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Analysis of a non-linear structure by considering two non-linear formulations

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Abstract

In recent years, modal synthesis methods have been extended for solving non-linear dynamic problems subjected to harmonic excitation. These methods are based on the notion of non-linear or linearized modes and exploited in the case of structures affected by localized non-linearity. Actually, the experimental tests executed on non-linear structures are time consuming, particularly when repeated experimental tests are needed. It is often preferable to consider new non-linear methods with a view to decrease significantly the number of attempts during prototype tests and improving the accuracy of the dynamic behaviour.

This article describes two fundamental non-linear formulations based on two different strategies. The first formulation exploits the eigensolutions of the associated linear system and the dynamics characteristics of each localized non-linearity. The second formulation is based on the exploitation of the linearized eigensolutions obtained using an iterative process. This article contains a numerical and an experimental study which examines the non-linear behaviour of the structure affected by localized non-linearities. The study is intended to validate the numerical algorithm and to evaluate the problems arising from the introduction of non-linearities. The complex responses are evaluated using the iterative Newton–Raphson method and for a series of discrete frequencies. The theory has been applied to a bi-dimensional structure and consists of evaluating the harmonic responses obtained using the proposed formulations by comparing measured and calculated transfer functions.

1. Introduction

The non-linear analysis of certain mechanical structures has been developed over several years, although the difficulties encountered in numerical computation or in practical acquisition system. Several methods have been developed for the dynamic analysis of non-linear structures [1–10].

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These structures are often characterized by the presence of some localized non-linearities which are restricted to a small parts of the structure. An experimental study shows that each non-linearity is characterized by non-linear parameters of stiffness and damping. Therefore, the analysis of the global behaviour of this type of structure can be only evaluated by a numerical algorithm.

The main purpose of this paper is to develop two non-linear formulations based on two different strategies in view of testing mechanical structures affected by localized non-linearities. In one approach, the non-linear system is formulated by exploiting the eigensolutions of the associated conservative linear system and the stiffness and the damping characteristics of each localized non-linearity. In the other approach, the non-linear system is obtained using the eigensolutions of a linearized eigenvalue problem in which the iterative process conducts, respectively, to a stationary coefficients of the linearized spectrum and modal matrices.

Many alternative methods of calculations can be envisaged in the resolution of the non-linear problems (Newton–Raphson method, the incremental harmonic balance (IHB), method, etc). Many authors have applied the IHB method to various problems in non-linear dynamics. Pierre et al. [1] proposed a multi-harmonic analysis of a dry friction-damped system using the IHB method. Ferri [2] showed the equivalence of the IHB method and the harmonic balance Newton–Raphson method. Cheung and Iu [3] presented a development of a simple algorithm for the implementation of the harmonic balance method for solving a non-linear dynamic systems. Jezequel et al. [4] proposed a non-linear synthesis in the frequency domain by using the Ritz–Galerkin–Newton–Raphson method and also the IHB method. Friswell and Penny [5] proposed the iterative Newton–Raphson method for solving the sets of non-linear equations.

Note that the different methods named above can be exploited to compare the solutions obtained with the two proposed non-linear formulations. In the following study, the stationary solution of each non-linear system is obtained using the Newton–Raphson method and by considering the fundamental harmonic of the solutions. This article describes the experimental technique for accurately evaluating the non-linear parameters of stiffness and damping. In this study, the calculation of the stationary solution and the iterative algorithms will be presented. The data acquisition system will also be described.

In order to illustrate the efficiency of the proposed methods, a numerical application to a bi-dimensional articulated beam system will be presented. The effect of a localized non-linearities will be characterized by performing an increasing and/or decreasing sine-sweep frequency and by a comparison between theoretical and experimental results.

Finally, the experimental set-up is trying to model. In fact, the comparison between numerical and experimental results is intended to choose and validate the proposed non-linear formulations and the numerical resolution algorithms and to evaluate the difficulties due to the introduction of localized non-linearities.

2. General formulation of the non-linear problem

The general equation representing the behaviour of a N-degree-of-freedom (d.o.f) non-linear structure is represented by the differential equation

$$[\mathbf{M}]\ddot{\mathbf{y}}(t) + [\mathbf{B}]\dot{\mathbf{y}}(t) + [\mathbf{K}]\mathbf{y}(t) + f_{nl}(\mathbf{y}(t), \dot{\mathbf{y}}(t)) = \mathbf{F}(t), \tag{1}$$

where

$[\mathbf{M}], [\mathbf{B}], [\mathbf{K}] \in \mathbf{R}^{N,N}$ are, respectively, the mass, damping and stiffness matrices which are real, symmetric, positive definite and $f_{nl}(\mathbf{y}(t), \dot{\mathbf{y}}(t))$ represents the non-linear forces of stiffness and damping in the time domain. The particular solution $\mathbf{y}(t)$ resulting from the exterior force $\mathbf{F}(t)$, is written in the harmonic form (approximation of the first harmonic) as

$$\mathbf{y}(t) = \mathbf{y}e^{st}, \quad \mathbf{F}(t) = \mathbf{F}e^{st}; \quad s = j\omega.$$

The differential system (1) is written in the frequency domain and given by the linearized form

$$(-\omega^2[\mathbf{M}] + j\omega[\mathbf{B}] + [\mathbf{K}])\mathbf{y}(\omega) + f_{nl}(\mathbf{y}, \omega) = F(\omega), \tag{2}$$

where $\mathbf{y}(\omega)$, and $\mathbf{F}(\omega)$ are, respectively, the frequency response vector and the force vector.

The non-linear force vector $f_{nl}(\mathbf{y}, \omega)$ is expressed as a function of the linearized stiffness and damping matrices (approximation of the first harmonic) as

$$f_{nl}(\mathbf{y}, \omega) = [\Delta\mathbf{K}_{nl}(\mathbf{y}, \omega)] + j\omega[\Delta\mathbf{B}_{nl}(\mathbf{y}, \omega)]. \tag{3}$$

System (2) is then written in the harmonic form

$$(-\omega^2[\mathbf{M}] + j\omega[\mathbf{B}_{nl}(\mathbf{y}, \omega)] + [\mathbf{K}_{nl}(\mathbf{y}, \omega)])\mathbf{y}(\omega) = \mathbf{F}(\omega). \tag{4}$$

Each coefficient of the matrix $[\mathbf{K}_{nl}(\mathbf{y}, \omega)]$, $[\mathbf{B}_{nl}(\mathbf{y}, \omega)]$ depends on the amplitude of each non-linear element and the pulsation ω.

2.1. Expression of the linearized stiffness matrix

2.1.1. Linearized stiffness matrix

The linearized stiffness matrix $[\mathbf{K}_{nl}(\mathbf{y}, \omega)]$ can be written in the form

$$[\mathbf{K}_{nl}(\mathbf{y}, \omega)] = [\mathbf{K}] + [\Delta\mathbf{K}_{nl}(\mathbf{y}, \omega)], \tag{5}$$

where $[\Delta\mathbf{K}_{nl}]$ represents the linearized stiffness matrix.

The matrix $[\Delta\mathbf{K}_{nl}(\mathbf{y}, \omega)]$ is expressed as a function of both the linearized stiffness coefficients $(k_{nl}^{10}, k_{nl}^{23})$ and the associated linear stiffness coefficients (k_{10}, k_{23}). The latter depends on the pulsation ω and the amplitudes of the beams $(\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3)$.

2.1.2. Linearized damping matrix

Similarly, one can now obtain the linearized damping matrix

$$[\mathbf{B}_{nl}(\mathbf{y}, \omega)] = [\mathbf{B}] + [\Delta\mathbf{B}_{nl}(\mathbf{y}, \omega)], \tag{6}$$

where the matrix $[\Delta\mathbf{B}_{nl}(\mathbf{y},\omega)]$ is expressed as a function of the linearized damping coefficients $(b_{nl}^{10}, b_{nl}^{23})$.The latter depends on the pulsation ω as well as the amplitudes of the beams $(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$.
The structure of the associated linear system is not damped ($[\mathbf{B}] = [0]$), this is

$$[\mathbf{B}_{nl}(\mathbf{y},\omega)] = [\Delta\mathbf{B}_{nl}(\mathbf{y},\omega)]. \tag{7}$$

2.2. Proposed non-linear formulations

Two types of formulations of the non-linear problem will be presented, one formulation based on the exploitation of the eigensolutions of the associated conservative linear system $(\mathbf{Y}, \mathbf{\Lambda})$ and the linearized parameters of each localized non-linearity, the second formulation based on the exploitation of the linearized eigensolutions $(\hat{\mathbf{Y}}, \hat{\mathbf{\Lambda}})$ of the associated conservative non-linear system.

2.2.1. Formulation based on the exploitation of the linear eigensolutions

The first technique proposed to formulate the non-linear problem contains the following steps:

(1) a projection of the solution of the non-linear system on the Ritz basis $\mathbf{Y}$ of the associated conservative system as $\mathbf{y}(\omega) = \mathbf{YC}(\omega)$;
(2) resolution of the non-linear system by the Newton–Raphson methods.

2.2.1.1. Formulation of the associated linear problem. The general equation representing the behaviour of the associated linear system is given by its harmonic form as

$$\lfloor(-\omega^2[\mathbf{M}] + \mathrm{j}\omega\mathbf{B} + \mathbf{K}\rfloor\mathbf{y}_1(\omega) = \mathbf{F}(\omega). \tag{8}$$

The general force vector $\mathbf{F}(\omega)$ and the displacement vector $\mathbf{y}_1(\omega)$ are, respectively, given by

$$\mathbf{F}(\omega) = [\mathbf{0\ 0\ 0\ af}(\omega)]^{\mathrm{T}} \quad \mathbf{y}_1(\omega) = [\theta_1\ \theta_2\ \theta_3\ \theta_4]^{\mathrm{T}}.$$

Particular case. Since the d.o.f. θ_4 is imposed in the proposed practical strategy, one can obtain a reduced linear system (8′) of unknowns $(\theta_1, \theta_2, \theta_3)$ and a constraint Eq. (9):

$$[-\omega^2\mathbf{M} + \mathrm{j}\omega\mathbf{B} + \mathbf{K}]\mathbf{y}_1(\omega) = \mathbf{F}(\omega), \tag{8′}$$

$$\mathrm{s}^2\mathbf{I}_{oz}\theta_4 + sl^2(-k_{34}\theta_3 + (k_{34} + k_{40} + k_p)\theta_4) = \mathbf{af}(\omega), \tag{9}$$

$[\mathbf{M}], [\mathbf{B}], [\mathbf{K}] \in \mathbf{R}^{N',N'}$ are the reduced mass, stiffness and damping matrices of size $(N', N'), N' = N - 1$ The displacement $\mathbf{y}_1(\omega)$ and the force vectors $\mathbf{F}(\omega)$ are given by

$$\mathbf{y}_1(\omega) = [\theta_1\ \theta_2\ \theta_3]^{\mathrm{T}}, \quad \mathbf{F}(\omega) = [\mathbf{0\ 0}\ l^2k_{34}\theta_4]^{\mathrm{T}}.$$

The resolution of the linear system (8′) leads to the angular displacement $(\theta_1, \theta_2, \theta_3)$. Eq. (9) leads to the force applied on the beam B_4.

Eigensolutions of the linear system

The eigensolutions $[\mathbf{\Lambda}], [\mathbf{Y}]$ are derived from the associated conservative linear system eigenvalue problem (10) defined by

$$\lfloor -\omega^2\mathbf{M} + \mathbf{K} \rfloor \mathbf{y} = 0. \tag{10}$$

These solutions verify the orthonormality relations

$$[\mathbf{Y}]^{\mathrm{T}}[\mathbf{M}][\mathbf{Y}] = [\mathbf{I}], \quad [\mathbf{Y}]^{\mathrm{T}}[\mathbf{K}][\mathbf{Y}] = [\mathbf{\Lambda}]. \tag{11}$$

2.2.1.2. Formulation of the non-linear system. The structure of the associated linear system is supposed undamped. From relationship (11) and the premultiplication by $\mathbf{Y}^{\mathrm{T}}$, the non-linear system (4) becomes

$$(-\omega^2[\mathbf{I}] + [\mathbf{\Lambda}] + \mathrm{j}\omega\mathbf{Y}^{\mathrm{T}}[\Delta\mathbf{B}_{nl}(\mathbf{y},\omega)]\mathbf{Y} + \mathbf{Y}^{\mathrm{T}}[\Delta\mathbf{K}_{nl}(\mathbf{y},\omega)]\mathbf{Y})\mathbf{C} = \mathbf{Y}^{\mathrm{T}}\mathbf{F}. \tag{12}$$

The expressions of the matrices $[\Delta\mathbf{B}_{nl}(\mathbf{y},\omega)]$, $[\Delta\mathbf{K}_{nl}(\mathbf{y},\omega)]$ can be transformed, respectively, to the linearized generalized damping matrix β_{nl} and the linearized generalized stiffness matrix $\tilde{\mathbf{K}}_{nl}$:

$$\boldsymbol{\beta}_{nl} = \mathbf{Y}^{\mathrm{T}}[\Delta\mathbf{B}_{nl}(\mathbf{y},\omega)]\mathbf{Y}, \tag{13}$$

$$\tilde{\mathbf{K}}_{nl} = \mathbf{Y}^{\mathrm{T}}[\Delta\mathbf{K}_{nl}(\mathbf{y},\omega)]\mathbf{Y}, \tag{14}$$

The generalized co-ordinate vector is expressed by its real and imaginary parts:

$$\mathbf{C} = \mathbf{C}_r + \mathrm{j}\mathbf{C}_i. \tag{15}$$

By separating real and imaginary parts, system (12) of three complex equations of three unknowns becomes a real system of six real unknowns $\mathbf{C}_r$ and $\mathbf{C}_i$

$$\begin{aligned}&(-\omega^2[\mathbf{I}] + [\mathbf{\Lambda}] + [\tilde{\mathbf{K}}_{nl}(\mathbf{y},\omega)])\mathbf{C}_r - [\omega\boldsymbol{\beta}_{nl}(\mathbf{y},\omega)]\mathbf{C}_i = \mathbf{Y}^{\mathrm{T}}\mathbf{F},\\ &[\omega\boldsymbol{\beta}_{nl}(\mathbf{y},\omega)]\mathbf{C}_r + (-\omega^2[\mathbf{I}] + [\mathbf{\Lambda}] + [\tilde{\mathbf{K}}_{nl}(\mathbf{y},\omega)])\mathbf{C}_i = 0.\end{aligned} \tag{16}$$

The non-linear system is presented by its matrix form

$$[\mathbf{L}_{nl}(\mathbf{y},\omega)] \cdot \mathbf{x} = \mathbf{b}, \tag{17}$$

where

$$[\mathbf{L}_{nl}(\mathbf{x},\omega)] = \begin{bmatrix} -\omega^2[\mathbf{I}] + [\mathbf{\Lambda}] + [\tilde{\mathbf{K}}_{nl}(\mathbf{y},\omega)] & -[\omega\boldsymbol{\beta}_{nl}(\mathbf{y},\omega)] \\ [\omega\boldsymbol{\beta}_{nl}(\mathbf{y},\omega)] & -\omega^2[\mathbf{I}] + [\mathbf{\Lambda}] + [\tilde{\mathbf{K}}_{nl}(\mathbf{y},\omega)] \end{bmatrix} \tag{18}$$

and

$$\mathbf{x} = [\,\mathbf{C}_r\ \mathbf{C}_i]^{\mathrm{T}} \quad \mathbf{b} = [\,\mathbf{Y}^{\mathrm{T}}\mathrm{F}\ \mathbf{0}]^{\mathrm{T}}. \tag{19}$$

Note that the non-linear response $\mathbf{y}$ is a function of the non-linear solution $\mathbf{x}$.

2.2.2. Formulation based on the exploitation of the basis $\hat{\mathbf{Y}}$

The general equation of behaviour of a non-linear structure (1) is considered by formulating and resolving the linearized conservative eigenvalues system. The eigensolutions obtained are used to reach the stationary harmonic solutions.

2.2.2.1. Eigensolutions of the associated conservative linearized system. Each iteration to the eigensolutions $(\hat{\mathbf{Y}}; [\hat{\mathbf{\Lambda}}_{nl}])$ resolves the linearized system.

$$\{-\omega^2[\mathbf{M}] + [\mathbf{K} + \Delta\mathbf{K}_{nl}(\mathbf{y}, \omega)]\}\hat{\mathbf{y}} = 0 \tag{20}$$

The linearized eigenvectors $\hat{\mathbf{Y}}$ are normalized at each iteration utilizing the mass matrix [**M**], and these eigensolutions satisfy the relations of norms

$$\hat{\mathbf{Y}}^{\mathrm{T}}[\mathbf{M}]\hat{\mathbf{Y}} = [\mathbf{I}], \quad [\hat{\mathbf{\Lambda}}_{nl}] = \hat{\mathbf{Y}}^{T}[\hat{\mathbf{K}}_{nl}(\mathbf{y}, \omega)]\hat{\mathbf{Y}}, \tag{21}$$

where

$$[\hat{\mathbf{K}}_{nl}(\mathbf{y}, \omega)] = [\mathbf{K} + \Delta\mathbf{K}_{nl}(\mathbf{y}, \omega)].$$

When $\hat{\mathbf{Y}}$ is not normalized using the mass matrix, the relations of norms then becomes

$$\hat{\mathbf{Y}}_n^{\mathrm{T}}[\mathbf{M}]\hat{\mathbf{Y}}_n = [\hat{\mathbf{m}}], \quad [\hat{\mathbf{\Lambda}}_n] = \hat{\mathbf{Y}}_n^{\mathrm{T}}[\hat{\mathbf{K}}_{nl}(\mathbf{y}, \omega)]\hat{\mathbf{Y}}, \tag{22}$$

where $[\hat{\mathbf{m}}]$ and $[\hat{\mathbf{\Lambda}}_n]$ are, respectively, the generalised mass and stiffness matrices.These generalized matrices $[\hat{\mathbf{\Lambda}}_{nl}]$ and $[\hat{\mathbf{\Lambda}}_n]$, verify the relation

$$[\hat{\mathbf{\Lambda}}_n] = [\mathbf{P}]^{\mathrm{T}}[\hat{\mathbf{\Lambda}}_{nl}][\mathbf{P}], \tag{23}$$

where the matrix [**P**] verifies the relation between the two basis $[\hat{\mathbf{Y}}] = [\hat{\mathbf{Y}}][\mathbf{P}]$

2.2.2.2. Formulation of the non-linear system. A projection of the stationary solution on the base of the eigenvectors of the non-linear system is given by

$$\mathbf{y}(\omega) = \hat{\mathbf{Y}}\hat{\mathbf{C}}(\omega), \tag{24}$$

where

$$\hat{\mathbf{C}}(\omega) = \hat{\mathbf{C}}_r(\omega) + j\hat{\mathbf{C}}_i(\omega).$$

Explicitly one obtains:

$$\mathbf{y}^i(\omega) = \sum_{l=1}^{m} \hat{\mathbf{Y}}_{il}\hat{\mathbf{C}}_l(\omega). \tag{25}$$

$\hat{\mathbf{Y}}$, the linearized base of the eigenvectors is expressed at each iteration as a function of the excitation frequency and the amplitude of movement of the non-linear elements, $\hat{\mathbf{C}}(\omega)$ is the generalised co-ordinate vector.

The substitution of Eq. (24) in relationship (4) and the premultiplication by $\hat{\mathbf{Y}}^{\mathrm{T}}$ leads to the non-linear system written in accordance with the generalized co-ordinate system, as

$$\{-\omega^2\hat{\mathbf{Y}}^{\mathrm{T}}[\mathbf{M}]\hat{\mathbf{Y}} + \mathrm{j}\omega\hat{\mathbf{Y}}^{\mathrm{T}}[\mathbf{B} + \Delta\mathbf{B}_{nl}]\hat{\mathbf{Y}} + \hat{\mathbf{Y}}^{\mathrm{T}}[\mathbf{K} + \Delta\mathbf{K}_{nl}]\hat{\mathbf{Y}}\}\hat{\mathbf{C}}(\omega) = \hat{\mathbf{Y}}^{\mathrm{T}}\mathbf{F}. \tag{26}$$

As the structure of the associated linear system is conservative, the non-linear system becomes

$$\{-\omega^2[\mathbf{I}] + \mathrm{j}\omega\hat{\mathbf{Y}}^{\mathrm{T}}[\Delta B_{nl}]\hat{\mathbf{Y}} + \hat{\mathbf{Y}}^{\mathrm{T}}[\mathbf{K} + \Delta\mathbf{K}_{nl}]\hat{\mathbf{Y}}\}\hat{\mathbf{C}} = \hat{\mathbf{Y}}^{\mathrm{T}}\mathbf{F}. \tag{27}$$

The non-linear system (27) can be written in the form

$$[\mathbf{L}_{nl}(\hat{\mathbf{x}}, \omega)]\hat{\mathbf{x}} = \mathbf{d}(\omega), \tag{28}$$

where

$$\hat{\mathbf{x}} = [\hat{\mathbf{C}}_r\ \hat{\mathbf{C}}_i]^{\mathrm{T}}, \quad \mathbf{d}(\omega) = [\hat{\mathbf{Y}}^{\mathrm{T}}\mathbf{F}\ \mathbf{0}]^{\mathrm{T}}. \tag{29}$$

The linearized function $[\mathbf{L}_{nl}(\hat{\mathbf{x}}, \omega)]$ defined by expression (30) is evaluated at each iteration and for each value of ω with respect to the relations of norms:

$$[\mathbf{L}_{nl}(\hat{\mathbf{x}}, \omega)] = \begin{bmatrix} -\omega^2[\mathbf{I}] + [\hat{\mathbf{\Lambda}}_{nl}(\mathbf{y}, \omega)] & -\omega[\hat{\beta}_{nl}(\mathbf{y}, \omega)] \\ \omega[\hat{\beta}_{nl}(\mathbf{y}, \omega)] & -\omega^2[\mathbf{I}] + [\hat{\mathbf{\Lambda}}_{nl}(\mathbf{y}, \omega)] \end{bmatrix}, \tag{30}$$

where

$$[\hat{\beta}_{nl}(\mathbf{y}, \omega)] = \hat{\mathbf{Y}}^{\mathrm{T}}[\Delta\mathbf{B}_{nl}(\mathbf{y}, \omega)]\hat{\mathbf{Y}}. \tag{31}$$

$$[\hat{\mathbf{\Lambda}}_{nl}(\mathbf{y}, \omega)] = \hat{\mathbf{Y}}^{\mathrm{T}}[\hat{\boldsymbol{K}}_{nl}(\mathbf{y}, \omega)]\hat{\mathbf{Y}}. \tag{32}$$

The non-linear response $\mathbf{y}(\omega)$ is function of the solution $\hat{\mathbf{x}}$. In order to obtain a stationary solution, two iterative resolution algorithms of resolution will be applied.

3. Characteristics of the non-linear elements

The experimental technique consists of applying the displacement $\boldsymbol{D}$ (relation 33) at one side of the non-linear support to obtain a sinusoidal force f defined by expression (34). This force depends on ω and is characterized by the amplitude fo and a phase φ toward

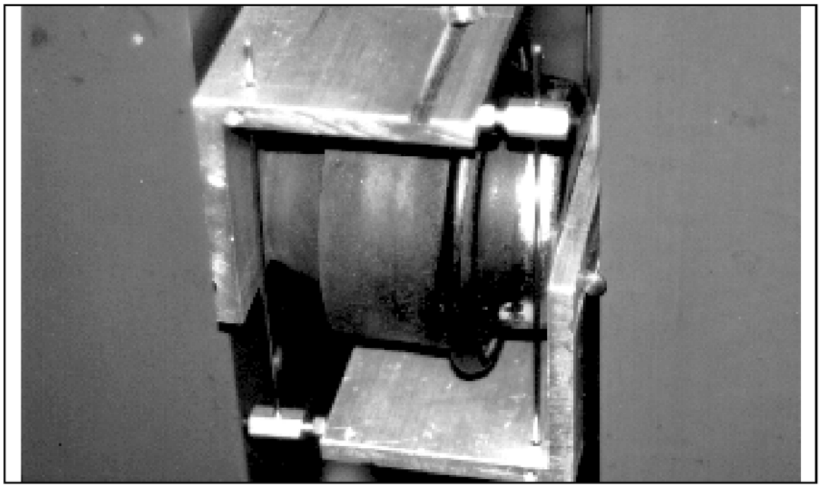


Fig. 1. Photograph of a non-linear support.

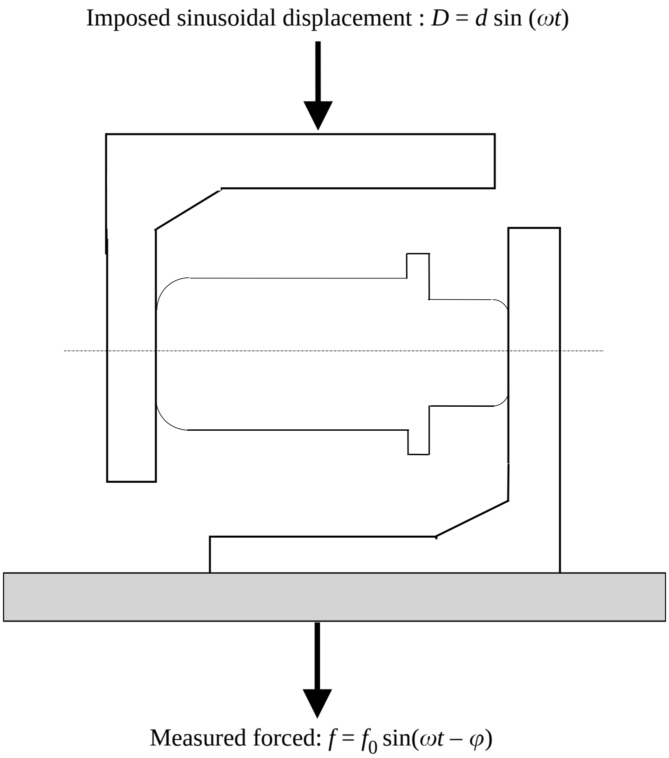


Fig. 2. Experimental model of a non-linear support.

$\boldsymbol{D}$ (Figs. 1 and 2).

$$\boldsymbol{D} = \mathbf{d} \sin(\omega t), \tag{33}$$

$$f = f_o \sin(\omega t - \varphi). \tag{34}$$

The stiffness complex method leads respectively, to the expression of the stiffness coefficient k_{nl} and that of the damping coefficient b_{nl} of the non-linear support. These are a function of the amplitude of the force f_0 and the phase φ

$$k_{nl} = (f_o/d)\cos(\varphi), \quad b_{nl} = (1/\omega)\frac{f_o}{d}\sin(\varphi). \tag{35}$$

4. Resolution of the non-linear system

Each non-linear system is resolved iteratively. One procedure consists of exploiting the eigen solutions of the original structure, the stiffness and the damping coefficients of the modified parts and the exterior force due to an imposed displacement. At each iteration step, the linearized stiffness and damping coefficients can be evaluated for each of the excitation frequencies. This estimation is achieved through a linear interpolation using the amplitudes established at the end of the previous iteration step. The generalized matrices ($[\boldsymbol{\beta}_{nl}], [\tilde{\mathbf{K}}_{nl}]$) are then evaluated and the non-linear system is resolved. A stationary solution is obtained if the convergence criteria of the iterative process is met. If this is not the case, the linearized parameters are corrected and the iterative process is continued to convergence.

If the second non-linear formulation is considered, the non-linear system is also resolved iteratively. The coefficients of the linearized eigensolutions are corrected with an iterative process identical to the first one.

4.1. Iterative procedure

The non-linear system obtained by the first or the second non-linear formulation is given by

$$\mathbf{f}(\mathbf{x}, \omega) = [\mathbf{L}_{nl}(\mathbf{x}, \omega)] \cdot \mathbf{x} - \mathbf{b}(\omega). \tag{36}$$

This non-linear system (36) is of the type

$$\left\{ \begin{array}{l} f_1(x_1, \cdots x_v, \cdots x_{2m}) = 0 \\ f_{2m}(x_1, \cdots x_v, \cdots x_{2m}) = 0 \end{array} \right\}, \quad v = 1 \rightarrow 2m.$$

$(x_1, \cdots x_v, \cdots x_{2m}), (f_1, \cdots f_v, \cdots f_{2m})$ are, respectively, the real unknowns and the real functions of $2m$ variables.

The iterative method of Newton–Raphson is exploited to solve the resulting algebraic non-linear system. A Taylor expansion of the first order gives the iterative solution $\mathbf{x}^{t+1}$ at the iteration $t+1$ which verifies the relation

$$f_i(\mathbf{x}^t) + \sum_{j=1}^{2m} (\mathbf{x}_j^{t+1} - \mathbf{x}_j^t) \frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_j}(\mathbf{x}^t) = 0 \quad (i = 1 \rightarrow 2m). \tag{37}$$

From an initial estimation of the solution $\mathbf{x}^t$, the new estimation $\mathbf{x}^{t+1}$ given at the iteration $t+1$ is expressed by $\mathbf{x}^{t+1} = \mathbf{x}^t + \Delta\mathbf{x}^t$. The non-linear system is then written in the form

$$[\nabla \mathbf{f}(\mathbf{x}^t, \omega)]\Delta x^t + \mathbf{f}((\mathbf{x}^t, \omega) = 0, \tag{38}$$

where $\mathbf{f}(\mathbf{x}^t, \omega) = [\mathbf{L}_{nl}(\mathbf{x}^t, \omega] \cdot \mathbf{x}^t - \mathbf{b}(\omega)$ (first non-linear formulation), $\mathbf{f}(\hat{\mathbf{x}}^t, \omega) = [\hat{\mathbf{L}}_{nl}(\hat{\mathbf{x}}^t, \omega] \cdot \mathbf{x}^t - \mathbf{d}(\omega)$ (second non-linear formulation). This system has solutions, if the Jacobian matrix $[\nabla \mathbf{f}(\mathbf{x}^t, \omega)]$ is regular and well conditioned around the point $\mathbf{x}^t$. The solution of the non-linear system given at the iteration $t+1$ and expressed by relation (39), represents the standard form of the iterative Newton–Raphson method and is given by

$$\{\mathbf{x}^{t+1}\} = \mathbf{x}^t = [\nabla \mathbf{f}(\mathbf{x}^t, \omega)]^{-1}\, \mathbf{f}(\mathbf{x}^t, \omega). \tag{39}$$

The solution obtained at the iteration $t+1$ from the previous solution $\mathbf{x}^t$ given below represents a first approximation of the Newton–Raphson method:

$$\{\mathbf{x}^{t+1}\} = [\mathbf{L}_{nl}(\mathbf{x}^t, \omega)]^{-1}\mathbf{b}(\omega). \tag{40}$$

4.2. Convergence criteria

The iteration is continued until convergence to a required response accuracy is obtained. When the solution $\{\mathbf{x}^{t+1}\}$ gives displacements outside of the defined domain, the step must be reduced by considering the new estimation of the solution, namely,

$$\tilde{\mathbf{x}}^{t+1} = (1-\alpha)\mathbf{x}^{t+1} + \alpha\, \mathbf{x}^t; 0 < \alpha < 1. \tag{41}$$

The convergence is checked by comparing the relative error (42) with a control value ε_r Consequently, a stationary solution is obtained, when the convergence criteria

$$\|\hat{\mathbf{x}}^{t+1} - \mathbf{x}^t\|/\mathbf{x}^t\| \leqslant \varepsilon_r \tag{42}$$

is verified.

4.3. Algorithm of resolution

The first algorithm of resolution based on the exploitation of the first non-linear formulation is provided in Appendix A while the second algorithm of resolution based on the exploitation of the second non-linear formulation is presented in Appendix B.

5. Theoretical and experimental simulation

5.1. Description of the proposed structures

5.1.1. Description of the non-linear structure

The methods presented have been applied to a non-linear structure (Figs. 3 and 4). This structure is composed of four rigid beams $\mathbf{B}_i, (i = 1, \cdots, 4)$. Each beam is characterized by a moment of inertia $\mathbf{I}_{oz} = 4.16$ kg m^2 and articulated at the position points $O_i, (i = 1, \cdots, 4)$. The pivot joining at each articulation points O_i of each beam $\mathrm{B_i}$ induces a couple $(C_p) = 2250$ Nm. which is modelled by an equivalent linear stiffness $(k_p) = 2.5\,\mathrm{E}4\,\mathrm{N/m}$. The four beams of the structure are connected between them by three linear stiffness $(k_{12}), (k_{34}), (k_{40})$ with a constant numerical value: $k_1 = 1.731$ E5 N/m, and two localized non-linear supports. The first, placed between the grounded left side and the beam B_1 is modelled simultaneously by a linearized stiffness coefficient (k_{nl}^{10}) and a linearized damping coefficient b_{nl}^{10}. The second, placed between the beams B_2 and B_3 is modelled by linearized stiffness and damping coefficients (k_{nl}^{23}) and (b_{nl}^{23}). The characteristics of the stiffness and damping non-linearities obtained experimentally are also presented in Fig. 5. The three linear stiffnesses and the two non-linear supports are compressed by a static force of 1000 N. This preconstraint is realized by the displacement of the right and left clamped points of the structure to ensure non-linear working conditions. One now assumes that:

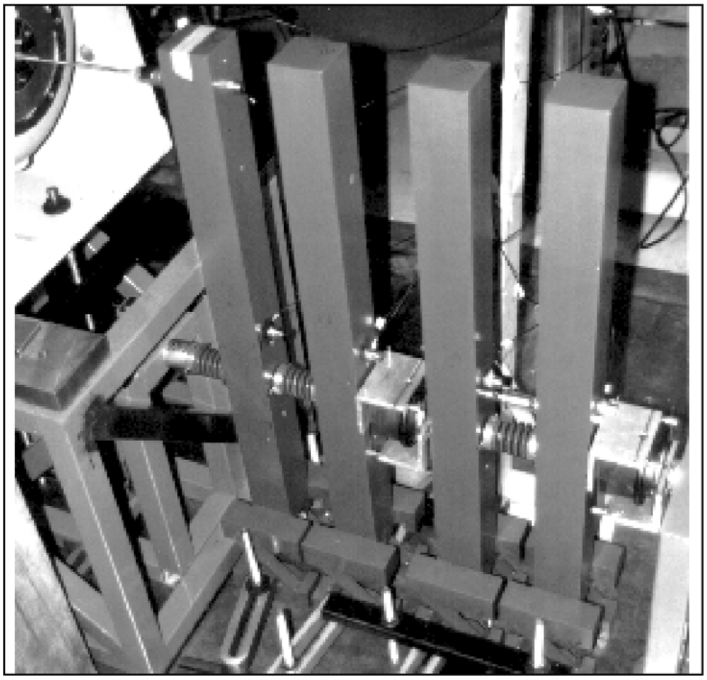


Fig. 3. Photograph of the proposed non-linear system.

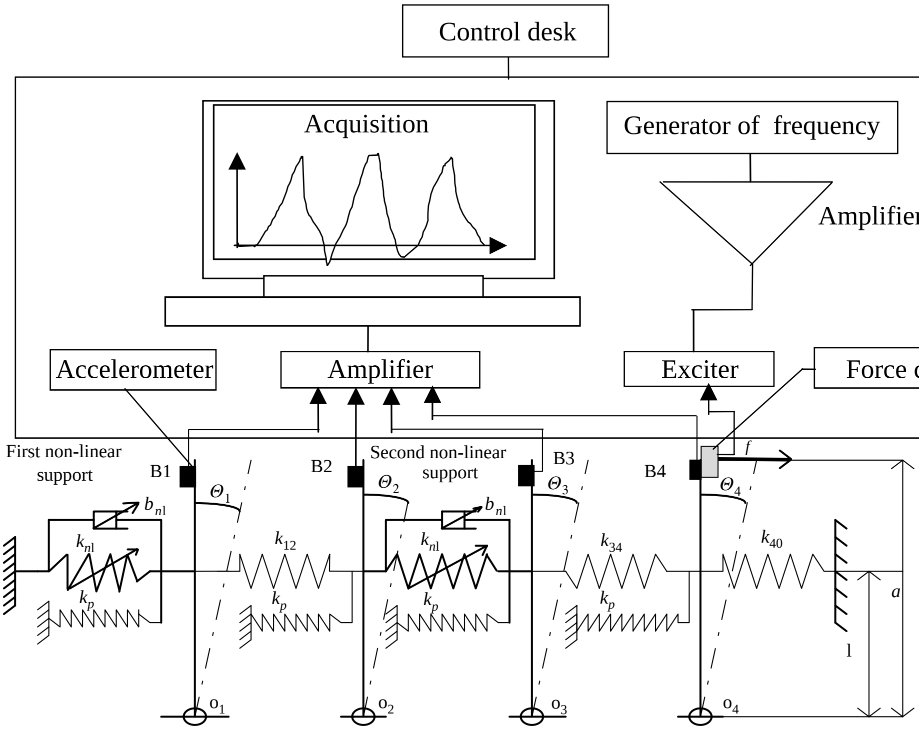


Fig. 4. The data acquisition system of the non-linear transfer functions.

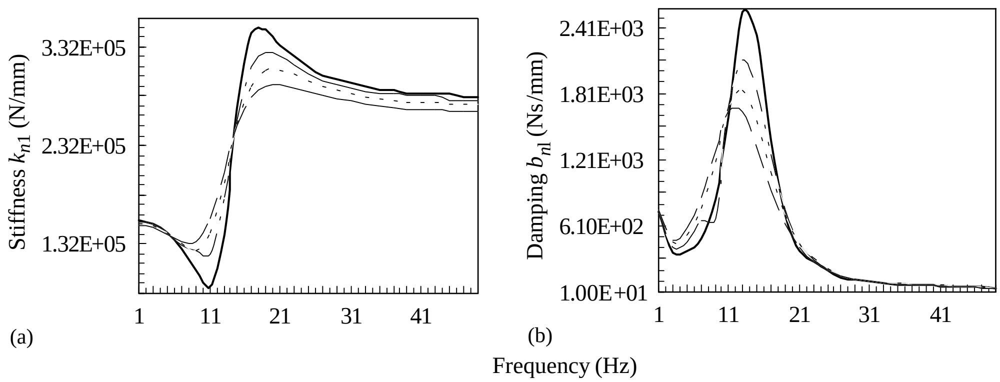


Fig. 5. Non-linear characteristics of stiffness (a), and damping (b) of the imposed displacements. Key for displacements, d,: – , 0.25 mm; ——, 0.5 mm; – – –, 0.75 mm; —, 1.0 mm.

$(k_{nl}^{10}) = (k_{nl}^{23}) = (k_{nl})$, $(b_{nl}^{10}) = (b_{nl}^{23}) = (b_{nl})$. The linear stiffnesses and the two non-linear supports are situated at a distance $\mathbf{l} = 0.3$ m from the rotation axle O_i.

The experimental data acquisition system (Fig. 4) incorporated one accelerometer for each beam $\mathbf{B}_i$ placed at 0.36 m from the axle rotation $O_i, (i = 1, \cdots, 4)$; a force cell placed on the beam

B_4 at a distance $a = 0.56\,\mathrm{m}$ from the position O_4; a single sinusoidal force $f = f(\omega)$ applied on the beam B_4. This force is controlled in order to obtain a constant amplitude θ_4 of the beam B_4 For each frequency excitation, calculated and experimental values are compared for the amplitude displacement and the phase of each beam, and the force applied on the beam B_4 when the displacement is imposed on the beam B_4.

5.1.2. Description of the associated linear structure

The linear structure and with linear are identical. Two localized non-linear elements are replaced by the linear stiffness $((k_{10}), (k_{23}))$ with a same constant numerical value $k_l = 1.731E5\ N/m$.

5.2. Case of the stiffness non-linearity

First, the model of the articulated beams is only affected by stiffness non-linearities. A comparison between the harmonic responses obtained by the following methods is presented.

5.2.1. Methods based on the first formulation and applied to the Newton–Raphson method

(1) Exploits a first approximation of the iterative Newton–Raphson with an increasing and (method 1i) and decreasing (method 1d) sweep frequencies.

(2) Uses Newton–Raphson with an increasing (method 2i) and decreasing (method 2d) sweep frequencies.

5.2.2. Method based on the second formulation and applied to the Newton–Raphson method

(3) A non-linear method which exploits the second non-linear formulation with an increasing method 3d) and decreasing (method 3d) sweep frequencies.

One now can consider for example the frequency bands around the first, second and third modes. The effect of a stiffness non-linearities on the response for both increasing or (and) decreasing sine-sweep frequencies is considered.

Fig. 6, shows that the results obtained using the proposed methods compare well with an increasing sine-sweep frequency, around the first mode. An identical numerical treatment around second and third modes, provides the same conclusions (Figs. 7 and 8).

Note that increasing and decreasing sine-sweep frequencies around the second and third modes show a non-linear phenomena characterized by a jump, due to the variation of the stiffness with the frequency and the amplitude (Figs. 7 and 8).

5.3. Stiffness and damping non-linearities

The variation of phase and amplitude of the beam response and the force amplitude has been studied, when an imposed displacement d is applied to the beam B_4.

5.4. Comparison between the theoretical and experimental results

The results of the proposed methods were compared to the experimental results. Figs. 9 and 10 show the variation of the phase and the displacement of the beams (B_1, B_2, B_3) as a function of the

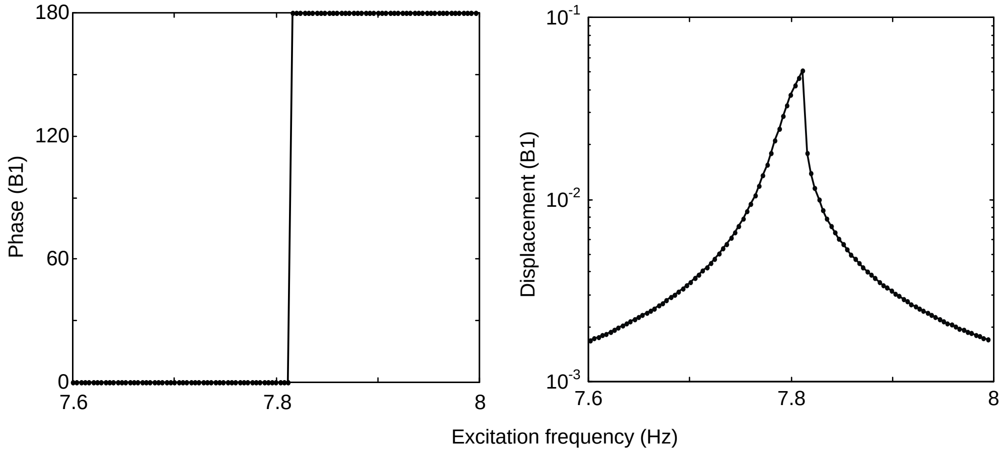


Fig. 6. Phase and displacement of the beam B1 with a stiffness non-linearity effect and imposed displacement $d = 0.25$ mm, for increasing sine-sweep frequency around the first mode. Key: …, method 3i; –, Method 1i.

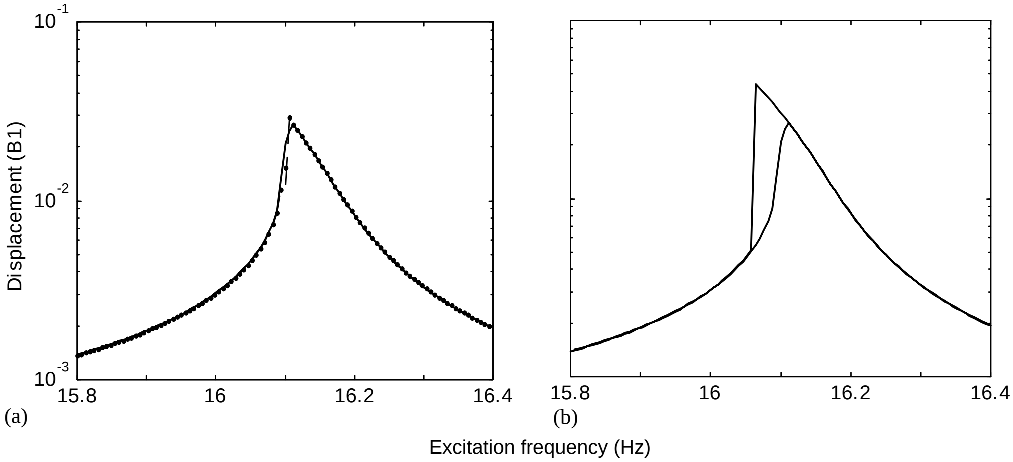


Fig. 7. Superposition of the displacement of the beam B1 with a stiffness non-linearity, imposed displacement, $d = 0.25$ mm; for (a) increasing sine-sweep frequency around the second (mode …), method 1i; , method 2i; – – –, method 3i) and (b) increasing and decreasing sine-sweep frequency around the second mode –, method 2i; , method 2d.

excitation frequency. Comparison is made between theoretical and experimental results, respectively, for the imposed displacements $d = 0.5$ and $d = 1.5$ mm with both stiffness and damping non-linearities effects.

Figs. 9 and 10 confirm also the presence of a first linearized mode around the frequency 8.5 Hz (first peak displacement). The second linearized mode is identified only by the phase functions around the frequency 14.5 Hz. This mode does not appear on the displacements functions of the beams (B_1, B_2, B_3) because it is lightly excited.

Fig. 11 shows the variation of the calculated and measured amplitude of the force as a function of the excitation frequency for the imposed displacements $d = 0.5$ and $d = 1.5$ mm. The results

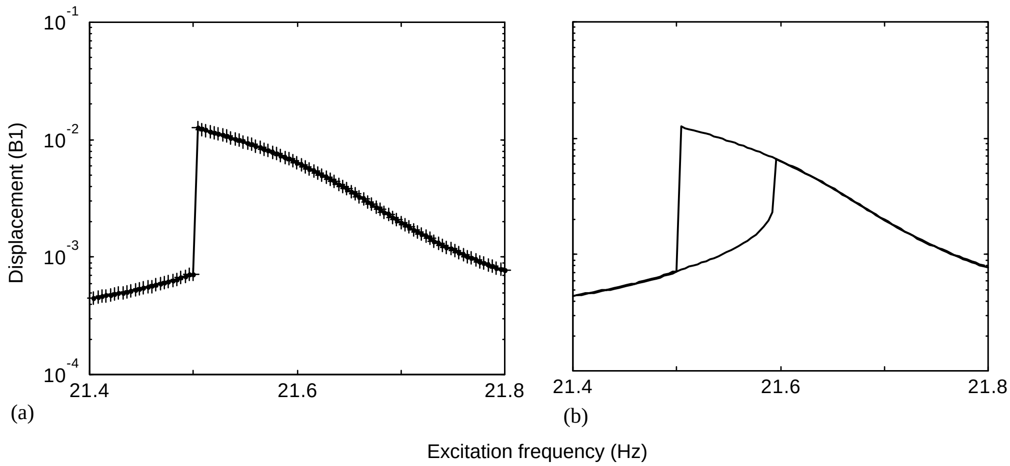


Fig. 8. Superposition of the displacement of the beam B1 with a stiffness non-linearity, imposed displacement $d = 0.25$ mm for (a) decreasing sine-sweep frequency around the third mode. –, method 1d; + +, method 2d;… method 3d and (b) increasing and decreasing sine-sweep frequency the second mode –, method 2i; –, method 2d.

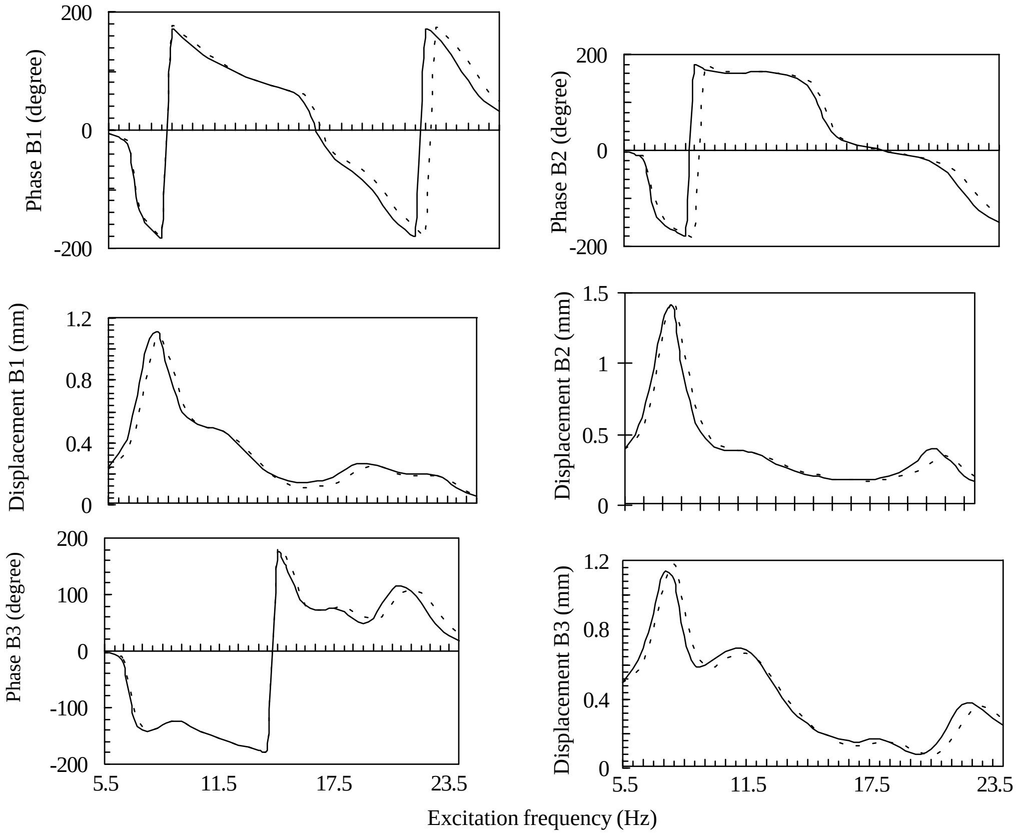


Fig. 9. Phase and displacements of the beams (B_1, B_2, B_3) with imposed displacement $d = 0.5$ mm for increasing sine-sweep frequency Key. —, calculated phase; – – –, measured phase;—, calculated response; – – –, measured response.

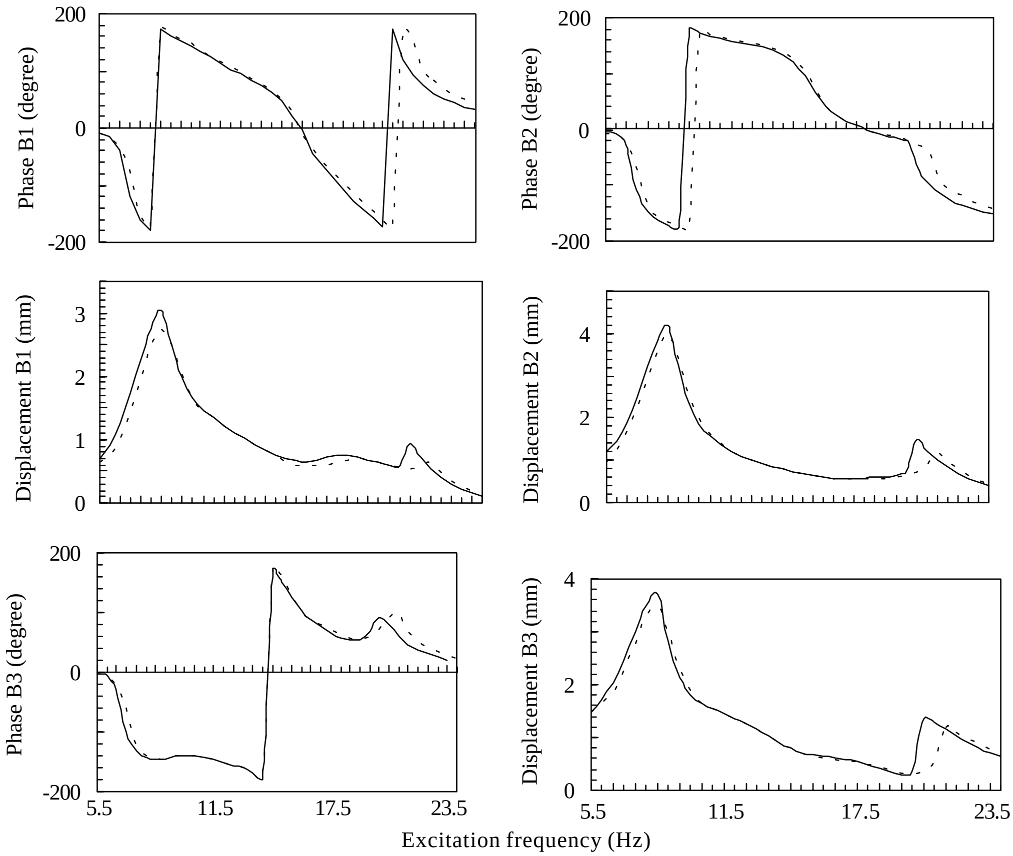


Fig. 10. Phase and displacements of the beams ($\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$) with imposed displacement $d = 1.5$ mm for increasing sine-sweep frequency. Key: —, calculated phase; ---, measured phase; —, Calculated response; ---, measured response.

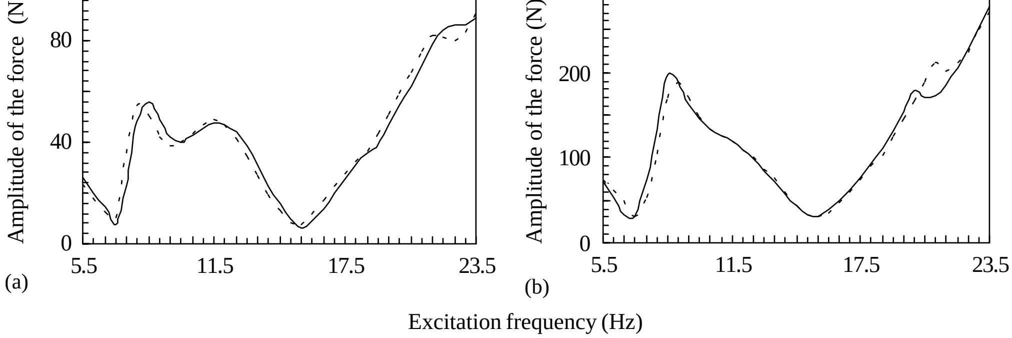


Fig. 11. Amplitude of the force for (a) imposed displacement $d = 0.5$ mm and (b) imposed displacement $d = 1.5$ mm. Key: —, calculated force; ---, measured force.

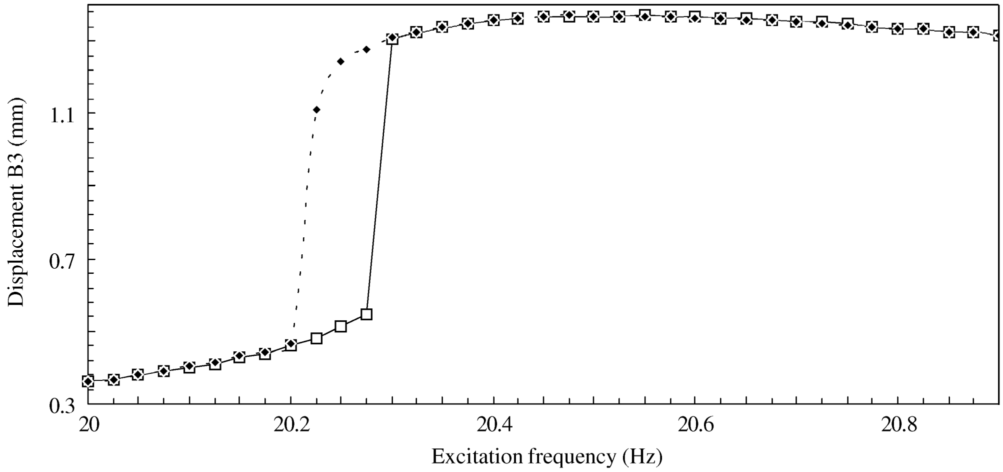


Fig. 12. Superposition of the displacements of the beam $\mathbf{B}_3$ with an increasing and decreasing sweep frequencies, changing an imposed displacement of $d = 1.5$ mm. Key: —□—, increasing sweep frequency; – – ♦ – –, decreasing sweep frequency.

show that through the measured and calculated transfers and phases functions due to harmonic excitation, one obtains a good agreement between the measured and calculated points. It was observed that there is also a good coincidence between the measured and calculated amplitudes forces.

Fig. 12 shows that a third linearized mode can be identified around the frequency 20.2 Hz through a superposition of an increasing and a decreasing sine-sweep frequency for the beam B_3. A non-linear behaviour characterized a jump phenomenon is the most important characteristic of this mode.

6. Conclusions

Two non-linear formulations have been proposed in order to study a structural behaviour that includes the effect of some localized non-linearities.

The first formulation is based on the exploitation of the eigensolutions of the associated conservative linear system and the characteristics of local non-linearities. The second formulation was developed using the linearized eigensolutions which are calculated with an iterative process.

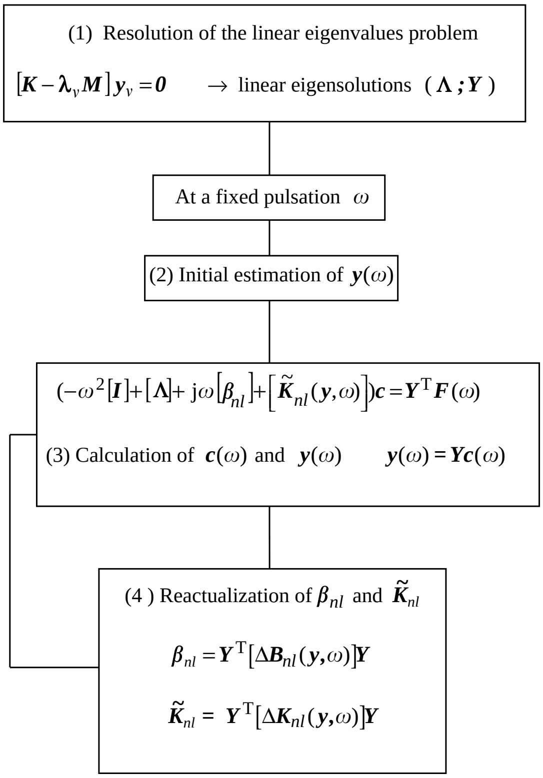
A harmonic solution of the proposed non-linear systems was obtained by using the iterative Newton–Raphson method. A stationary solution is obtained, if the convergence criteria is satisfied. Note that in the two non-linear formulations, the presence of local non-linearities are modelled by a variation of the stiffness and damping matrix as a function of the excitation frequency and the amplitude of each non-linear element. An iterative strategy based on the correction of the linearized stiffness and damping coefficients at each frequency and for each amplitude of the non-linear element is used.

The proposed formulations lead to coherent results. The increasing of the stability of the solution around the resonance frequency depends on several parameters (sine-sweep frequency, [weighting] coefficient, convergence criteria).

This method was validated on different cases. The first, study was undertaken to analyze only the stiffness non-linearity effects on the transfer functions. A second study treated stiffness and damping non-linearity effects on transfer functions. The comparison between theoretical and experimental results confirm that the proposed non-linear model yields good coherent results.

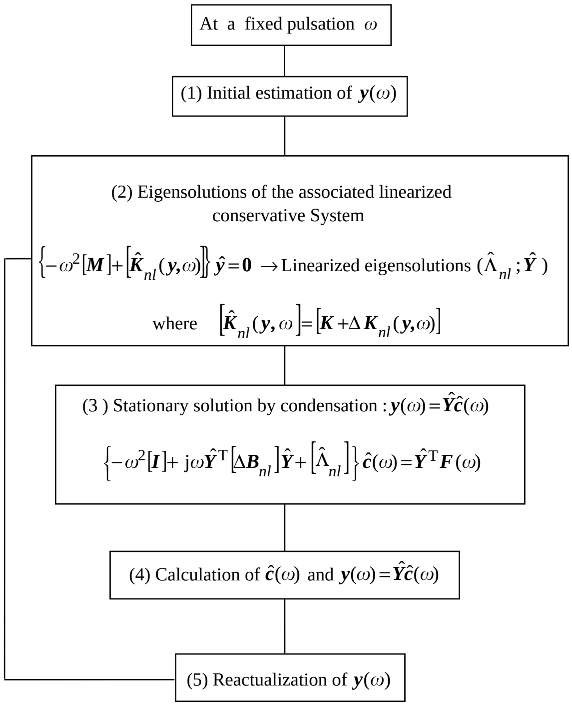
Appendix A. Algorithm of the first procedure

Condensation of relation (1) based on the exploitation of the eigensolutions of the associated conservative linear system is achieved by following the different steps of the algorithm used in the calculation of the frequencies responses.



Appendix B. Algorithm of the second procedure

Condensation, frequency by frequency, on a reactualized modal base $\hat{\mathbf{Y}}$. The iterative process allowing the calculation of a stationary solution, is outlined.



Appendix C. Nomenclature

$[\mathbf{M}],[\mathbf{B}],[\mathbf{K}]\in\mathbf{R}^{N,N,}$	mass, damping and stiffness matrices of the associated linear system
(N,N')	the number of degrees of freedom of the general and the reduced linear system (imposed θ_4), respectively,
f	excitation force due to the imposed displacement,
$Y,\mathbf{F}$	respectively the displacement vector and the excitation force vector,
$(k_{10}),\cdots(k_{34})$	linear stiffness coefficient,
$[\mathbf{K}_{nl}],[\mathbf{B}_{nl}]$	linearized stiffness and damping matrices,
d,f_0	the amplitudes of the imposed displacement $\boldsymbol{D}$ and the force $\boldsymbol{f}$, respectively,
φ	phase between the displacement $\boldsymbol{D}$ and the force f,
$(k_{nl}^{10});(b_{nl}^{10})$	linearized stiffness, damping coefficients between the beam B_1 and the reference 0
$(k_{nl}^{23}),(b_{nl}^{23})$	linearized stiffness and damping coefficients between the beams B_2 and B_3.
$[\Delta\mathbf{K}_{nl}],[\Delta\mathbf{B}_{nl}]$	the linearized stiffness and damping correction matrices, respectively.
$[\tilde{\mathbf{K}}_{nl}],[\beta_{nl}]$	the linearized generalised stiffness and damping matrices, respectively.
$[\beta_{nl}]$	linearized generalised damping matrix,
$f_{nl}(y,\omega)$	force vector of stiffness and damping, non-linear.

$[\mathbf{\Lambda}], [\mathbf{Y}]$	eigensolutions of the associated conservative linear system,
$[\hat{\mathbf{\Lambda}}_{nl}], [\hat{\mathbf{Y}}]$	eigensolutions of the linearized eigenvalues problem,
$[\mathbf{L}_{nl}(\mathbf{x}, \omega)], [\hat{\mathbf{L}}_{nl}(\hat{\mathbf{x}}, \omega)]$	linearized functions, respectively, for the first and second formulations
$\nabla \mathbf{f}(\mathbf{x}^t, \omega)]$	Jacobian matrix.

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