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Letter to the Editor

Vibration isolation using buckled struts

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1. Vibration isolation

The concept of vibration isolation, i.e., how to minimize the force or motion transmitted to a device from a source of vibration, is well established (see for example, Ref. [1]). An outline of the theory is based on the following.

Consider a translational mechanical system consisting of a mass supported by a spring and damper, which are themselves supported by a base as shown in Fig. 1. If the motion of the base is harmonic, e.g., $y(t) = Y \sin \omega t$, then it can be shown that the steady state displacement transmissibility, X/Y (where X is the amplitude of the mass), is given by the familiar expression

$$\frac{X}{Y} = \left[\frac{1 + (2\zeta\Omega)^2}{(1 - \Omega^2)^2 + (2\zeta\Omega)^2}\right]^{1/2}. \tag{1}$$

This is plotted in Fig. 2 as a function of the frequency ratio $\Omega = \omega/\omega_n$, where $\omega_n = \sqrt{k/m}$ for a number of representative damping ratios, ζ.

Thus, the transmissibility is small for relatively high-frequency ratios, i.e., for $\Omega > \sqrt{2}$, $X/Y < 1.0$. Given a forcing frequency, ω, a typical design option would be to mount the device on a soft spring to induce a low natural frequency, ω_n. However, the spring should obviously be stiff enough to support the mass statically (since $mg = kx$), and this often places practical limits on the spring stiffness.

2. Post-buckling of a strut

Axially loaded structures typically possess non-linear characteristics, especially close to, or beyond, initial buckling. In most cases, buckling is viewed as an undesirable occurrence, in particular when this precipitates a total loss of stiffness and collapse. However, certain structural forms possess a significant amount of post-buckled strength which can be utilized in design. For

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example, in aeronautics the weight of a flight vehicle is of supreme importance and hence allowing a panel to buckle (elastically) may be admissible.

In this work we develop the notion of using post-buckled struts as the spring components in a vibration isolation system, as indicated schematically in Fig. 1.

Consider a thin elastic structural element, pinned at both ends and subject to constant axial loading as shown in Fig. 3. This is the classic Euler strut (elastica) with flexural rigidity EI,

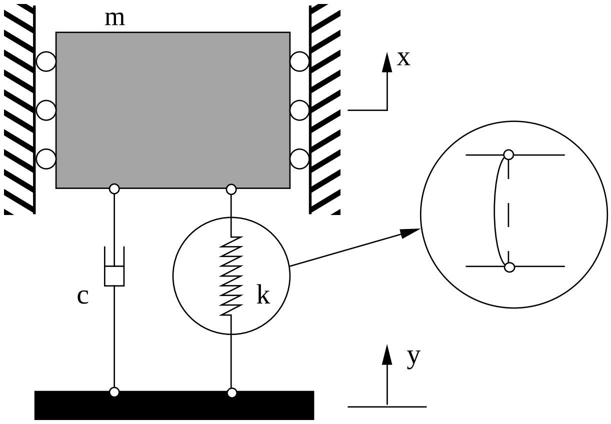

Fig. 1. Schematic of a mass isolated from the motion of a foundation.

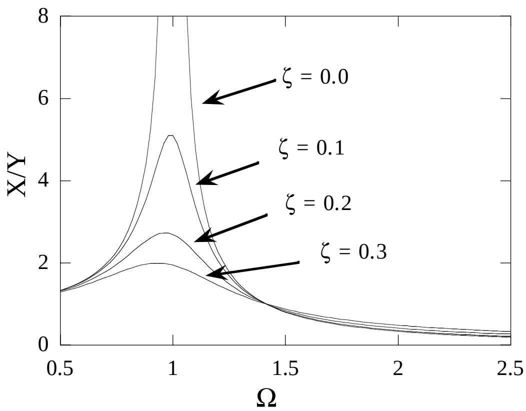

Fig. 2. Displacement transmissibility.

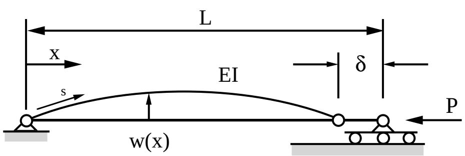

Fig. 3. Schematic of a continuous buckled strut.

length L, and P is the axial load. An exact (but cumbersome) solution to the governing equation can be obtained using elliptic integrals [2]. Since we will be focusing on mildly buckled behavior certain truncations can be performed or an approximate energy analysis can be used [3].

For this type of thin prismatic beam, it can be shown, assuming a half-sine wave as the buckling mode shape, i.e.,

$$w = Q \sin \frac{\pi x}{L}, \tag{2}$$

that the postbuckled equilibrium configuration is described by

$$\frac{P}{P_e} = 1 + \frac{\pi^2}{8}\left(\frac{Q}{L}\right)^2, \tag{3}$$

where

$$P_e = EI\left(\frac{\pi}{L}\right)^2. \tag{4}$$

P_e is the classical Euler critical load [4,5], and provided the axial load remains less than this value, the trivial equilibrium state is stable. Beyond the critical value, the strut deflects into one of two symmetrically located equilibria as given by Eq. (3).

The geometric relation between the lateral deflection Q and the end shortening δ can also be established:

$$\left(\frac{\delta}{L}\right) = \frac{\pi^2}{4}\left(\frac{Q}{L}\right)^2 + \frac{3\pi^4}{64}\left(\frac{Q}{L}\right)^4, \tag{5}$$

i.e., the end shortening is approximately related to the square of the lateral deflection [4]. Eliminating Q in Eqs. (3) and (5) leads to

$$\frac{P}{P_e} = \frac{1}{3}\left[2 + \sqrt{1 + 3\left(\frac{\delta}{L}\right)}\right] \approx 1 + \frac{1}{2}\left(\frac{\delta}{L}\right). \tag{6}$$

This is a result good for lateral deflections up to about 20% of the length, [2]. Eqs. (3) and (6) are shown by the solid curves in Fig. 4(a) and (b), respectively.

It is well known that initial geometric imperfections have a relatively large influence on the behavior of axially loaded structures [5,6]. This effect can be incorporated into the analysis by assuming an initial equilibrium configuration described, for example, by

$$w_0 = Q_0 \sin \frac{\pi x}{L}. \tag{7}$$

An expression analogous to Eq. (3) but incorporating the initial curvature of the strut is given by [2]

$$\frac{P}{P_e} = \left(1 - \frac{Q_0}{Q}\right)\left(1 + \frac{\pi^2}{8L^2} Q^2\right). \tag{8}$$

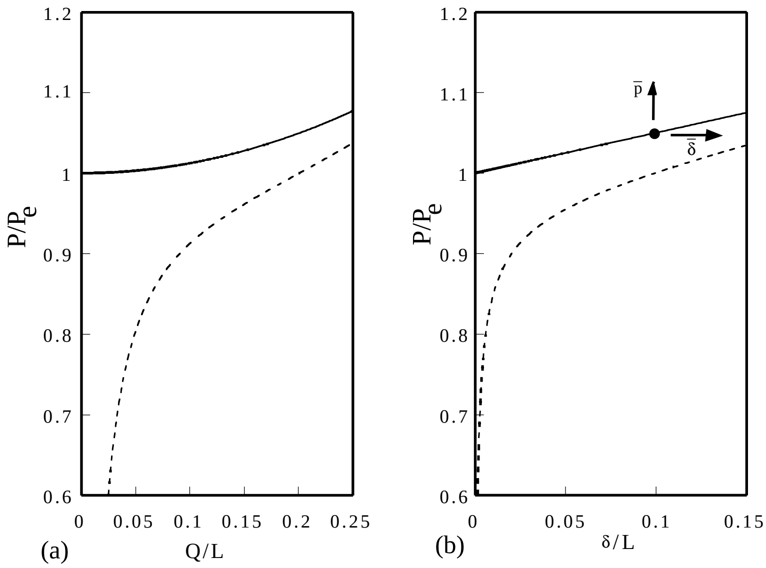

Fig. 4. Deflection of the strut as a function of axial load: (a) midpoint lateral deflection, (b) end shortening.

The corresponding imperfect load-deflection curves are also plotted in Fig. 4 as dashed lines for a representative imperfection amplitude of $Q_0/L = 0.01$. The relation between axial load and end shortening for moderately large lateral deflections is then given by

$$\frac{P}{P_e} = \left[1 - \pi\left(\frac{Q_0}{L}\right)\left(\pi^2\left(\frac{Q_0}{L}\right)^2 + 4\left(\frac{\delta}{L}\right)\right)^{-1/2}\right]\left[1 + \frac{\pi^2}{8}\left(\frac{Q_0}{L}\right)^2 + \frac{1}{2}\left(\frac{\delta}{L}\right)\right], \tag{9}$$

where we recover Eq. (6) for $Q_0/L = 0$.

Hence, if we load the strut axially to just above its elastic critical load, for example,

$$\frac{P}{P_e} = 1.05, \tag{10}$$

then (for $Q_0 = 0$):

$$\frac{Q}{L} \approx 0.2, \quad \frac{\delta}{L} \approx 0.1. \tag{11}$$

If we use this as our basic equilibrium, we observe a (low slope) linear relation for small oscillations about this point based on new local co-ordinates:

$$\bar{p} = \frac{P}{P_e} - 1.05, \quad \bar{\delta} = \frac{\delta}{L} - 0.1, \tag{12}$$

and indicated in Fig. 4(b). Again, for the imperfect strut we would expect an equilibrium datum for a slightly lower axial load but with practically the same stiffness. This strut then, is able to support a relatively high axial load (sufficient to cause buckling) but exhibits the desirable soft spring characteristic.

3. Experimental verification

3.1. The shaker

In order to test this approach to vibration isolation the experimental system shown in Fig. 5 was constructed. The vertical shaker was based on a slider-crank attached to a variable speed motor. The forcing amplitude was fixed at 3 mm, and Fig. 6(a) shows a typical time series of the vertical motion of the platform (which is not quite simple harmonic). Part (b) of this figure is a frequency spectrum of this signal. We observe the higher harmonics typical of this kind of shaker. Furthermore, we also note some slight variability in the amplitude of the shaker platform. These effects are not modelled in this paper, but a more thorough description of their effect on dynamic response can be found in Ref. [7]. It is anticipated that a more consistent harmonic input would result in improved isolation. The shaker then imparts motion through the isolation system to the

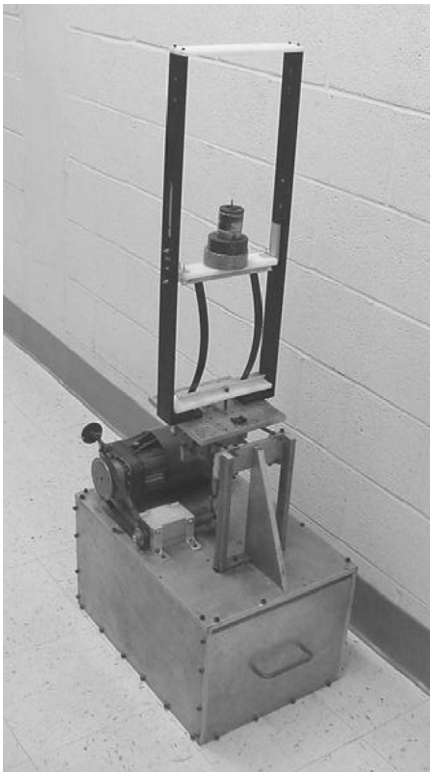

Fig. 5. Photograph of the experimental set-up.

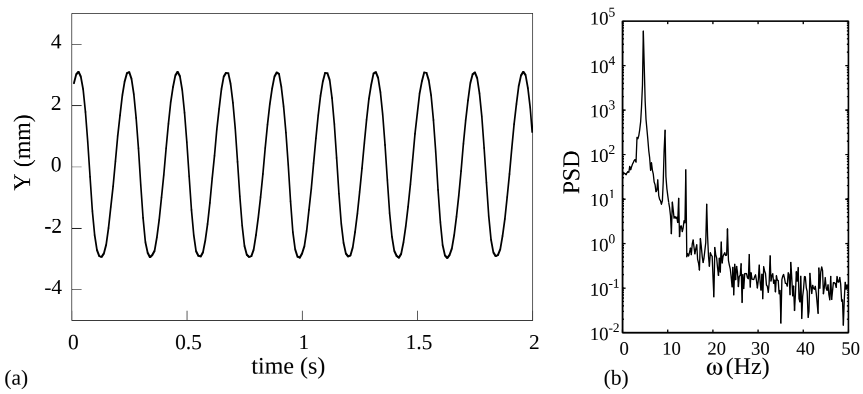

Fig. 6. The characteristics of the shaker: (a) input time series, (b) corresponding frequency spectrum.

mass, which moves vertically and is guided by low-friction linear bearings. The frequency of excitation was then varied over an appropriate range and the motion of the mass was measured.

3.2. Spring characteristics

Two struts made of spring steel were used as the support system. The experimental results are presented in terms of dimensional units. Plots of axial load versus lateral deflection and end shortening are shown in Fig. 7, where the deflections were measured using an LVDT. The two struts were 268 mm long, 19 mm wide, 0.66 mm thick (and thus $I = 4.55 \times 10^{-13}$ m^4), and taking a typical value for Young's modulus of 200 GPa, we anticipate an Euler load (using Eq. (4)) in the

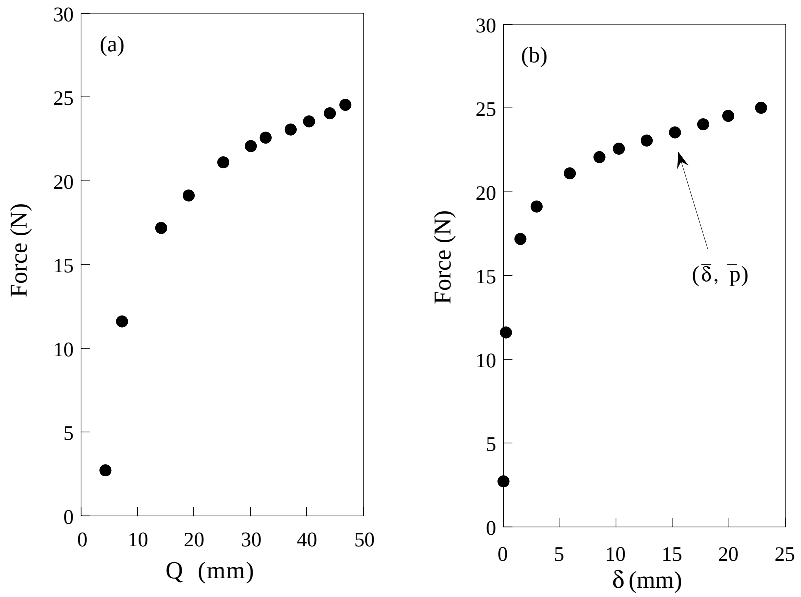

Fig. 7. Axial load versus (a) midpoint lateral deflection and (b) end shortening for the experimental system.

vicinity of 25 N $\equiv$ 2.55 kg. The experimental result indicates a critical load in this vicinity, and a Southwell plot [2] can be used to recast the data from Fig. 7(a) to estimate a critical load of approximately 23 N. Experimental systems necessarily include initial geometric imperfections, and hence the "critical load" is manifest as a relatively rapid increase in the deflection due to additional load. The axial load versus end shortening results are shown in Fig. 7(b). These data are based on the average results of three tests, although the actual variation between runs was negligible. These results can be compared with the corresponding theoretical curves given in Fig. 4. Also plotted in this figure is the corresponding lateral deflection at the center of the strut (Fig. 7(a)). The quadratic relation between lateral deflection and end shortening established by Eq. (5) is thus confirmed.

We choose a point on this (imperfect) curve (Fig. 7(b)) as the (buckled) equilibrium position:

$$P = 23.5\ \text{N} \rightarrow (P/P_e \approx 1.0), \tag{13}$$

$$\delta = 15.2\ \text{mm} \rightarrow (\delta/L = 0.057), \tag{14}$$

and this is the point about which transmissibility is measured. Locally the stiffness is approximately 195 N/m (i.e., the slope of P/P_e versus δ/L about the chosen operating point) and since the mass is 2.4 kg we would thus expect a natural frequency of free vibration close to 9 rad/s $\equiv$ 1.43 Hz. A representative free decay result is shown in Fig. 8. This illustrates a damped oscillation started from an offset and released, giving a natural period of approximately 0.68 s (and hence $\omega_n = 1.47$ Hz), which corresponds quite closely to the estimated value, although we would expect damping to reduce the measured frequency somewhat. However, we anticipate effective isolation for forcing frequencies $\omega > \sqrt{2}\omega_n \approx 2.2$ Hz. It is worth noting that replacing the buckled struts with conventional linear springs of the same stiffness would result in a static deflection of approximately 60 mm, i.e., four times the deflection of the struts.

3.3. *The forced response*

When the base excitation is applied to the strut-supported mass, we desire the transmissibility ratio (given by Eq. (1)) to be small. This should be the case for $\Omega > \sqrt{2}$. Experimental results are shown in Fig. 9 for a range of frequency ratios. The low transmissibility illustrated in Fig. 2 is thus confirmed. Three responses are shown as insets for different frequency ratios. The time series occurring when the forcing frequency is exactly twice the natural frequency (indicated by the star)

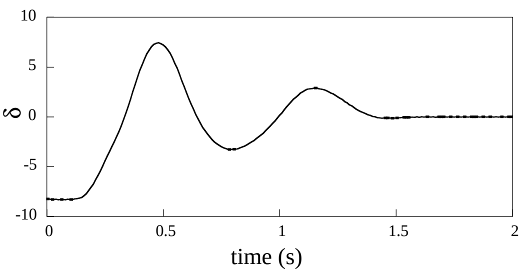

Fig. 8. Free vibration of the system about a post-buckled equilibrium configuration. Note, in this figure the origin corresponds to a static deflection of 15.2 mm.

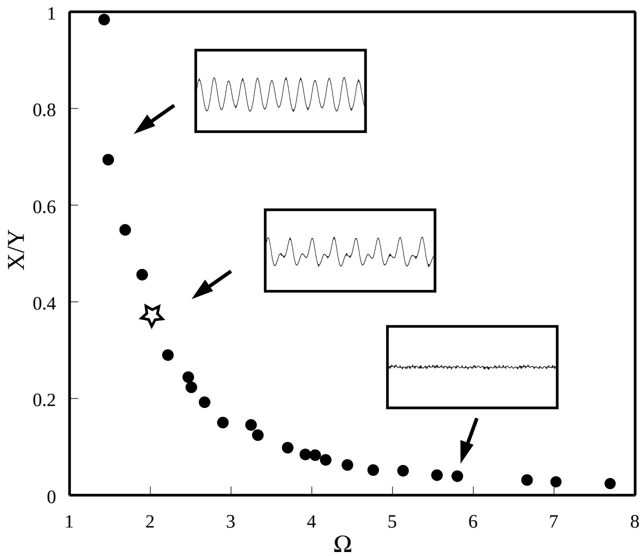

Fig. 9. Transmissibility for the displacement of the strut-supported mass.

shows an interesting subharmonic of order two. This is a consequence of combined parametric and external forcing terms in the governing dynamics equations and has been the subject of an analytic and numerical study [8]. In general we see a highly attenuated response for higher frequencies.

4. Conclusions

This paper has considered a novel use of buckled struts for the purpose of vibration isolation. Experimental verification was achieved without the need for fine tuning, although of course there are a number of practical limitations involved with this concept. It is important that the axial load does not reduce below the critical buckling value, since this would result in a relatively hard stroke-out (i.e., the axial stiffness in the pre-buckled state is very high, although the initial imperfection helps to mitigate this effect). Also, for large deformations the “linearity” in the post-buckled stiffness is lost and various non-linear effects may be encountered [9]. The transmissibility can also be reduced by *reducing* the damping present in the system. Although, of course, a high damping is desirable in the event that the system forcing tends to cause resonance. In a practical situation it may also be quite difficult to maintain simply-supported boundary conditions and clamped ends may be easier to achieve. However, changing the number (or thickness) of the struts provides useful means of changing the critical load without compromising the low stiffness. This approach has been shown to have promise in isolating very sensitive systems subject to very high frequency excitation [10]. However, these initial results are encouraging, and indeed, there may be other types of post-buckled structures which would fulfil the requirements of vibration isolation even more effectively.

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