ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Sound and Vibration 260 (2003) 889–900

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Triply coupled vibrations of thin-walled open cross-section beams including rotary inertia effects

A. Arpaci*, S.E. Bozdag, E. Sunbuloglu

Faculty of Mechanical Engineering, Istanbul Technical University, Gumussuyu 80191, Istanbul, Turkey

Received 17 December 2001; accepted 15 April 2002

Abstract

An exact analytical method is presented for predicting the undamped natural frequencies of beams with thin-walled open cross-sections having no axis of symmetry. The governing differential equations give a characteristic equation of the 12th order with real coefficients. The roots are found numerically and the exact boundary conditions are considered especially for free ends to obtain natural frequencies. The simpler cases of neglecting cross-sectional warping and/or rotary inertia are also dealt with. It is seen that when the effect of rotary inertia is neglected significant errors incur for some boundary conditions, cross-section thicknesses and mode numbers. This is more profound when the warping effect is taken into account.
© 2002 Elsevier Science Ltd. All rights reserved.

1. Introduction

Beams of thin-walled open cross-sections are widely used in structural design. The most common application areas are aerospace engineering, civil engineering and related sites. Due to the reliability requirements of the advanced constructions in such fields, vibration characteristics of open-section beams have always been of great concern for engineers.

For beams that have the shear centers and the geometric centers of the cross-sections coincident, the vibration characteristics for lateral and torsional vibrations have been extensively studied by many researchers. Nevertheless, in general practice, the centroid and the shear center of cross-sections are not coincident; hence, the flexural and the torsional vibrations are coupled, and the vibration characteristics of such beams should be investigated in detail.

The case where the cross-section has a single axis of symmetry, and consequently the flexural vibrations in one direction are coupled with the torsional vibrations, has been extensively studied

*Corresponding author. Tel.: +90-212-293-1300X2415; fax: +90-212-245-0795.
E-mail address: arpacial@itu.edu.tr (A. Arpaci).

0022-460X/03/$ - see front matter © 2002 Elsevier Science Ltd. All rights reserved.
doi:10.1016/S0022-460X(02)00935-5

by many researchers using Bernoulli–Euler theory [1–5]. This is called double coupling. However, the number of studies dealing with doubly coupled flexural–torsional vibrations of monosymmetric Timoshenko beams is rather limited. Bishop and Price [6], and Banerjee and Williams [7] have taken into account the effects of rotary inertia and transverse shear. Warping stiffness has been neglected in both works.

A more recent study carried on by Bercin and Tanaka [8] has included the effect of warping in doubly-coupled vibration analysis of Timoshenko beams with monosymmetric open cross-sections. Hence, one supposes the doubly coupled vibrations of thin-walled open cross-section beams to have been fully investigated.

In the case of triple coupling, where the flexural vibrations in two mutually perpendicular directions and the torsional vibration are all coupled is dealt with by Yaman [9] and Arpaci and Bozdag [10]. This is experienced when the cross-sections of no symmetry are of concern. But to the authors' knowledge, no work appears in open literature studying even one of the Timoshenko effects on triple coupling.

The objective of the present study is to include the effect of rotary inertia in triply coupled vibration analysis of thin-walled open cross-section beams, and to reflect the effect of rotary inertia over the natural frequencies with and without the warping effect taken into account. The basis on which the present theory depends is consistent with the Vlasov beam theory partially accounting for the Timoshenko effects by including rotary inertia but not shear deformation. Thus it is expected that this will enable the researchers to compare their results with the present ones if they study the same problem by using the Vlasov beam theory.

2. Theory

2.1. Equations of motion

Fig. 1 represents a typical cross-section of no axial symmetry where the x- and y-axis are taken through the shear center S and parallel to the principal centroidal axes ξ and η. Under the assumptions of uniform thin-walled open cross-sections for beams with no planar distortions and the location of the shear center being a function of cross-section only, and homogeneous and isotropic Hooke material, the governing equations of motion can be derived by considering the equilibrium of an infinitesimal beam segment.

Fig. 2(a), (b) and (c) show the elastic curve in the η–z plane, an infinitesimal rotated beam element and a beam segment with bending moments M_ξ, and shear forces F_y acting through the shear center S, respectively.

The condition for rotational equilibrium of the beam segment requires that

$$F_y = \frac{\partial M_\xi}{\partial z} + \rho I_\xi \frac{\partial^2 \alpha}{\partial t^2}, \tag{1}$$

and the condition for translational equilibrium requires that

$$\frac{\partial F_y}{\partial z} = \rho A \frac{\partial^2}{\partial t^2}(v + \phi e_1), \tag{2}$$

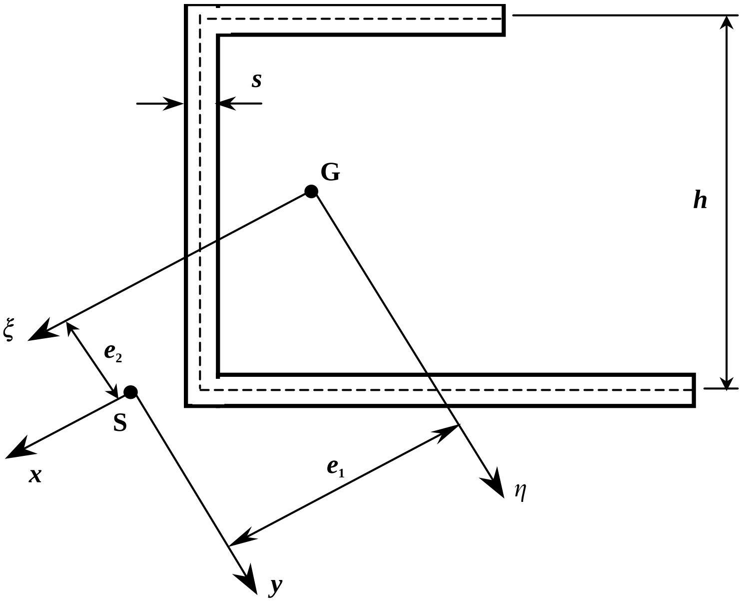


Fig. 1. Co-ordinate systems on a sample non-symmetrical cross-section.

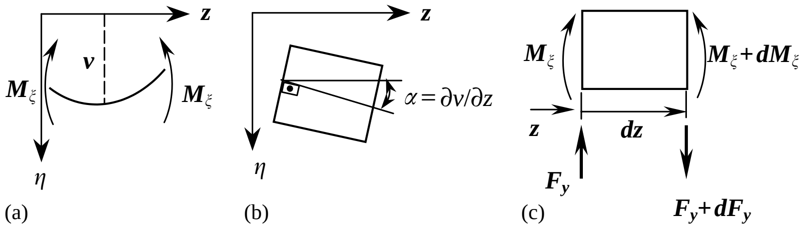


Fig. 2. Equilibrium diagrams for an infinitesimal beam segment in the η–z-plane.

where ρ is the mass density, A is the cross-sectional area, v is the deflection of shear center in the y direction, I_ξ is the principal centroidal area moment of inertia, ϕ is the angle of rotation of the cross-section, α is the bending slope and t is the time. Combining Eqs. (1) and (2), and considering that $M_\xi = -EI_\xi \partial\alpha/\partial z$ and $\alpha = \partial v/\partial z$, the final form of the equation is found to be

$$EI_\xi \frac{\partial^4 v}{\partial z^4} - \rho I_\xi \frac{\partial^4 v}{\partial z^2 \partial t^2} + \rho A \frac{\partial^2}{\partial t^2}(v + \phi e_1) = 0. \tag{3}$$

Similarly, Fig. 3 shows the case for the (ξ–z) plane.

To have rotational equilibrium of the segment, the sum of all torques must be zero:

$$F_x = -\frac{\mathrm{d}M_\eta}{\mathrm{d}z} - \rho I_\eta \frac{\partial^2 \beta}{\partial t^2}. \tag{4}$$

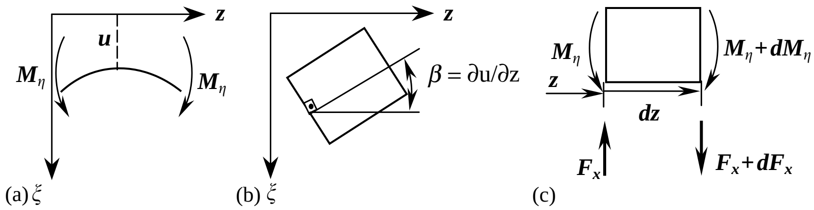


Fig. 3. Equilibrium diagrams for an infinitesimal beam segment in the ξ–z-plane.

The condition for translational equilibrium requires that

$$\frac{\partial F_x}{\partial z} = \rho A \frac{\partial^2}{\partial t^2}(u - \phi e_2), \tag{5}$$

where I_η is the principal centroidal area moment of inertia, u is the deflection of the shear center in the x direction and β is the bending slope.

Substituting Eq. (4) into Eq. (5) and assuming that $M_\eta = EI_\eta \partial\beta/\partial z$ and $\beta = \partial u/\partial z$ yields

$$EI_\eta \frac{\partial^4 u}{\partial z^4} + \rho I_\eta \frac{\partial^4 u}{\partial z^2 \partial t^2} + \rho A \frac{\partial^2}{\partial t^2}(u - \phi e_2) = 0. \tag{6}$$

To establish the governing equation for the torsional vibration, let m_t be the intensity of the distributed torque assumed to act along the shear center axis. In Saint-Venant theory, cross-sectional warping is assumed to be uniform along the beam and consequently no longitudinal stress is produced. This assumption is valid only for members in which all cross-sections are free to warp. For bars of solid cross-section, if one or more cross-sections of the beam are caused to remain plane, the warping constraint will produce negligible effect on the angle of twist. But in the case of thin-walled members of open cross-section, the prevention of warping during twist is accompanied by bending of the flanges and may have considerable effect on the angle of twist. The given torque is then balanced at any cross-section partially by shearing stresses due to twist and partially by shearing stresses due to bending of the flanges. Hence,

$$m_t = EI_w \frac{\mathrm{d}^4\phi}{\mathrm{d}z^4} - GJ \frac{\mathrm{d}^2\phi}{\mathrm{d}z^2}, \tag{7}$$

where EI_w is the torsional rigidity associated with non-uniform warping and GJ is the Saint-Venant torsional rigidity [11]. According to d'Alembert's principle, the governing differential equations of torsional vibration may be derived by substituting the inertia torque into the equations of static equilibrium. Because the inertia forces of translation act through the centroid, they must be transferred to the shear center. Hence, d'Alembert's principle leads to

$$m_t = -\rho I_p \frac{\partial^2\phi}{\partial t^2} + \rho A \frac{\partial^2}{\partial t^2}(u - e_2\phi)e_2 - \rho A \frac{\partial^2}{\partial t^2}(v + e_1\phi)e_1. \tag{8}$$

Substituting Eq. (7) into Eq. (8) yields

$$EI_w \frac{\partial^4 \phi}{\partial z^4} - GJ \frac{\partial^2 \phi}{\partial z^2} + \rho A e_1 \frac{\partial^2 v}{\partial t^2} - \rho A e_2 \frac{\partial^2 u}{\partial t^2} + \rho I_0 \frac{\partial^2 \phi}{\partial t^2} = 0. \tag{9}$$

Restating the equations above, the system of equations for triply coupled vibrations of beams with thin-walled open cross-sections can be expressed as follows:

$$EI_\xi \frac{\partial^4 v}{\partial z^4} - \frac{\partial^2}{\partial t^2}\left(\rho I_\xi \frac{\partial^2 v}{\partial z^2} - \rho A(v + \phi e_1)\right) = 0, \tag{10}$$

$$EI_\eta \frac{\partial^4 u}{\partial z^4} + \frac{\partial^2}{\partial t^2}\left(\rho I_\eta \frac{\partial^2 u}{\partial z^2} + \rho A(u - \phi e_2)\right) = 0, \tag{11}$$

$$EI_w \frac{\partial^4 \phi}{\partial z^4} - GJ \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2}{\partial t^2}(\rho A e_1 v - \rho A e_2 u + \rho I_0 \phi) = 0. \tag{12}$$

Let

$$u = U(z)\mathrm{e}^{\mathrm{i}\omega t}, \quad v = V(z)\mathrm{e}^{\mathrm{i}\omega t}, \quad \phi = \Phi(z)\mathrm{e}^{\mathrm{i}\omega t},$$

$$\lambda_\xi = \omega^2 \rho A / EI_\xi, \quad \lambda_\eta = \omega^2 \rho A / EI_\eta, \quad \lambda_w = \omega^2 \rho A / EI_w,$$

$$\lambda_b = GJ / EI_w, \quad \lambda_0 = \omega^2 \rho I_0 / EI_w, \quad \mu = \rho \omega^2 / E, \tag{13}$$

where ω is the radian frequency. Assuming $\mathbf{U}(z) = C_u \mathrm{e}^{rz}$, $V(z) = C_v \mathrm{e}^{rz}$ and $\mathbf{\Phi}(z) = C_\Phi \mathrm{e}^{rz}$ for amplitudes, the governing differential Eqs. (10)–(12) yield

$$\begin{aligned} &(r^4 - \mu r^2 - \lambda_\eta) C_U + \lambda_\eta e_2 C_\Phi = 0, \\ &(r^4 + \mu r^2 - \lambda_\xi) C_V - \lambda_\xi e_1 C_\Phi = 0, \\ &\lambda_w e_2 C_U - \lambda_w e_1 C_V + (r^4 - \lambda_b r^2 - \lambda_0) C_\Phi = 0. \end{aligned} \tag{14}$$

The equations can be expressed in matrix form as

$$\begin{bmatrix} r^4 - \mu r^2 - \lambda_\eta & 0 & +\lambda_\eta e_2 \\ 0 & r^4 + \mu r^2 - \lambda_\xi & -\lambda_\xi e_1 \\ \lambda_w e_2 & -\lambda_w e_1 & (r^4 - \lambda_b r^2 - \lambda_0) \end{bmatrix} \begin{bmatrix} C_U \\ C_V \\ C_\Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{15}$$

Setting the determinant of the linear system equal to zero yields an ordinary polynomial of the 12th order in terms of r for each function as

$$\{r^{12} + a_1 r^{10} + a_2 r^8 + a_3 r^6 + a_4 r^4 + a_5 r^2 + a_6\} = 0, \tag{16}$$

where

$$\begin{aligned} &a_1 = -\lambda_b, \quad a_2 = -(\lambda_0 + \mu^2 + \lambda_\eta + \lambda_\xi), \quad a_3 = \lambda_b(\mu^2 + \lambda_\eta + \lambda_\xi) + \mu(-\lambda_\eta + \lambda_\xi), \\ &a_4 = \lambda_0 \mu^2 + \mu \lambda_b (\lambda_\eta - \lambda_\xi) + \lambda_\xi(\lambda_0 - e_1^2 \lambda_w^2) + \lambda_\eta(\lambda_0 + \lambda_\xi - e_2^2 \lambda_w^2), \\ &a_5 = -\lambda_b \lambda_\eta \lambda_\xi + \mu[\lambda_\xi(-\lambda_0 + e_1^2 \lambda_w) + \lambda_\eta(\lambda_0 - e_2^2 \lambda_w)], \\ &a_6 = \lambda_\eta[e_2^2 \lambda_\xi \lambda_w + \lambda_\xi(-\lambda_0 + e_1^2 \lambda_w)]. \end{aligned}$$

Upon defining a new variable $p = r^2$, polynomial (16) becomes

$$\{p^6 + a_1 p^5 + a_2 p^4 + a_3 p^3 + a_4 p^2 + a_5 p + a_6\} = 0. \tag{17}$$

The solution to polynomial (17) yields six values as roots after the numerical calculations, and thereafter, the general solution of $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{\Phi}$ can be formulated as

$$\mathbf{U} = \sum_{i=1}^{12} C_{Ui} z^{k_i} \mathrm{e}^{r_i z}, \quad \mathbf{V} = \sum_{i=1}^{12} C_{Vi} z^{k_i} \mathrm{e}^{r_i z} \quad \text{and} \quad \mathbf{\Phi} = \sum_{i=1}^{12} C_{\Phi i} z^{k_i} \mathrm{e}^{r_i z}. \tag{18}$$

In set (18), the possibility of repeated roots has been included in the form of z^{k_i} when writing the homogeneous solution formulae. As an example, if the third and fourth roots of polynomial (17) are equal, as a consequence k_3 would equal 0 and k_4 would equal 1.

C_U, C_V and C_Φ in Eq. (18) indicate three different sets of constants. Substituting u into Eq. (11) yields the relations between the vector of C_U and C_Φ. Similarly, another set of relations between the vectors of C_V and C_Φ can be derived by substituting v into Eq. (10). By imposing the boundary conditions, the three vectors of constants can be calculated and exact formulae can be established.

2.2. Boundary conditions

2.2.1. Boundary conditions for clamped end

The well-known boundary conditions for a clamped end are that the translations, rotations and slopes at a clamped end are identically zero. Hence,

$$\mathbf{U} = 0, \quad \mathbf{V} = 0, \quad \mathbf{\Phi} = 0, \quad \mathbf{U}' = 0, \quad \mathbf{V}' = 0, \quad \mathbf{\Phi}' = 0. \tag{19}$$

Notice that the condition $\Phi' = 0$ is due to the warping effect bundled in the current theory, implied by the restricted longitudinal translation of the beam [12], and is not considered if only the Saint-Venant torsion theory is assumed to be valid.

2.2.2. Boundary conditions for free end

The boundary conditions for a free end are much more complex, and derived accordingly to clarify the details. For a free end, the moments and the shear forces equal to zero. Thus for $M_\xi = 0$,

$$-EI_\xi \frac{\partial \alpha}{\partial z} = -EI_\xi \frac{\partial^2 v}{\partial z^2} = 0 \text{ and for } \mathbf{v} = V(z)\mathrm{e}^{\mathrm{i}\omega t}, \quad \mathbf{V}'' = 0. \tag{20}$$

Similarly for $M_\eta = 0$,

$$EI_\eta \frac{\partial \beta}{\partial z} = EI_\eta \frac{\partial^2 u}{\partial z^2} = 0 \text{ and for } \mathbf{u} = U(z)\mathrm{e}^{\mathrm{i}\omega t}, \quad \mathbf{U}'' = 0. \tag{21}$$

For $F_x = 0$, $-\mathrm{d}M_\eta/\mathrm{d}z - \rho I_\eta \partial^2\beta/\partial t^2 = 0$. Since $M_\eta = EI_\eta \partial\beta/\partial z$ and $\beta = \partial u/\partial z$, $EI_\eta \partial^3 u/\partial z^3 + \rho I_\eta \partial^3 u/\partial z \partial t^2 = 0$. Assuming $\mathbf{u} = U(z)\mathrm{e}^{\mathrm{i}\omega t}$,

$$\rho I_\eta \omega^2 \mathbf{U}' - EI_\eta \mathbf{U}''' = 0. \tag{22}$$

Following the same steps for $F_y = 0$, $\mathrm{d}M_\xi/\mathrm{d}z - \rho I_\xi \partial^2\alpha/\partial t^2 = 0$. Since $M_\xi = -EI_\xi \partial\alpha/\partial z$ and $\alpha = \partial v/\partial z$, $-EI_\xi \partial^3 v/\partial z^3 + \rho I_\xi \partial^3 v/\partial z \partial t^2 = 0$. Proposing $v = V(z)\mathrm{e}^{\mathrm{i}\omega t}$,

$$\rho I_\xi \omega^2 V' + EI_\xi \mathbf{V}''' = 0. \tag{23}$$

For zero torsional moment at a free end,

$$GJ\mathbf{\Phi}' - EI_w \mathbf{\Phi}''' = 0. \tag{24}$$

Also, since there is no constraint on the warping freedom of the cross-section, normal stresses due to warping are zero [12], and accordingly

$$\mathbf{\Phi}'' = 0. \tag{25}$$

2.2.3. *Boundary conditions for hinged end*

For a hinged end, the boundary conditions are stated as the translations and rotations are prohibited, but the bending moments and normal stresses are zero. Thus, initially

$$\mathbf{U} = 0, \quad \mathbf{V} = 0, \quad \mathbf{\Phi} = 0. \tag{26}$$

Because of restrictions on translation and rotation,

$$\mathbf{U}'' = 0, \quad \mathbf{V}'' = 0. \tag{27}$$

Since bending moments about the ξ- and η-axis are zero, respectively, and

$$\mathbf{\Phi}'' = 0, \tag{28}$$

the normal stresses at a hinged end are zero, providing allowance for warping.

3. Numerical evaluation technique

Application of boundary conditions to Eqs. (18) at $z = 0$ and l yields 12 linear homogeneous equations. The characteristic equation, which can be numerically solved to give the natural frequencies, is obtained by setting the determinant of the linear system of equations for any function $\mathbf{U}$, $\mathbf{V}$ or $\mathbf{\Phi}$ equal to zero.

The first step is to form the sixth order characteristic polynomial (17). To realize this, the material and the geometric properties are fed into a computer program and the coefficients a_i of the polynomial are formed for an initial value of ω and the roots p_i and consequently r_i are numerically calculated.

Being of the fourth order with real coefficients, Bishop et al. [4] prove, for doubly coupled vibrations, that the characteristic equation has four non-zero real roots, two of them negative and two positive. Thus they form the mode shape functions as combinations of trigonometric and hyperbolic functions. But in the present theory, it is not possible to determine the nature of the roots because the order of the polynomial is higher than four and the coefficients are too

Table 1
The cross-section studied in Example 1

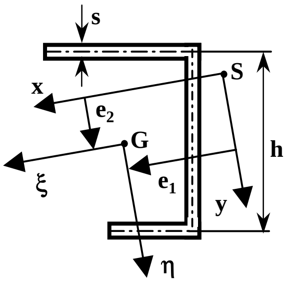
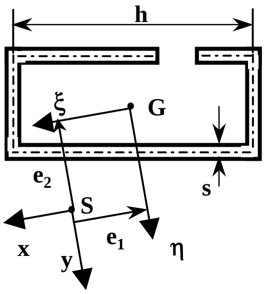
Cross-section layout for Example 1	Beam properties					
	s (mm)	1	2	3	4	5
	EI_ξ (N m^2)	3.42E+3	6.86E+3	10.33E+3	13.85E+3	17.42E+3
	EI_η (N m^2)	352.80E+0	711.68E+0	1.08E+3	1.47E+3	1.89E+3
	GJ (N m^2)	1.86E+0	14.87E+0	50.03E+0	118.24E+0	230.24E+0
	EI_w (N m^4)	53.50E−3	109.37E−3	169.88E−3	237.10E−3	312.82E−3
	I_0 (m^4)	29.82E−9	59.44E−9	88.67E−9	117.35E−9	145.39E−9
	ρ (kg/m^3)	7.86E+3	7.86E+3	7.86E+3	7.86E+3	7.86E+3
	A (m^2)	70.00E−6	140.00E−6	210.00E−6	280.00E−6	350.00E−6
	e_1 (m)	8.68E−3	8.61E−3	8.51E−3	8.36E−3	8.17E−3
	e_2 (m)	8.99E−3	8.93E−3	8.84E−3	8.70E−3	8.53E−3
	l (m)	1.00E+0	1.00E+0	1.00E+0	1.00E+0	1.00E+0
	h (m)	40E−3	40E−3	40E−3	40E−3	40E−3

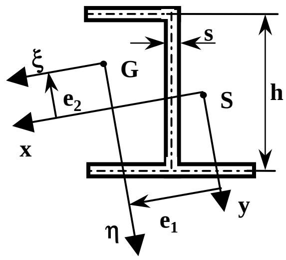
Table 2
The cross-section studied in Example 2

Cross-section layout for Example 2	Beam properties					
	s (mm)	1	2	3	4	5
	EI_ξ (N m^2)	1.80E+3	3.61E+3	5.47E+3	7.38E+3	9.37E+3
	EI_η (N m^2)	5.26E+3	10.55E+3	15.90E+3	21.33E+3	26.88E+3
	GJ (N m^2)	3.12E+0	24.99E+0	84.50E+0	200.66E+0	392.59E+0
	EI_w (N m^4)	2.01E+0	4.03E+0	6.03E+0	8.03E+0	10.02E+0
	I_0 (m^4)	118.49E−9	235.63E−9	350.19E−9	461.02E−9	567.22E−9
	ρ (kg/m^3)	7.86E+3	7.86E+3	7.86E+3	7.86E+3	7.86E+3
	A (m^2)	116.50E−6	233.00E−6	349.50E−6	466.00E−6	582.50E−6
	e_1 (m)	−16.77E−3	−16.65E−3	−16.46E−3	−16.19E−3	−15.85E−3
	e_2 (m)	−20.80E−3	−20.74E−3	−20.63E−3	−20.47E−3	−20.28E−3
	l (m)	1.00E+0	1.00E+0	1.00E+0	1.00E+0	1.00E+0
	h (m)	40E−3	40E−3	40E−3	40E−3	40E−3

complicated to analyze. Instead, the mode shape functions are expressed in exponential form and the possibility of repeated roots is included in the numerical evaluation algorithm. However, the numerical values have not demonstrated such a behaviour and all the roots of polynomial (17) have been found as distinct and real.

The second step is application of boundary conditions to Eqs. (18) and arrangement of the characteristic determinant considering the relations among the unknown coefficients C_{Ui}, C_{Vi} and $C_{\Phi i}$. The numerical value of the determinant is computed. The next step is to add a small increment to the initial value of ω and to calculate the roots of polynomial (17) and the value of the determinant. The natural frequencies are obtained by repeating many steps over and over

Table 3
The cross-section studied in Example 3



Cross-section layout for Example 3	Beam properties					
	s (mm)	2	4	6	8	10
	EI_ξ (N m^2)	53.47E+3	106.20E+3	158.41E+3	210.29E+3	262.05E+3
	EI_η (N m^2)	2.83E+3	5.74E+3	8.79E+3	12.06E+3	15.60E+3
	GJ (N m^2)	29.75E+0	236.99E+0	795.54E+0	1.88E+3	3.64E+3
	EI_w (N m^4)	3.39E+0	6.85E+0	10.42E+0	14.15E+0	18.06E+0
	I_0 (m^4)	349.05E−9	691.18E−9	1.02E−6	1.34E−6	1.64E−6
	ρ (kg/m^3)	7.86E+3	7.86E+3	7.86E+3	7.86E+3	7.86E+3
	A (m^2)	278.00E−6	552.00E−6	822.00E−6	1.09E−3	1.35E−3
	e_1 (m)	4.46E−3	4.41E−3	4.31E−3	4.18E−3	4.01E−3
	e_2 (m)	−14.94E−3	−14.79E−3	−14.40E−3	−13.81E−3	−13.04E−3
	l (m)	1.00E+0	1.00E+0	1.00E+0	1.00E+0	1.00E+0
	h (m)	80E−3	80E−3	80E−3	80E−3	80E−3

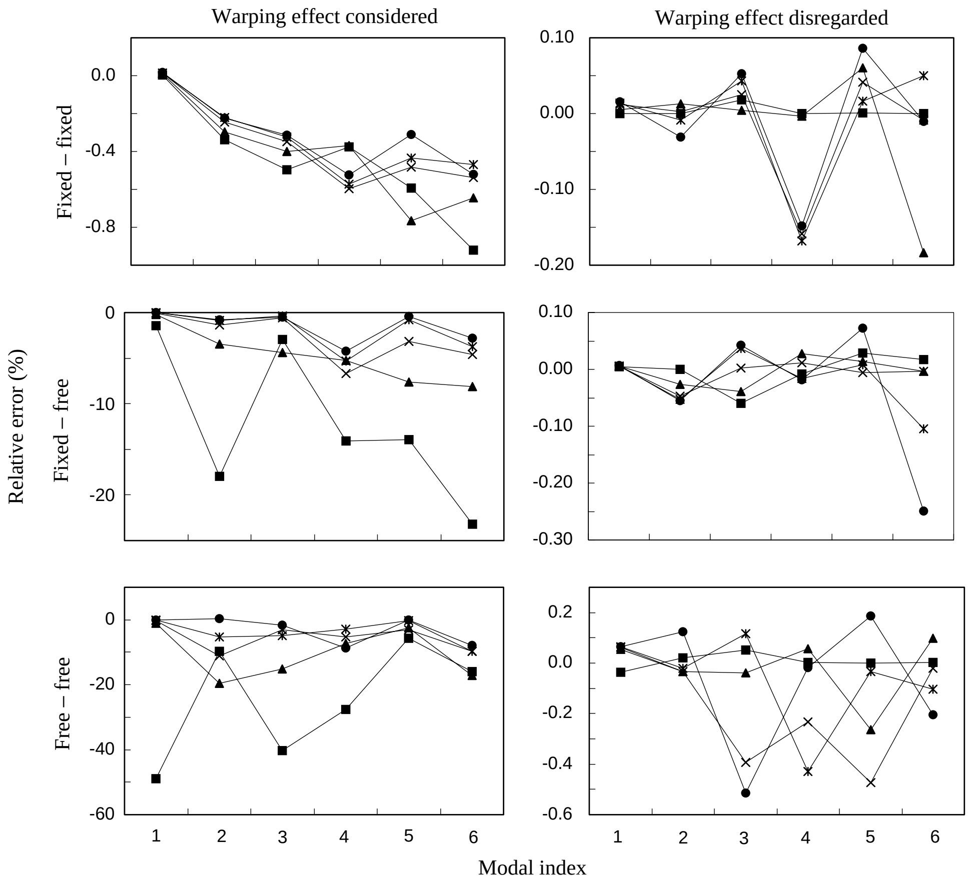


Fig. 4. Plot of relative errors on natural frequencies versus modal index for cross-sections in Table 1. Key for cross-sections, s, (mm): −◆−, 1; −■−, 2; −▲−, 3; − × −, 4; −●−, 5.

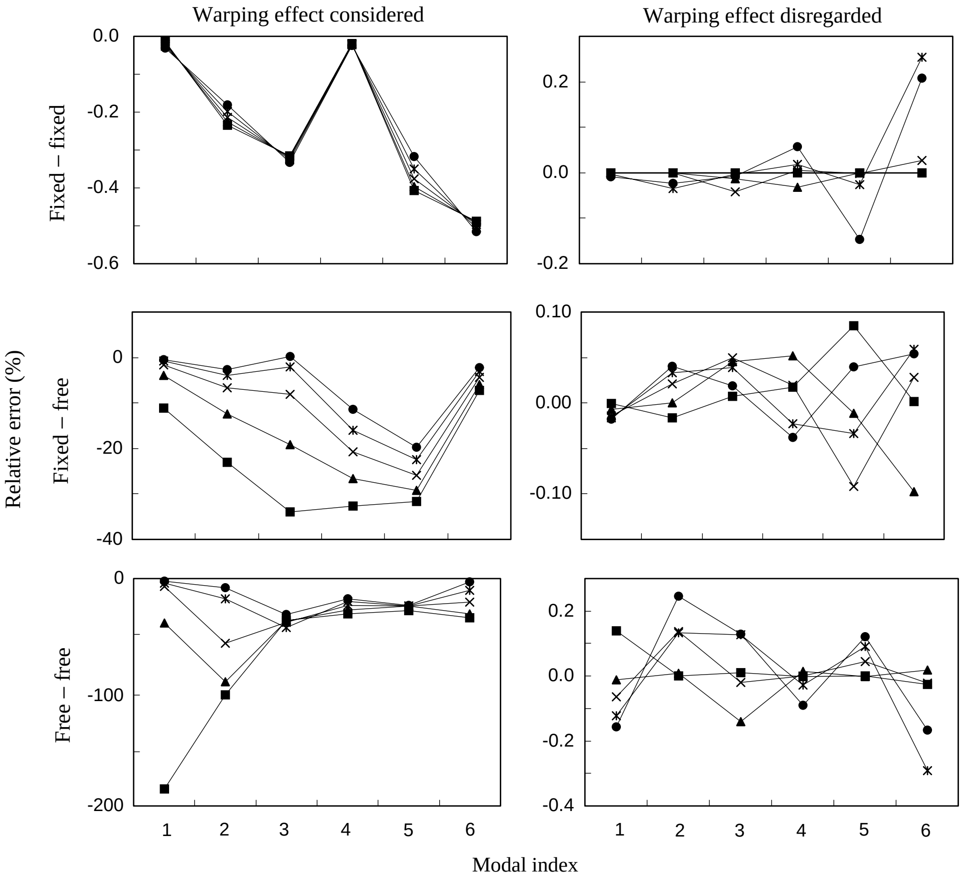


Fig. 5. Plot of relative errors on natural frequencies versus modal index for cross-sections in Table 2. Key as for Fig. 4.

trying to improve the initial guess so that the values of ω nullifying the characteristic determinant are approached.

4. Numerical examples

In this section the natural frequency analysis is applied to three example problems, and the effect of rotary inertia on the coupled bending/bending/torsional frequencies of thin-walled beams are investigated. The material and the geometric properties of the beams are listed in Tables 1–3 together with their cross-sectional layouts.

The first six natural frequencies are obtained by including/excluding the effect of rotary inertia for several cross-section thicknesses (s). The relative errors ($\varepsilon = 1 - \omega_0/\omega_r$) due to omission of the rotary inertia effect are shown in Figs. 4–6 for the cases in which the warping effect regarded or disregarded. Here, ω_0 is the natural frequency calculated using Bernoulli–Euler beam theory and ω_r is the one obtained by including the rotary inertia effect.

Three boundary conditions, namely fixed–fixed, fixed–free and free–free beams are considered.

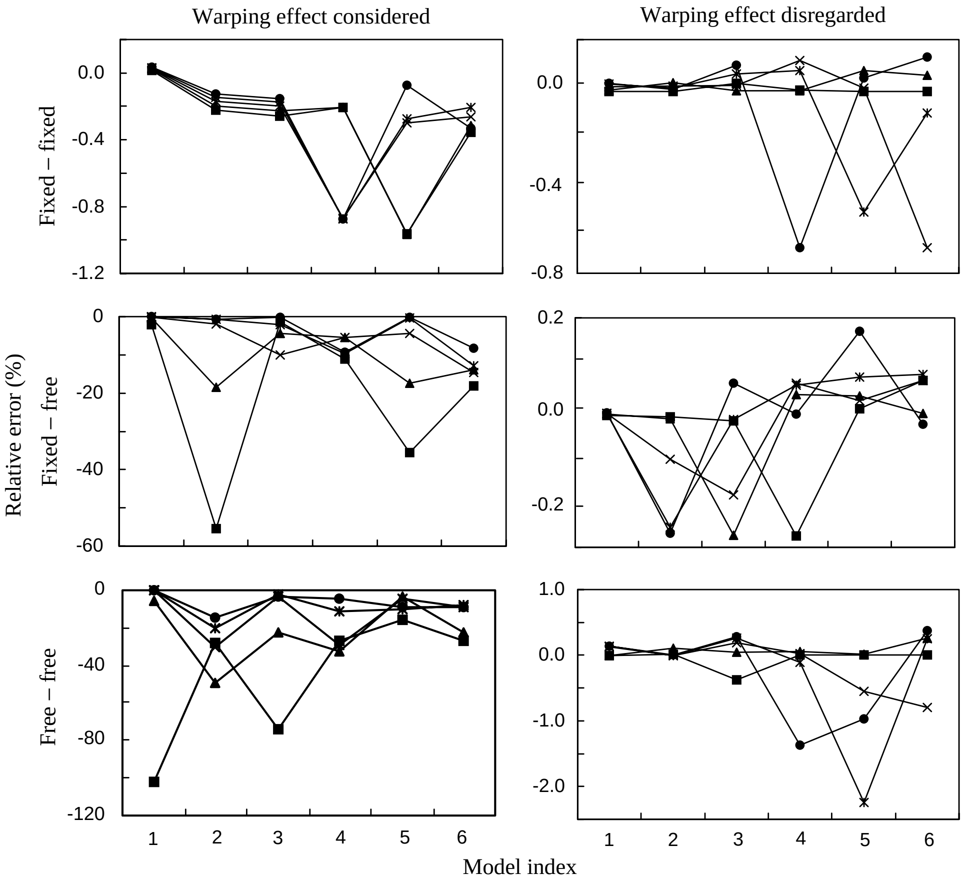


Fig. 6. Plot of relative errors on natural frequencies versus modal index for cross-sections in Table 3. Key as for Fig. 4.

5. Conclusions

Figs. 4–6 reveal that when triply coupled vibrations of thin-walled open cross-section beams are of concern, the rotary inertia effect may greatly alter the natural frequencies, the relative error associated with the neglecting of it, for some conditions, reaching 170%, unless the warping effect is also neglected.

This can be attributed to the fact that excluding warping decreases the torsional rigidity of beams and consequently the natural frequencies. Thus the modes, which are originally bending dominated, become strongly coupled or even torsion dominated ones and the rotary inertia effect gets weaker on them.

Similarly, the cross-section thickness alter the relative error incurred by ignoring the rotary inertia being considerable in some modes of vibration which are predominantly bending or at least strongly coupled.

As shown in all figures, existence of free ends increases the errors because their cross-sectional rotation is obviously expected to be larger. A ship's hull is an industrial example of this case.

The general effect of rotary inertia is to decrease the natural frequencies. For beams where warping is not negligible, the rotary inertia effect should absolutely be taken account of especially if free ends are of concern.

References

[1] S.P. Timoshenko, D.H. Young, W. Weaver Jr., Vibration Problems in Engineering, Wiley, New York, 1974.

[2] P.O. Friberg, Coupled vibrations of beams—an exact dynamic element stiffness matrix, International Journal for Numerical Methods in Engineering 19 (1983) 479–493.

[3] E. Dokumaci, An exact solution for coupled bending and torsion vibrations of uniform beams having single cross-sectional symmetry, Journal of Sound and Vibration 119 (3) (1987) 443–449.

[4] R.E.D. Bishop, M. Cannon, S. Miao, On coupled bending and torsional vibration of uniform beams, Journal of Sound and Vibration 131 (1) (1989) 457–464.

[5] J.R. Banerjee, Coupled bending—torsional dynamic stiffness matrix for beam elements, International Journal for Numerical Methods in Engineering 28 (1989) 1283–1298.

[6] R.E.D. Bishop, W.G. Price, Coupled bending and twisting of a Timoshenko beam, Journal of Sound and Vibration 50 (4) (1977) 469–477.

[7] J.R. Banerjee, F.W. Williams, Coupled bending—torsional dynamic stiffness matrix for Timoshenko beam elements, Journal of Sound and Vibration 42 (3) (1992) 301–310.

[8] A.N. Bercin, M. Tanaka, Coupled flexural–torsional vibrations of Timoshenko beams, Journal of Sound and Vibration 207 (1) (1997) 47–59.

[9] Y. Yaman, Vibration of open-section channels: a coupled flexural and torsional wave analysis, Journal of Sound and Vibration 204 (1) (1997) 131–158.

[10] A. Arpaci, E. Bozdag, On free vibration analysis of thin-walled beams with nonsymmetrical open cross-sections, Computers and Structures 80 (2002) 691–695.

[11] S.P. Timoshenko, Strength of Materials, Van Nostrand, New York, 1960.

[12] J.M. Gere, Torsional vibrations of beams of thin-walled open section, Journal of Applied Mechanics 21 (1954) 381–387.