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Letter to the Editor

On the periods of the periodic solutions of the non-linear oscillator equation $\ddot{x} + x^{1/(2n+1)} = 0$

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In a series of papers Mickens and his co-authors [1–3], and Awrejcewicz and Andrianov [4] considered non-linear oscillator equations of the form

$$\ddot{x} + f(x) = 0. \quad (1)$$

In particular, the case $f(x) = x^{1/(2n+1)}$ with n a positive integer has been studied in Refs. [2–4]. Using a generalized harmonic balance method (see Ref. [1]) approximations of the periodic solutions have been constructed in Refs. [2,3] in the form

$$x(t) \simeq \frac{A\cos(\omega t)}{(1 + B\cos(2\omega t))}, \quad (2)$$

where A, B, and ω are to be determined as functions of the special initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. The ultimate procedure used in Ref. [3] to calculate A, B, and ω is based on the numerical integration of the differential equation subject to $x(0) = x_0$ and $\dot{x}(0) = 0$. The "angular frequency" $\omega_n(x_0)$ of the periodic solution of (1) with $f(x) = x^{1/(2n+1)}$ was approximated in Ref. [2] by

$$\omega_n(x_0) \simeq \left[\frac{2^{2n}}{\binom{2n+1}{n} x_0^{2n}}\right]^{1/(4n+2)}. \quad (3)$$

From Eq. (3) the period $T_n(x_0)$ of the periodic solution of Eq. (1) with $f(x) = x^{1/(2n+1)}$ can be approximated by $2\pi/\omega_n(x_0)$. In Ref. [4] the so-called small δ-method has been applied to approximate $T_n(x_0)$.

In this paper an exact, analytical expression for $T_n(x_0)$ will be given, which can easily be approximated numerically (up to any desired accuracy). First of all it should be observed that

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Eq. (1) with $f(x) = x^{1/(2n+1)}$ has as an integrating factor $\dot{x}$. Using this integrating factor the following first integral is obtained:

$$\frac{1}{2}(\dot{x})^2 + \frac{(2n+1)}{(2n+2)} x^{(2n+2)/(2n+1)} = c, \tag{4}$$

where c is a non-negative constant of integration. It follows from Eq. (4) that all orbits in the phase-plane (that is, in the $(x, \dot{x})$-plane) are closed, and are symmetric with respect to the x-axis, and are symmetric with respect to the $\dot{x}$-axis. So, all solutions of Eq. (1) with $f(x) = x^{1/(2n+1)}$ are periodic. Without loss of generality, it can be assumed that a periodic solution starts at $t = 0$ in $(x(0), \dot{x}(0)) = (x_0, 0)$ with $x_0 > 0$, and so c in Eq. (4) is equal to $((2n+1)/(2n+2))x_0^{(2n+2)/(2n+1)}$.

Let $T_n(x_0)$ be the period of this periodic solution. Since the orbits in the phase-plane are symmetric with respect to the x-axis and the $\dot{x}$-axis it follows that $(x(\frac{T_n(x_0)}{2}), \dot{x}(\frac{T_n(x_0)}{2})) = (-x_0, 0)$. From Eq. (4) it then follows that

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \pm\sqrt{\frac{2(2n+1)}{2n+2}}\sqrt{x_0^{(2n+2)/(2n+1)} - x(t)^{(2n+2)/(2n+1)}},$$

or equivalently

$$\frac{1}{\sqrt{(2n+1)/(n+1)}\sqrt{x_0^{(2n+2)/(2n+1)} - x^{(2n+2)/(2n+1)}}}\frac{\mathrm{d}x}{\mathrm{d}t} = \pm 1. \tag{5}$$

Then, integrating Eq. (5) with respect to t from $t = 0$ to $T_n(x_0)/2$ yields

$$\frac{T_n(x_0)}{2} = \frac{1}{\sqrt{(2n+1)/(n+1)}}\int_{-x_0}^{x_0} \frac{1}{\sqrt{x_0^{(2n+2)/(2n+1)} - x^{(2n+2)/(2n+1)}}}\,\mathrm{d}x,$$

and after introducing the new dimensionless variable $s = x/x_0$ instead of x, the period $T_n(x_0)$ of the vibration becomes

$$T_n(x_0) = \frac{4x_0^{n/(2n+1)}}{\sqrt{(2n+1)/(n+1)}}\int_0^1 \frac{1}{\sqrt{1 - s^{(2n+2)/(2n+1)}}}\,\mathrm{d}s. \tag{6}$$

To avoid computational difficulties when the integral in Eq. (6) is integrated numerically, it should be observed that after some integrations by parts this integral can be rewritten in

$$\int_0^1 \frac{1}{\sqrt{1 - s^{(2n+2)/(2n+1)}}}\,\mathrm{d}s = \frac{(3n+2)}{n+1}\int_0^1 \sqrt{1 - s^{(2n+2)/(2n+1)}}\,\mathrm{d}s. \tag{7}$$

From Eqs. (6) and (7) it then follows that the period $T_n(x_0)$ of a periodic solution of $\ddot{x} + x^{1/(2n+1)} = 0$ (with $x(0) = x_0 > 0$, $\dot{x}(0) = 0$, and n a positive integer) is given by

$$T_n(x_0) = \frac{4(3n+2)x_0^{n/(2n+1)}}{\sqrt{(2n+1)(n+1)}}\int_0^1 \sqrt{1 - s^{(2n+2)/(2n+1)}}\,\mathrm{d}s. \tag{8}$$

For $n = 0$ (the harmonic oscillator case) it follows from Eq. (8) that $T_0(x_0)$ is equal to the well-known value 2π. For large values of n (and for finite, n-independent, and fixed values of x_0) it also follows that $x_0^{n/(2n+1)} \to x_0^{1/2}$ and $\sqrt{1 - s^{(2n+2)/(2n+1)}} \to \sqrt{1-s}$, and so, it follows from Eq. (8) that $T_n(x_0) \to 4\sqrt{2}\,x_0^{1/2}$ for $n \to \infty$. For other values of n the integral in Eq. (8) has to be calculated

Table 1
The period $T_n(x_0)$ and the "angular frequency" $\omega_n(x_0)$ accurate up to five decimal places, and the approximations of $\omega_n(x_0)$ as given in Ref. [2]

n	$T_n(x_0)$	$\omega_n(x_0)$	Approximation of $\omega_n(x_0)$ as given in Ref. [2] (see also Eq. (3))
0	2π	1.00000	1.00000
1	$x_0^{1/3}$ 5.86966	$x_0^{-1/3}$ 1.07045	$x_0^{-1/3}$ 1.04912
2	$x_0^{2/5}$ 5.78495	$x_0^{-2/5}$ 1.08613	$x_0^{-2/5}$ 1.04812
3	$x_0^{3/7}$ 5.74847	$x_0^{-3/7}$ 1.09302	$x_0^{-3/7}$ 1.04405
4	$x_0^{4/9}$ 5.72816	$x_0^{-4/9}$ 1.09689	$x_0^{-4/9}$ 1.04017
5	$x_0^{5/11}$ 5.71522	$x_0^{-5/11}$ 1.09938	$x_0^{-5/11}$ 1.03684
6	$x_0^{6/13}$ 5.70626	$x_0^{-6/13}$ 1.10110	$x_0^{-6/13}$ 1.03403
7	$x_0^{7/15}$ 5.69968	$x_0^{-7/15}$ 1.10238	$x_0^{-7/15}$ 1.03164
8	$x_0^{8/17}$ 5.69465	$x_0^{-8/17}$ 1.10335	$x_0^{-8/17}$ 1.02960
9	$x_0^{9/19}$ 5.69067	$x_0^{-9/19}$ 1.10412	$x_0^{-9/19}$ 1.02783
10	$x_0^{10/21}$ 5.68745	$x_0^{-10/21}$ 1.10474	$x_0^{-10/21}$ 1.02628
50	$x_0^{50/101}$ 5.66322	$x_0^{-50/101}$ 1.10947	$x_0^{-50/101}$ 1.00919
500	$x_0^{500/1001}$ 5.65750	$x_0^{-500/1001}$ 1.11059	$x_0^{-500/1001}$ 1.00149
∞	$x_0^{1/2} 4\sqrt{2}$	$x_0^{-1/2} \frac{\pi}{2\sqrt{2}}$	$x_0^{-1/2}$ 1.00000

numerically. Using a standard numerical integration routine (as for instance available in the formula manipulation package Maple) the integral in Eq. (8) can easily be approximated numerically (up to any desired accuracy). For some values of n approximations of the period $T_n(x_0)$ and approximations of the "angular frequency" $\omega_n(x_0) = 2\pi/T_n(x_0)$ are given in Table 1 up to five decimals. Also in Table 1 the approximations of $\omega_n(x_0)$ (as obtained in Refs. [2,3] by using a harmonic balance/numerical method, and given by Eq. (3)) are listed. As can be seen from this table the fractional errors of the approximations as obtained in Refs. [2,3] for $n \geqslant 1$ range from approximately 2 to approximately 11 per cent. The approximations of $T_n(x_0)$ as obtained in Ref. [4] by using the small δ-method can also readily compared with the accurate (up to five decimals) results as given in Table 1.

Finally it should be remarked that the analysis (to obtain periods of periodic solutions) as presented in this paper is not only restricted to the non-linear oscillator given by Eq. (1) with $f(x) = x^{1/(2n+1)}$, but can be extended to more general, non-linear oscillator equations.

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