.5 + a_h)C(k)\{1 + \mathrm{i}k(0.5 - a_h)\} + \frac{k^2}{8} + k^2 a_h^2 + (a_h - 0.5)\mathrm{i}k].
\end{aligned} \tag{25}$$

In Eqs. (25), $U, b, \rho, k, C(k)$ and a_h are in the usual notation: the airspeed, semi-chord, density of air, reduced frequency parameter (defined as $k = \omega b/U$), Theodorsen function and elastic axis location from mid-chord, respectively [26–28]. Note that the signs of A_{11} and A_{21} as given in Refs. [26–28] have been reversed because h is considered positive upward in this paper.

Substituting Eqs. (24) and (21) into Eq. (23) gives

$$\begin{bmatrix} \dfrac{\delta W_1}{\delta q_1} \\ \dfrac{\delta W_2}{\delta q_2} \\ \vdots \\ \vdots \\ \dfrac{\delta W_n}{\delta q_n} \end{bmatrix} = \int_0^1 \begin{bmatrix} H_1 & \Psi_1 \\ H_2 & \Psi_2 \\ \vdots & \vdots \\ \vdots & \vdots \\ H_n & \Psi_n \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} H_1 & H_2 & \cdots & \cdots & H_n \\ \Psi_1 & \Psi_2 & \cdots & \cdots & \Psi_n \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ \vdots \\ q_n(t) \end{bmatrix} \mathrm{d}\xi$$

$$= \begin{bmatrix} QA_{11} & QA_{12} & \cdots & \cdots & QA_{1n} \\ QA_{21} & QA_{22} & \cdots & \cdots & QA_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ QA_{n1} & QA_{n2} & \cdots & \cdots & QA_{nn} \end{bmatrix}, \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}, \tag{26}$$

where **[QA]** is the generalized aerodynamic matrix with

$$QA_{ij} = \int_0^1 (A_{11} H_i H_j + A_{12} H_j \Psi_i + A_{21} H_i \Psi_j + A_{22} \Psi_i \Psi_j)\, \mathrm{d}\xi. \tag{27}$$

The elements of generalized aerodynamic matrix **[QA]** are complex terms because the terms A_{11}, A_{12}, etc., in Eq. (27) are complex (see Eq. (25)). By contrast, the generalized mass and stiffness terms (see Eqs. (18) and (17)) are both real. Expressions for each of the integrals in Eq. (27) are very simple both for $i = j$ and $i \neq j$ as a result of using simple support end conditions of the bridge. However, care should be exercised while writing down these integrals because each value of n in Eq. (9) gives two successive natural frequencies as a result of the $\pm$ sign before the square root expression. Thus, if H_1 and H_2 are the first two bending modes corresponding to $n = 1$, and H_3 and H_4 are the third and fourth bending modes corresponding to $n = 2$, then it follows from Eq. (6) that $\int_0^1 H_1 H_2\, \mathrm{d}\xi = C_1 C_2/2$, $\int_0^1 H_1 H_3\, \mathrm{d}\xi = 0$, $\int_0^1 H_1 H_4\, \mathrm{d}\xi = 0$, $\int_0^1 H_2 H_3\, \mathrm{d}\xi = 0$, $\int_0^1 H_3 H_4\, \mathrm{d}\xi = C_3 C_4/2$.

The real and imaginary parts of the complex terms A_{11}, A_{12}, A_{21} and A_{22} in Eq. (27) are dependent on the Theodorsen function $C(k)$ (see Eq. (25)), which can be expressed in the following form (see Ref. [26, p. 396]).

$$C(k) = F + \mathrm{i}G, \tag{28}$$

where F and G are real functions of the variable k given by [26–28].

$$F = \frac{\{J_1(J_1 + Y_0) + Y_1(Y_1 - J_0)\}}{\{(J_1 + Y_0)^2 + (Y_1 - J_0)^2\}}, \tag{29}$$

$$G = \frac{-(Y_1 Y_0 + J_1 J_0)}{\{(J_1 + Y_0)^2 + (Y_1 - J_0)^2\}}, \tag{30}$$

and

J_0, J_1, Y_0 and Y_1 are standard Bessel functions of first and second kinds of argument k [26–28].

With the help of Eq. (28) the real and imaginary parts of the terms A_{11}, A_{12}, A_{21} and A_{22} in Eq. (25) can be expressed as

$$A_{11R} = \pi\rho U^2(k^2 + 2kG),$$
$$A_{11I} = -2\pi\rho U^2 kF, \tag{31}$$

$$A_{12R} = \pi\rho U^2 b\{a_h k^2 + 2F - 2kG(0.5 - a_h)\},$$
$$A_{12I} = \pi\rho U^2 b\{k + 2G + 2kF(0.5 - a_h)\}, \tag{32}$$

$$A_{21R} = \pi\rho U^2 b\{kG(1 + 2a_h) + k^2 a_h\},$$
$$A_{21I} = -\pi\rho U^2 bkF(1 + 2a_h), \tag{33}$$

$$A_{22R} = \pi\rho U^2 b^2[2(0.5 + a_h)\{F - kG(0.5 - a_h)\} + \frac{k^2}{8} + k^2 a_h^2],$$
$$A_{22I} = \pi\rho U^2 b^2[2(0.5 + a_h)\{G + kF(0.5 - a_h)\} - k(0.5 - a_h)], \tag{34}$$

where the suffices R and I stand for the real and imaginary parts of the coefficients, respectively.

2.4. Formulation of the flutter problem

Using a classical approach, the flutter determinant is formed from the flutter matrix, and this is formed by summing algebraically the generalized mass, generalized stiffness and the generalized aerodynamic matrices. Thus for a system without structural damping the flutter matrix **[QF]** can be defined as given by Eq. (35). (Structural damping generally has a small effect on the free vibrational mode shapes, and it is not included here.)

$$[\mathbf{QF}]\{\mathbf{q}\} = [-\omega^2[\mathbf{M}] + [\mathbf{k}] - [\mathbf{QA}]]\{\mathbf{q}\}, \tag{35}$$

where **[QA]** is the complex $(n \times n)$ generalized aerodynamic matrix defined in Eqs. (26) and (27), **[M]** and **[K]** are $(n \times n)$ diagonal matrices of generalized mass and generalized stiffness, respectively (with the ith diagonal representing the generalized mass M_i and generalized stiffness K_i), $\{\mathbf{q}\}$ is the column vector of n generalized co–ordinates and ω is the circular frequency in rad/s.

For flutter to occur, the determinant of the complex flutter matrix must be zero so that from Eq. (35)

$$|\mathbf{QF}| = |-\omega^2[\mathbf{M}] + [\mathbf{K}] - [\mathbf{QA}]| = 0. \tag{36}$$

The solution of the flutter determinant can be sought by expanding the above determinant in algebraic form because each of the terms of **[M]**, **[K]** and **[QA]**, and hence each of the elements of **[QF]** is now available in an analytical form.

2.5. Application to classical bending–torsion flutter problem

In the case of classical bending–torsion (binary) flutter of a simply supported suspension bridge, usually two modes are chosen, one of them is bending dominated and the other is torsion dominated. For simplicity it is assumed that these modes are the first two appearing in succession as a result of using $n = 1$ in Eq. (9). (Note that such an assumption is not always necessary.) Thus the two modes which results from $n = 1$ are given as suffices 1 and 2 so that (H_1, Ψ_1) and (H_2, Ψ_2) are the chosen normal modes of vibration.

Thus Eq. (18) gives

$$M_1 = \tfrac{1}{2}\{mC_1^2 + I_\alpha D_1^2 - 2mbx_\alpha C_1 D_1\},$$
$$M_2 = \tfrac{1}{2}\{mC_2^2 + I_\alpha D_2^2 - 2mbx_\alpha C_2 D_2\} \quad (37)$$

and from Eq. (17)

$$K_1 = \omega_1^2 M_1,\ K_2 = \omega_2^2 M_2. \quad (38)$$

The real and imaginary parts of the elements of **[QA]** can be expressed with the help of Eqs. (27), (31)–(34) and (6) as

$$QA_{11R} = \tfrac{1}{2}[A_{11R}C_1^2 + (A_{12R} + A_{21R})C_1 D_1 + A_{22R}D_1^2],$$
$$QA_{11I} = \tfrac{1}{2}[A_{11I}C_1^2 + (A_{12I} + A_{21I})C_1 D_1 + A_{22I}D_1^2], \quad (39)$$

$$QA_{12R} = \tfrac{1}{2}[A_{11R}C_1C_2 + A_{12R}C_2D_1 + A_{21R}C_1D_2 + A_{22R}D_1D_2],$$
$$QA_{12I} = \tfrac{1}{2}[A_{11I}C_1C_2 + A_{12I}C_2D_1 + A_{21I}C_1D_2 + A_{22I}D_1D_2], \quad (40)$$

$$QA_{21R} = \tfrac{1}{2}[A_{11R}C_1C_2 + A_{12R}C_1D_2 + A_{21R}C_2D_1 + A_{22R}D_2D_1],$$
$$QA_{21I} = \tfrac{1}{2}[A_{11I}C_1C_2 + A_{12I}C_1D_2 + A_{21I}C_2D_1 + A_{22I}D_2D_1], \quad (41)$$

$$QA_{22R} = \tfrac{1}{2}[A_{11R}C_2^2 + (A_{12R} + A_{21R})C_2 D_2 + A_{22R}D_2^2],$$
$$QA_{22I} = \tfrac{1}{2}[A_{11I}C_2^2 + (A_{12I} + A_{21I})C_2 D_2 + A_{22I}D_2^2]. \quad (42)$$

Finally, with the help of Eqs. (31)–(34) and (35), the real and imaginary parts of the elements of the 2×2 flutter matrix $[QF]$ can be written as

$$QF_{11R} = K_1 - \omega^2 M_1 - QA_{11R},$$
$$QF_{11I} = -QA_{11I}, \quad (43)$$

$$QF_{12R} = -QA_{12R},$$
$$QF_{12I} = -QA_{12I}, \quad (44)$$

$$QF_{21R} = -QA_{21R},$$
$$QF_{21I} = -QA_{21I}, \quad (45)$$

$$QF_{22R} = K_2 - \omega^2 M_2 - QA_{22R},$$
$$QF_{22I} = -QA_{22I}. \quad (46)$$

For flutter to occur, the real and imaginary parts $F_R(U, \omega)$ and $F_I(U, \omega)$ of the flutter function $F(U, \omega)$, which is in fact the flutter determinant $|\mathbf{QF}|$, must be zero yielding the flutter speed and flutter frequency. Thus for the present case, the condition for flutter is given by

$$F_R(U,\omega)=0,\quad F_I(U,\omega)=0, \tag{47}$$

where

$$F_R(U,\omega)=QF_{11R}QF_{22R}-QF_{11I}QF_{22I}-QF_{12R}QF_{21R}+QF_{12I}QF_{21I} \tag{48}$$

and

$$F_I(U,\omega)=QF_{11R}QF_{22I}+QF_{11I}QF_{22R}-QF_{12R}QF_{21I}-QF_{12I}QF_{21R}. \tag{49}$$

With the analytical expressions for QF_{11R}, QF_{12R}, QF_{21R}, QF_{22R} and QF_{11I}, QF_{12I}, QF_{21I}, QF_{22I} now known, the flutter problem is solved analytically without performing any prior numerical manipulation.

A simple solution technique would be to compute $F_R(U,\omega)$ and $F_I(U,\omega)$ for a range of airspeeds and a range of frequencies and to locate the condition for flutter when $F_R(U,\omega)$ and $F_I(U,\omega)$ are identically zero. In actual practice an airspeed is first chosen and $F_R(U,\omega)$ and $F_I(U,\omega)$ are computed for a range of frequencies and then the process is repeated for a range of airspeeds until both $F_R(U,\omega)$ and $F_I(U,\omega)$ become zero. A computer program is developed to locate the flutter speed and flutter frequency using the above theory. A significant further development of this work that is now possible is to automate the search for U and ω in order to establish the flutter condition. The solution can best be achieved by defining a single (real) flutter function $V(U,\omega)=F_R^2+F_I^2$ and using a suitable algorithm to search for $V(U,\omega)=0$.

3. Scope and limitations of the theory

The literature shows that a wide range of analytical models with varying degrees of complexity has been used to investigate the free vibration and flutter characteristics of suspension bridges. On the one hand, simple representative section models of a suspension bridge having motion in only two degrees of freedom have been employed by both past [29,30] and present [31,32] investigators, which provide considerable insight into the problem. On the other hand, a more detailed analysis using finite element methods enables a suspension bridge to be modelled in its full complexity [3–5,8,12]. This paper is a compromise between these extremes of simplified and comprehensive analysis. In this paper, the structural idealization of the bridge has been modelled as a simply supported beam coupled in bending and torsion whilst Theodorsen-type unsteady aerodynamics are assumed in the aerodynamic idealisation. The structural idealization is simpler than the aerodynamic one in the sense that the beam model is considered to possess equivalent, but representative, values of the bending, torsional and warping rigidities of the complete bridge. Thus the influence of individual components such as cables, hangers and towers are not considered separately. As a consequence the method will not describe exactly the behaviour of a suspension bridge, but its application is reasonably simple because it leads to a system of two coupled governing differential equations of motion which have exact solutions. Such an analysis is useful for parametric studies and for establishing trends, but is restricted in its value because it assumes that the deck, as a beam, alone contributes to the dynamic behaviour of the complete bridge. (Note that the mass and stiffness properties of the component structures like cables and towers can be partially taken into account and incorporated into the deck properties.) It has been

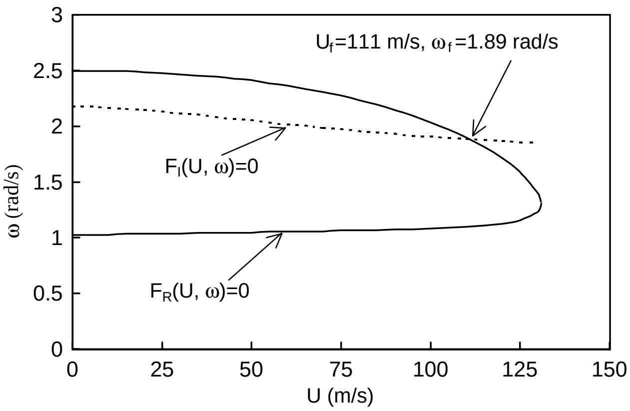

Fig. 3. Solution of the flutter function $F(U,\omega) = 0$ for Innoshima bridge [7,18].

established by a number of investigators [4,5,7,11,12,17,18] that in the complete analysis of a suspension bridge, the interaction between the cables, towers and suspended structures may be important. Although the present analysis is deficient in this respect, it is inherently conservative because it ignores the stiffness contributions from the towers and cables. In future work, the coupled bending–torsion beam theory presented in this paper can be extended further to include the cable equations [33,34], and in the preliminary stage of the theoretical development, the horizontal component of the cable tension induced in the deck can be taken into account.

In the aerodynamic idealization of a bridge the expressions for unsteady lift and moment can be obtained using flutter derivatives [35] as used in studying aircraft flutter. For more reliable and realistic representations of bridge aerodynamics it may be appropriate to obtain these derivatives experimentally [14,29] so that the expressions for unsteady lift and moment can be reformulated accurately. Interested readers are referred to the work of Scanlan who made a detailed comparison of flutter derivatives of a thin aerofoil with those of bridge decks of different cross-sections (see Ref. [35], Fig. 3). Thus the Theodorsen expressions for unsteady lift and moment used in this paper (see Eqs. (24)–(25)) can be replaced by suitably modified expressions based on experimental flutter derivatives. The subsequent flutter analysis can then be carried out by applying the rest of the theory.

4. Numerical results and discussion

In order to demonstrate the effectiveness of the analytical method presented in this paper, three numerical examples are given and the results are discussed as follows.

Example 1. The first example used to illustrate the theory is that of the Innoshima suspension bridge [7,18] located between Honshu and Shikoku in Japan. The data used for the bridge idealized as a beam are as follows:

$$EI = 8.129 \times 10^{13}\ \mathrm{Nm}^2, \qquad GJ = 6.457 \times 10^{11}\ \mathrm{Nm}^2, \qquad E\Gamma = 9.695 \times 10^{15}\ \mathrm{Nm}^4,$$

$$m = 20667\ \mathrm{kg/m}, \qquad I_\alpha = 2.136 \times 10^6\ \mathrm{kgm}, \qquad x_\alpha = 0.0, \qquad a_h = 0.0, \qquad L = 770\ \mathrm{m}, \qquad b = 13\ \mathrm{m}.$$

Using the above data and with the help of Eq. (9), the fundamental bending and torsional natural frequencies were established as $\omega_{B1} = 1.044$ rad/s and $\omega_{T1} = 2.5079$ rad/s, which agree very closely with the corresponding computed natural frequencies quoted in Refs. [7,18], respectively. (Table 2 of Ref. [7] gives the natural period of the first symmetric mode of the bridge in vertical vibration for the hinged connection as 6.0169 s whereas Table 2 of Ref. [18] gives the natural period of the first symmetric mode in torsional vibration for the hinged connection without the effect of gravitational stiffness as 2.5760 s. These periods corresponds to natural frequencies of 1.044 and 2.439 rad/s) The corresponding mode shapes for both these frequencies are half sine waves as given by Eq. (6) with $n = 1$.

The real and imaginary parts of the flutter function $F_R(U,\omega)$ and $F_I(U,\omega)$ given by Eqs. (48) and (49) were computed for a range of frequencies and airspeeds. The density of air is taken to be at its sea level value of 1.225 kg/m^3. The loci of the roots of $F_R(U,\omega)$ and $F_I(U,\omega)$ are shown in Fig. 3. The intersecting point at which both $F_R(U,\omega)$ and $F_I(U,\omega)$, and hence $F(U,\omega) = 0$, gives the flutter speed (U_f) and the flutter frequency (ω_f). These calculated values of U_f and ω_f for the Innoshima Suspension Bridge are 111 m/s and 1.89 rad/s as shown in Fig. 3.

The calculated flutter speed of the Innoshima suspension bridge at 111 m/s (248 mph) is reasonable and safe. (For a suspension bridge, quoted flutter speeds [8] range from 70 to 104 m/s, for example, the calculated flutter speed of Severn bridge in the UK is around 77 m/s [8,11].)

Example 2. The second example used is that of the Jiangyin suspension bridge over the Yangtze River in China [36,37]. The data used for the analysis are taken from [36] and are as follows:

$$EI = 8.365 \times 10^9\ \mathrm{Nm}^2, \qquad GJ = 6.949 \times 10^9\ \mathrm{Nm}^2, \qquad m = 26680\ \mathrm{kg/m},$$

$$I_\alpha = 3.688 \times 10^6\ \mathrm{kgm}, \qquad x_\alpha = 0.0, \qquad a_h = 0.0, \qquad L = 81.2\ \mathrm{m}, \qquad b = 18.45\ \mathrm{m}.$$

The effect of the warping stiffness was not included in the analysis because the data were not available. The fundamental bending and torsional natural frequencies are established at 0.838 and 1.68 rad/s, respectively. The computed flutter speed and frequency using the present theory are 72.5 m/s and 1.28 rad/s. Ref. [37] does not quote the flutter frequency, but gives the flutter speed of the bridge using a full aeroelastic model as 74.4 m/s, which is in close agreement with the result obtained using present theory.

Example 3. The final example used is that of the deck model of the Jiangyin suspension bridge used by the authors of Refs. [36,37]. The data used were taken from Ref. [37, p. 1610] where the data for model 1 are given, and are as follows:

$$EI = 13.52\ \mathrm{Nm}^2, \qquad GJ = 10.54\ \mathrm{Nm}^2, \qquad m = 5.44\ \mathrm{kg/m}, \qquad I_\alpha = 0.1536\ \mathrm{kg\,m},$$

$$x_\alpha = 0.0, \qquad a_h = 0.0, \qquad b = 0.2635\ \mathrm{m}, \qquad L = 1.16\ \mathrm{m}.$$

The computed natural frequencies in fundamental bending and torsional vibration are 11.6 and 22.4 rad/s, respectively. The flutter speed and frequency are obtained as 13.7 m/s and 17.2 rad/s, respectively. The flutter speed of the bridge deck model was established experimentally by wind tunnel test [37] as 13 m/s which agrees very well with the result using present theory.

4.1. The effects of the location of elastic and mass axes on the flutter speed

The theory developed in this paper applies to those cases where the elastic and mass axes are offset from the central geometric axis of the bridge cross-section (see Fig. 1). The non-dimensional parameters a_h and x_α denote the elastic and mass axes locations and are generally expressed as fractions of the semi-width (b) of the bridge as shown in Fig. 1. In all the three numerical examples given above these parameters were, of course, set to zero because the bridge cross-sections were symmetric which is usually the case. However, in order to illustrate the theory for the unsymmetric cases, the effects of these parameters on the flutter speed of a bridge deck are studied. For convenience in the presentation of results, three additional non-dimensional parameters are defined using the usual notation namely, the mass ratio $M = m/(\pi\rho b^2)$, the frequency ratio ω_B/ω_T of the uncoupled fundamental bending (ω_B) and torsional (ω_T) natural frequencies of the bridge deck, and the non-dimensional flutter speed $U_f/(b\omega_T)$. (Note that the uncoupled fundamental bending (ω_B) and torsional (ω_T) natural frequencies of the bridge deck can be calculated from Eqs. (10) and (11)). The variation of the non-dimensional flutter speed $U_f/(b\omega_T)$ with the elastic axis location a_h is shown in Fig. 4 for a bridge deck with $M = 20$, $\omega_B/\omega_T = 0.5$ and $x_\alpha = 0.1$. As in the case of aircraft wings when a_h assumes negative values so that

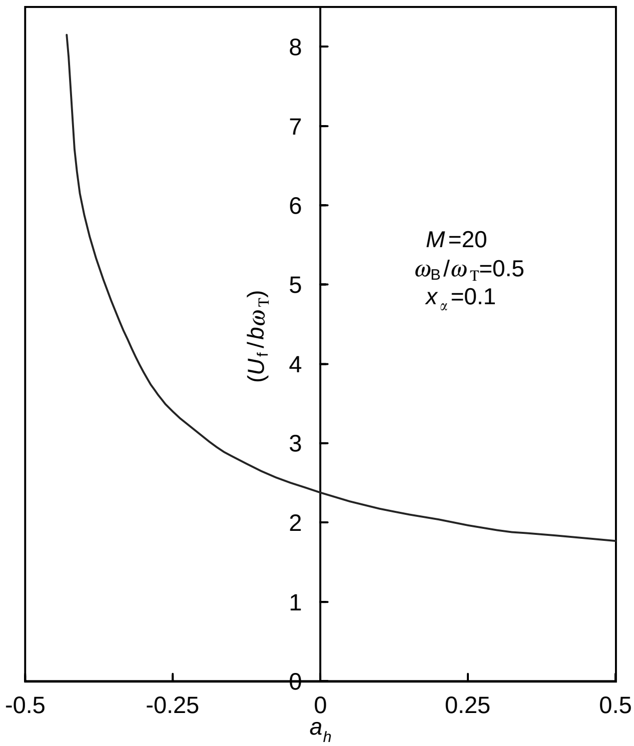

Fig. 4. The effects of elastic and mass axes offset on the flutter speed of a bridge deck.

the elastic axis is forward of the mid-chord, the flutter speed increases. Such a configuration results in a reduction of aerodynamic coupling. When a_h approaches -0.5 the elastic axis moves close to the quarter-chord point which is effectively the aerodynamic centre of the bridge deck so that the aerodynamic coupling reduces almost to zero, resulting in exceptionally high flutter speeds. In contrast, when a_h increases by assuming positive values, the flutter speed decreases because of the increased aerodynamic coupling. This is clearly evident in Fig. 4.

5. Conclusions

An analytical method for the free vibration and flutter analysis of bridge decks is presented by deriving explicitly each term required for the whole analysis. This was achieved by extensive algebraic simplifications to relate the expressions for generalized mass, generalized stiffness and generalized unsteady aerodynamic terms. The method is free from ill-conditioning problems usually associated with complex (numerical) matrix manipulation. Using the proposed method, the flutter speed and frequency of three illustrative examples of bridge decks have been demonstrated. In future work the current theory can be developed further to include the effects of cables, towers and other supporting structures.

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References

[1] D.B. Steinman, Modes and natural frequencies of suspension bridge oscillations, Journal of the Franklin Institute 268 (3) (1959) 148–174.
[2] J. Vellozzi, Vibration of suspension bridges under moving loads, Journal of the Structural Division, American Society of Civil Engineers 93 (ST4) (1967) 123–138.
[3] A.M. Abdel-Ghaffar, Free lateral vibrations of suspension bridges, Journal of the Structural Division, American Society of Civil Engineers 104 (ST3) (1978) 503–525.
[4] A.M. Abdel-Ghaffar, Free torsional vibrations of suspension bridges, Journal of the Structural Division, American Society of Civil Engineers 105 (ST4) (1979) 767–788.
[5] A.M. Abdel-Ghaffar, Vertical vibration analysis of suspension bridges, Journal of the Structural Division, American Society of Civil Engineers 106 (ST10) (1980) 2053–2075.
[6] F. Van Der Woude, Natural oscillations of suspension bridges, Journal of the Structural Division, American Society of Civil Engineers 108 (ST8) (1982) 1815–1830.
[7] T. Hayashikawa, N. Watanabe, Vertical vibration in Timoshenko beam suspension bridges, Journal of Engineering Mechanics 110 (3) (1984) 341–356.
[8] T.J.A. Agar, The analysis of aerodynamic flutter of suspension bridges, Computers and Structures 30 (3) (1988) 593–600.
[9] T. Miyata, H. Yamada, Coupled flutter estimate of a suspension bridge, Journal of Wind Engineering and Industrial Aerodynamics 33 (1990) 341–348.

[10] R.H. Scanlan, N.P. Jones, Aeroelastic analysis of cable-stayed bridges, Journal of Structural Engineering 116 (2) (1990) 279–297.
[11] V. Tinh, Analysis of damped flutter of suspension bridges, Computers and Structures 41 (5) (1991) 875–886.
[12] T.J.A. Agar, Dynamic instability of suspension bridges, Computers and Structures 41 (6) (1991) 1321–1328.
[13] H. Kobayashi, H. Nagaoka, Active control of Flutter of a suspension bridge, Journal of Wind Engineering and Industrial Aerodynamics 41-44 (1992) 143–151.
[14] A. Zasso, A. Cigada, S. Negri, Flutter derivatives identification through full bridge aeroelastic model transfer function analysis, Journal of Wind Engineering and Industrial Aerodynamics 60 (1996) 17–33.
[15] L. Singh, N.P. Jones, R.H. Scanlan, O. Lorendeaux, Identification of lateral flutter derivatives of bridge decks, Journal of Wind Engineering and Industrial Aerodynamics 60 (1996) 81–89.
[16] S. Preidikman, D.T. Mook, A new method for actively suppressing flutter of suspension bridges, Journal of Wind Engineering and Industrial Aerodynamics 69-71 (1997) 955–974.
[17] J.M.W. Brownjohn, Vibration characteristics of a suspension footbridge, Journal of Sound and Vibration 202 (1) (1997) 29–46.
[18] T. Hayashikawa, Torsional vibration analysis of suspension bridges with gravitational stiffness, Journal of Sound and Vibration 204 (1) (1997) 117–129.
[19] J.G. Beith, A practical engineering method for the flutter analysis of long span bridges, Journal of Wind Engineering and Industrial Aerodynamics 77-78 (1998) 357–366.
[20] Y.A. Rossikhin, M.V. Shitikova, Application of fractional calculus for analysis of nonlinear damped vibrations of suspension bridges, Journal of Engineering Mechanics 124 (9) (1998) 1029–1036.
[21] G. Holubova-Tajcova, Mathematical Modeling of suspension bridges, Mathematics and Computers in Simulation 50 (1999) 183–197.
[22] H. Katsuchi, N.P. Jones, R.H. Scanlan, Multimode coupled flutter and buffeting analysis of the Akashi-Kaikyo bridge, Journal of Structural Engineering 125 (1) (1999) 60–70.
[23] T. Theodorsen, General theory of aerodynamic instability and mechanisms of flutter, NACA Technical Report 496 (1934).
[24] J.R. Banerjee, S. Guo, W.P. Howson, Exact dynamic stiffness matrix of a bending-torsion coupled beam including warping, Computers and Structures 59 (1996) 613–621.
[25] R.E.D. Bishop, W.G. Price, Coupled bending and twisting of a Timoshenko beam, Journal of Sound and Vibration 50 (1977) 469–477.
[26] R.H. Scanlan, R. Rosenbaum, Introduction to the Study of Aircraft Vibration and Flutter, Dover, New York, 1968.
[27] Y.C. Fung, An Introduction to the Theory of Aeroelasticity, Dover, New York, 1969.
[28] R.L. Bisplinghoff, H. Ashley, R.L. Halfman, Aeroelasticity, Dover, New York, 1996.
[29] R.H. Scanlan, J.J. Tomko, Aerofoil and bridge deck flutter derivatives, Journal of the Engineering Mechanics Division, American Society of Civil Engineers 97 (EM6) (1971) 1717–1737.
[30] R.H. Scanlan, R.H. Gade, Motion of suspension bridge spans under gusty wind, Journal of the Structural Division, American Society of Civil Engineers 103 (ST9) (1977) 1867–1883.
[31] P.P. Sarkar, N.P. Jones, R.H. Scanlan, Indentification of aeroelastic parameters of flexible bridges, Journal of the Engineering Mechanics, American Society of Civil Engineers 120 (8) (1994) 1718–1742.
[32] K. Wilde, Y. Fujino, J. Masukawa, Time domain modeling of bridge deck flutter, Structural Engineering/Earthquake Engineering, JSCE 13 (2) (1996) 93s–104s.
[33] H.M. Irvine, T.K. Caughey, The linear theory of free vibrations of a suspended cable, Proceedings of the Royal Society, London, Series A 341 (1974) 299–315.
[34] J. Kim, S.P. Chang, Dynamic stiffness matrix of an inclined cable, Engineering Structures 23 (2001) 1614–1621.
[35] R.H. Scanlan, The action of flexible bridges under wind, I: flutter theory, Journal of Sound and Vibration 60 (2) (1978) 187–199.
[36] M. Gu, R. Zhang, H. Xiang, Identification of flutter derivatives of bridge decks, Journal of Wind Engineering and Industrial Aerodynamics 84 (2000) 151–162.
[37] M. Gu, R. Zhang, H. Xiang, Parametric study on flutter derivatives of bridge decks, Engineering Structures 23 (2001) 1607–1613.