ACADEMIC PRESS

Available online at www.sciencedirect.com
SCIENCE DIRECT

Journal of Sound and Vibration 26 (2003) 109–122

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

The R-function method for the free vibration analysis of thin orthotropic plates of arbitrary shape

Lidia Kurpa[a], Vladimir Rvachev[a], Eduard Ventsel[b,*]

[a] *Division of Applied Mathematics, Institute of Mechanical Engineering of the Ukrainian National Academy of Sciences, 2/10 Pozharsky St., 310146 Kharkov, Ukraine*

[b] *Department of Engineering Science & Mechanics, The Pennsylvania State University, 205A Earth & Engineering Sciences Building, University Park, PA 16802-1401, USA*

Received 2 August 2001; accepted 3 May 2002

Abstract

Free flexural vibrations of homogeneous, thin, orthotropic plates of an arbitrary shape with mixed boundary conditions are studied using the R-function method. The proposed method is based on the use of the R-function theory and variational methods. In contrast to the widely used methods of the network type (finite differences, finite element, and boundary element methods), in the R-function method all the geometric information given in the boundary value problem statement is represented in an analytical form. This allows one to seek a solution in a form of some formulas called a solution structure. These solution structures contain some indefinite functional components that can be determined by using any variational method. A method of constructing the solution structures satisfying the required mixed boundary conditions for eigenvalue plate bending problems is described. Numerical examples for the vibration analysis of orthotropic plates of complex geometry with mixed boundary conditions for illustrating the aforementioned R-function method and comparison against the other methods are made to demonstrate its merits.

1. Introduction

The vibration analysis of isotropic and orthotropic thin plates of an arbitrary shape with mixed boundary conditions is of primary importance for structural mechanics. At the same time, it presents considerable mathematical difficulties. Dynamic analysis of isotropic and orthotropic plates with arbitrary boundary conditions has been analyzed by many researchers [1–14].

*Corresponding author. Tel.: +1-814-865-4523; fax: +1-814-863-7967.

PII: S0022-460X(02)00946-X

Analytical solutions for such kinds of problems can be obtained for thin plates with relatively simple plan forms [1–3,7]. In all other cases, the dynamic analysis mentioned above can be done only by numerical methods. The most widely used numerical methods for this type of vibration problems are the finite element method (FEM) and boundary element method (BEM). Literature reviews of dynamic analysis of plates using BEM and FEM are given by Beskos [10] and Mackerle [11]. Free vibrations of plates of an arbitrary shape have also been studied by the boundary collocation method [12,13]. A comprehensive review of vibration analysis of plates by analytical and some numerical methods is given by Leissa [3].

Despite the success of FEM and BEM in dynamic analysis of plates of an arbitrary shape, there are serious drawbacks. FEM requires a large number of domain elements to obtain accurate results. BEM offers two basic approaches for treating plate dynamic analysis. The first approach employs the dynamic fundamental solution of the problem in its formulation and this results in an integral representation involving only boundary integrals. The efficiency of such an approach is greatly hampered by the complicated form of the dynamic fundamental solution, which involves Hankel functions. As a result, the free vibration problem involves frequency-dependent complex matrices. Computation of the natural frequencies, based on the determinant method, is not efficient. The second approach utilizes the static fundamental solutions in the formulation of the problem, which creates domain integrals in the integral representation of the solution. This circumstance also extends the body of numerical calculations and reduces the accuracy of the method. In addition, it should be noted that an automation of the discretization process of domains (or boundaries) in the above-mentioned methods of the network type is needed to use rather intricate algorithms. This often leads to the necessity of solving additional optimization problems [15].

In this paper, the R-function method (RFM) [16,17] based on the joint application of variational methods and the R-function theory is developed for solving vibration problems of thin, orthotropic plates of arbitrary plan forms with mixed boundary conditions. In contrast to widely used methods of the network type (BEM, FEM, and finite differences method), in RFM all the geometric information contained in the mathematical models of the fields is transformed to the analytical form. This allows one to seek a solution in the form of analytical expressions called solution structures and to construct approximation sequences with the use of variational or other methods. The solution structures represent a system of co-ordinate functions that satisfy exactly all of the boundary conditions for an arbitrary geometry of a domain involved. The above-mentioned allows one to use R-functions as a theoretical basis when developing software packages for solving various types of boundary value problems of solid mechanics, in particular for the dynamic analysis of thin, orthotropic plates of complex geometry. The authors have developed RFM for solving static and vibration problems of thin, isotropic plates previously [18]. In the present paper, the method is extended to include orthotropic plates of an arbitrary plan form in the dynamic analysis.

2. Governing equations

Consider a thin, orthotropic plate of constant thickness of surface Ω and boundary $\partial\Omega$. Let us assume that the axes of symmetry of an orthotropic material coincide with the x_1, x_2 axes in rectangular co-ordinates. The governing differential equation of free vibration bending problems of this plate, on the basis of Kirchhoff's classical small deflection theory, has the form [19]

$$D_1\frac{\partial^4 W}{\partial x_1^4}+2H\frac{\partial^4 W}{\partial x_1^2\partial x_2^2}+D_2\frac{\partial^4 W}{\partial x_2^4}-\rho\lambda^2 W=0, \tag{1}$$

where

$$D_1=\frac{E_1h^3}{12(1-\vartheta_1\vartheta_2)},\quad D_2=\frac{E_2h^3}{12(1-\vartheta_1\vartheta_2)},\quad H=D_1\vartheta_2+2D_k,\ \ D_k=\frac{Gh^3}{12}. \tag{2}$$

Here $W=W(\mathbf{x},t)$ is the lateral deflection and $E_1, E_2, \vartheta_1, \vartheta_2$ are the moduli of elasticity and the Poisson ratios in the direction of the axes Ox_1, Ox_2, respectively. G is the shear modules, h is the plate thickness, ρ is the mass density per unit plate area, and λ is the frequency.

The deflections of the plate must satisfy prescribed boundary conditions on the boundary $\partial\Omega=\bigcup_{i=0}^{M}\partial\Omega_i$. The homogeneous boundary conditions usually employed are as follows [20].

(1) Clamped edge $\partial\Omega_1$:

$$W=0,\quad \frac{\partial W}{\partial n}=0\quad \forall\mathbf{x}=(x_1,x_2)\in\partial\Omega_1; \tag{3}$$

(2) simply supported edge $\partial\Omega_2$:

$$W=0,\quad M_n=0\quad \forall\mathbf{x}=(x_1,x_2)\in\partial\Omega_2; \tag{4}$$

(3) free edge $\partial\Omega_3$:

$$M_n=0,\quad Q_n-\frac{\partial M_{n\tau}}{\partial s}=0\quad \forall\mathbf{x}=(x_1,x_2)\in\partial\Omega_3; \tag{5}$$

where

$$M_n=M_1\cos^2\alpha+M_2\sin^2\alpha-2M_{12}\cos\alpha\sin\alpha, \tag{6}$$

$$M_{n\tau}=(M_1-M_2)\sin\alpha\cos\alpha+M_{12}(\cos^2\alpha-\sin^2\alpha), \tag{7}$$

$$Q_n=Q_1\cos\alpha+Q_2\sin\alpha, \tag{8}$$

$$M_1=-D_1\left(\frac{\partial^2 W}{\partial x_1^2}+\vartheta_2\frac{\partial^2 W}{\partial x_2^2}\right),\quad M_2=-D_2\left(\frac{\partial^2 W}{\partial x_2^2}+\vartheta_1\frac{\partial^2 W}{\partial x_1^2}\right),$$

$$M_{12}=-2D_k\frac{\partial^2 W}{\partial x_1\partial x_2}, \tag{9}$$

$$Q_1=\frac{\partial M_1}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2},\quad Q_2=\frac{\partial M_2}{\partial x_2}-\frac{\partial M_{12}}{\partial x_1}. \tag{10}$$

In the aforementioned relations, M_n and $M_{n\tau}$ are the normal and twisting moments at a point on the boundary with the outward normal n and the tangent τ, respectively, and Q_n is the normal shear force defined also on the boundary; s denotes the arc length measured along the boundary; α is an angle between the outward normal n and the x_1 axis.

3. The general scheme of RFM in the vibration plate bending problems

According to RFM, a solution of boundary value problems is represented in the form of the solution structure. The latter has the form of a formula and for the vibration plate bending problems (1)–(5) can be sought as [16,17]

$$W = B(\Phi, \omega, \omega_i), \tag{11}$$

where Φ is an indefinite component of the solution structure and ω, ω_i are some functions that describe equation of a domain boundary $\partial\Omega$ or its separate parts $\partial\Omega_i$ and satisfy the prescribed boundary conditions (3)–(5). The indefinite component Φ can be found by using some general principles of approximate methods. So, in reducing an infinite-dimensional problem to a finite-dimensional one, let us represent the above-mentioned component in the form of

$$\Phi(\mathbf{x}) \cong \Phi_N = \sum_{j=1}^{N} C_j \varphi_j(\mathbf{x}), \tag{12}$$

where $\{\varphi_j\}$ are known elements of some functional space containing Φ and forming some complete sequences in this space. For $\{\varphi_j\}$ we may select, for instance, algebraic or Chebyshev polynomials, trigonometric functions, splines, or other approximation functions. The unknown coefficients C_j in representation (12) can be determined from conditions of the best satisfaction (in one sense or another) of the governing Eq. (1). As it follows from the theory of variational methods [21], such conditions are equivalent to the problem of minimizing the functional

$$I(W) = \int_{\Omega} \left[D_1 \left(\frac{\partial^2 W}{\partial x_1^2} \right)^2 + 2D_1 \vartheta_2 \frac{\partial^2 W}{\partial x_1^2} \frac{\partial^2 W}{\partial x_2^2} + D_2 \left(\frac{\partial^2 W}{\partial x_2^2} \right)^2 + 4D_k \left(\frac{\partial^2 W}{\partial x_1 \partial x_2} \right)^2 - \rho \lambda^2 W^2 \right] \mathrm{d}\Omega. \tag{13}$$

Notice that the second condition (4) and conditions (5) are natural for the functional (13) because it is known that a minimization of the functional can be carried out on a set of functions satisfying only the so-called principal or kinematic conditions (3) and (4) (first one) [21].

The general procedure of solving the boundary value problem (1)–(5) by RFM includes the following steps:

(1) Write equation $\omega = 0$ of the boundary $\partial\Omega$ or its separate parts $\partial\Omega_i$ if mixed boundary conditions are given.
(2) Construct the solution structure (11) that satisfies exactly all the prescribed boundary conditions or only principal boundary conditions and contains some indefinite, free components Φ.
(c) Determine these indefinite components by using any variational or other methods.

Let us consider the construction of the solution structure (11) for orthotropic plates with mixed boundary conditions. For example, consider the solution structure that satisfies exactly the mixed boundary conditions of the type partially clamped ($\partial\Omega_1$) and partially simply supported ($\partial\Omega_2$) parts of the boundary $\partial\Omega$. To develop this structure we can use the formula

$$W = \left(\frac{W_1}{\omega_1^2} + \frac{W_2}{\omega_2^3} \right) \left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^3} \right)^{-1}. \tag{14}$$

Here, W_1 is the solution structure that satisfies exactly the boundary conditions (3) on $\partial\Omega_1$ and W_2 is the structure that satisfies the boundary conditions (4) on $\partial\Omega_2$. It can be seen that W_1 can be represented as

$$W_1 = \omega_1^2 \Phi_1, \tag{15}$$

where $\omega_1 = 0$ is the equation of a clamped part $\partial\Omega_1$ of the boundary.

Let us consider in detail how to construct the function W_2. Substituting expressions (9) for M_1, M_2, and M_{12} into Eq. (6) and going from differentiation with respect to the variables x_1, x_2 to the differentiation with respect to the normal n and the tangent τ directions at point $\mathbf{x} \in \partial\Omega$, Eq. (6) becomes

$$M_n = A_1^0 \frac{\partial^2 W}{\partial n^2} + 2A_2^0 \frac{\partial^2 W}{\partial n \partial t} + A_3^0 \frac{\partial^2 W}{\partial \tau^2}, \tag{16}$$

where

$$\begin{aligned} A_1^0 &= -(D_1 \cos^4 \alpha + D_2 \sin^4 \alpha + 2H \cos^2 \alpha \sin^2 \alpha), \\ A_2^0 &= -((H - D_1)\cos^2\alpha + (D_2 - H)\sin^2 \alpha)\cos \alpha \sin \alpha, \\ A_3^0 &= -(D_1 \vartheta_2 + (D_1 + D_2 - H)\cos^2 \alpha \sin^2 \alpha). \end{aligned} \tag{17}$$

Let $\omega_2 = 0$ be a normalized equation of the simply supported part $\partial\Omega_2$ of the boundary. This means that ω_2 must satisfy the conditions [16]:

$$\omega_2 = 0, \quad \forall \mathbf{x} = (x_1, x_2) \in \partial\Omega_2, \ \ |\nabla \omega_2(\mathbf{x})| = 1, \quad \forall \mathbf{x} = (x_1, x_2) \in \partial\Omega_2, \ \ \omega_2(\mathbf{x}) > 0 \quad \forall \mathbf{x} = (x_1, x_2) \in \Omega. \tag{18}$$

One can see that the functions *cos* α and $\sin \alpha$ can be continued into the domain Ω in the following manner:

$$\cos \alpha = -\frac{\partial \omega_2}{\partial x_1}, \quad \sin \alpha = -\frac{\partial \omega_2}{\partial x_2}.$$

Then formula (17) for the coefficients $A_i^0 (i = 1, 2, 3)$ becomes

$$A_1^0 = -\left(D_1 \left(\frac{\partial \omega_2}{\partial x_1} \right)^4 + D_2 \left(\frac{\partial \omega_2}{\partial x_2} \right)^4 + 2H \left(\frac{\partial \omega_2}{\partial x_1} \right)^2 \left(\frac{\partial \omega}{\partial x_2} \right)^2 \right),$$

$$A_2^0 = -\left((H - D_1) \left(\frac{\partial \omega_2}{\partial x_1} \right)^2 + (D_2 - H) \left(\frac{\partial \omega_2}{\partial x_2} \right)^2 \right) \left(\frac{\partial \omega_2}{\partial x_1} \right) \left(\frac{\partial \omega_2}{\partial x_2} \right),$$

$$A_3^0 = -\left(D_1 \vartheta_2 + (D_1 + D_2 - H) \left(\frac{\partial \omega_2}{\partial x_1} \right)^2 \left(\frac{\partial \omega_2}{\partial x_2} \right)^2 \right). \tag{19}$$

Let us denote the curvature of the boundary $\partial\Omega_2$ at point $\mathbf{x}$ by $1/\rho(\mathbf{x})$. Assume also that the function A_1^0 satisfies the conditions

$$A_1^0 \neq 0 \quad \forall \mathbf{x} \in \partial\Omega_2, \qquad A_1^0 \leqslant 0 \ \ \forall \mathbf{x} \in \Omega.$$

Using the differential operators of the special type D_k^j and T_k^j which have the form [16]:

$$D_k^j f = (-1)^k (\nabla\omega_j, \nabla)^k f, \quad T_1^j f = \frac{\partial f}{\partial x_1}\frac{\partial \omega_j}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial \omega_j}{\partial x_1}, \tag{20}$$

we can prove the following theorem.

Theorem. If the function $\omega_2(x)$ satisfies conditions (18) and functions Φ_2 and Φ_3 are chosen as $\Phi_2(\mathbf{x}) \in C^3(\Omega \cup \partial\Omega)$, $\Phi_3(\mathbf{x}) \in C^2(\Omega \cup \partial\Omega)$, then the function

$$W_2 = \omega_2\Phi_2 + \frac{\omega_2^2 A_3^0 \Phi_2}{2\rho(A_1^0 - \omega_2)} - \frac{\omega_2^2}{2(A_1^0 - \omega_2)}\,((2D_1^{(2)}\Phi_2 + \Phi_2 D_2^{(2)}\omega_2)A_1^0 + 2A_2^0 T_1^{(2)}\Phi_2) + \omega_2^3\Phi_3, \tag{21}$$

is a general and complete solution structure that satisfies the boundary conditions (4).

Let the function $\omega_2(\mathbf{x})$ be a normalized one to the second degree, that is, in addition to Eq. (18), it satisfies the condition

$$\frac{\partial^2 \omega_2}{\partial n^2} = 0, \quad \forall \mathbf{x} \in \partial\Omega_2. \tag{22}$$

To construct such function (denoted by $\omega_2^{(2)}$) we can use the formula

$$\omega_2^{(2)} = \omega_2(\mathbf{x}) + \omega_2(\mathbf{x} + t)/3, \tag{23}$$

where

$$t = (h_1, h_2),$$

and

$$h_1 = -\omega_2(x_1 + \tfrac{1}{2}\omega_2(\mathbf{x}), x_2) + \omega_2(x_1 - \tfrac{1}{2}\omega_2(\mathbf{x}), x_2),$$

$$h_2 = -\omega_2(x_1, x_2 + \tfrac{1}{2}\omega_2(\mathbf{x})) + \omega_2(x_1, x_2 - \tfrac{1}{2}\omega_2(\mathbf{x})).$$

Using the formulas

$$\omega_2 \frac{\partial \omega_2}{\partial x_1} = -h_1 + 0(\omega_2^2), \quad \omega_2 \frac{\partial \omega_2}{\partial x_2} = -h_2 + 0(\omega_2^2), \tag{24}$$

which are valid in the vicinity of the boundary $\partial\Omega_2$, it is possible to transform Eq. (21) to the form

$$\begin{aligned} W_2(\mathbf{x}) = {} & \omega_2\Phi_2 + \frac{\omega_2^2 A_3^0 \Phi_2}{2\rho(A_1^0 - \omega_2)} \\ & + \frac{\omega_2}{A_1^0 - \omega_2}\left(\frac{\partial \Phi_2}{\partial x_1}(A_1^0 h_1 - A_2^0 h_2) + \frac{\partial \Phi_2}{\partial x_2}(A_1^0 h_2 + A_2^0 h_1)\right) + \omega_2^3 \Phi_3. \end{aligned} \tag{25}$$

This formula also represents the solution structure. It satisfies exactly the boundary conditions (4) on $\partial\Omega_2$. Thus, the structures W_1 and W_2 have been constructed and formula (14) represents the general solution structure that satisfies exactly the mixed boundary conditions (3) and (4) prescribed on $\partial\Omega_1$ and $\partial\Omega_2$, respectively. The indefinite functions Φ_i $(i = 1, 2, 3)$ can be represented in the form of Eq. (12).

Since the second condition (4) is a natural one for functional (13), the solution structure may be simplified to the form

$$W = \omega_1 \omega \Phi. \tag{26}$$

This is sometimes more efficient for the numerical realization of the RFM.

The problem of free vibrations of orthotropic plates (1)–(5) can now be reduced to a standard matrix eigenvalue problem of classical dynamic analysis. Substituting Φ_i in the form of Eq. (12) into functional (13), and using conditions of the minimum of this functional, one can obtain a system of homogeneous algebraic equations for the unknown coefficients $C_j^{(i)}$. The conditions for a non-trivial solution of this system lead to the frequency equation which provides all the infinitely many natural frequencies of the plate being analyzed. For every natural frequency one can obtain the corresponding modal shape.

4. Numerical results

The general procedure of the RFM and numerical algorithms described above for dynamic analysis of thin orthotropic plates of complex geometry with mixed boundary conditions has been incorporated into a software package Polye-Plast system based on a general RFM formulation for various problems of solid mechanics. This package includes programs for executing the operations of construction of the solution structure (11), differentiation, integration, solving systems of linear algebraic equations, determining eigenvalues, presentation and output of results and other kinds of software support. The databases of the Polye-Plast system contain a wide set of domains, solution structures for various types of boundary conditions, and complete systems of approximate polynomials, etc.

Below we present a small selection of numerical results obtained with the software package Polye-Plast to illustrate the performance of the method, its relative accuracy, and overall effectiveness. The testing of the proposed method was carried out for the problems of vibration of isotropic and orthotropic square plates with mixed boundary conditions as shown in Figs. 1(a) and (b). These problems have also been analyzed in Refs. [1,2,3]. In these figures, clamped parts of plates are marked by shading and simply supported parts by dotted lines. For all the examples given below, the solution structures can be given by either Eqs. (14) or (26).

4.1. Example 1

For the plate shown in Fig. 1(a) a normalized equation of the clamped edge can be presented as the unification of the line segments GK and EF, namely in the form

$$\omega_1 = F_1 \vee_0 F_2 = 0,$$

where $F_1 = (\sigma_1 \equiv (x_1^2 - a_1^2)/(2a_1)$ is the part of the plane placed outside the vertical strip $|x_1| \geqslant a_1$, $F_2 = (\sigma_2 \equiv (a_2^2 - x_2^2)/(2a)$ is the part of the plane placed inside the horizontal strip $|x_2| \leqslant a$, and $\vee_0$ is the symbol of the R-disjunction:

$$X \vee_0 Y = X + Y + \sqrt{X^2 + Y^2}.$$

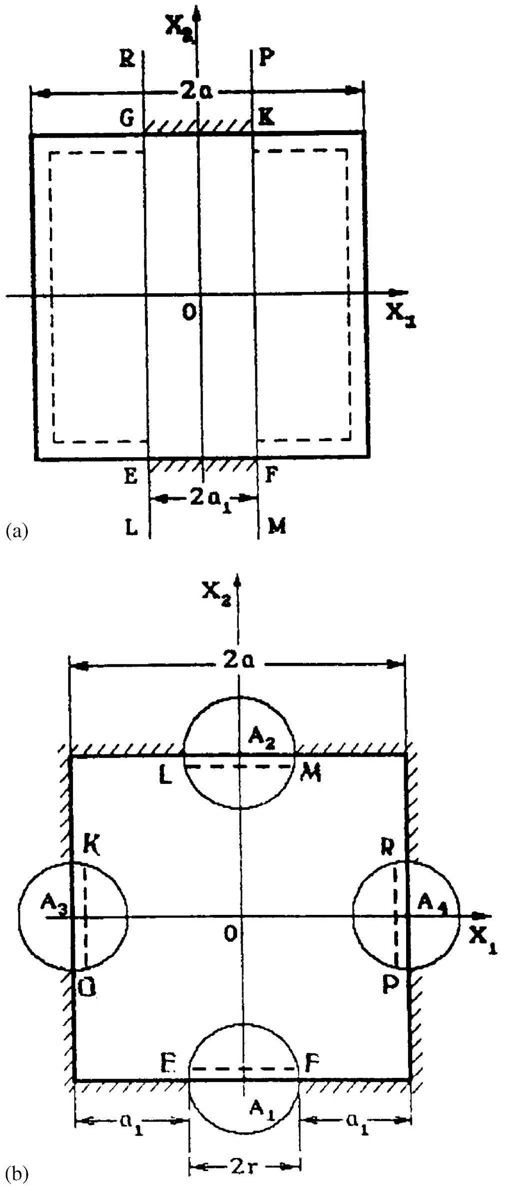

Fig. 1. Numerical examples: (a) Example 1 and (b) Example 2.

The function ω_1 equals zero over the line segments EF, GK, and rays EL, FM, GR, and KP that are not included in the domain Ω. Inside the data domain, the function $\omega_1 > 0$ and its normal derivative equals unity over the line segments EF and GK. The function ω_2 can be represented as

$$\omega_2 = -F_1 \vee_0 (F_2 \wedge_0 F_3) = 0,$$

where $F_3 = (\sigma_3 \equiv (a^2 - x_1^2)/(2a) \geqslant 0)$ is a part of the plane placed inside the vertical strip $|x_1| \leqslant a$. The functions F_1 and F_2 have the same sense as above, $\wedge_0$ is the symbol of R-conjunction. The equation of the boundary is given by

$$\omega \equiv F_2 \wedge_0 F_3 = 0.$$

4.2. Example 2

For the plate shown in Fig. 1b, the normalized equation of the clamped segment of the boundary can be represented in the form

$$\omega_1 = (F_2 \wedge_0 F_3) \vee_0 ((F_5 \vee F_6) \vee_0 (F_7 \vee_0 F_8)) = 0, \tag{27}$$

where the function F_2, F_3 have the same sense as in Example 1 and $F_i (i = 5, 6, 7, 8)$ are the parts of the plane placed inside the circles of radius r. The centers of these circles are placed at the points $A_1(0, -a), A_2(0, a), A_3(-a, 0), A_4(a, 0)$, respectively:

$$F_5 = (\sigma_5 \equiv (r^2 - x_1^2 - (x_2 + a)^2)/2/r \geqslant 0), \quad F_6 = (\sigma_6 \equiv (r^2 - x_1^2 - (x_2 - a)^2)/2/r \geqslant 0),$$

$$F_7 = (\sigma_7 \equiv (r^2 - (x_1 - a)^2 - x_2^2)/2/r \geqslant 0), \quad F_8 = (\sigma_8 \equiv (r^2 - (x_1 + a)^2 - x_2^2)/2/r \geqslant 0).$$

The function ω_1 equals zero on the whole boundary except the parts EF, LM, GK, and PR. Its normal derivative is equal to unity over the above-mentioned clamped boundary. The normalized equation of the simply supported parts of the boundary $\partial\Omega_2$ can be represented in the form

$$\omega_2 = (F_9 \vee_0 F_2) \wedge_0 (F_{10} \vee_0 F_3),$$

where $F_9 = (\sigma_9 \equiv (x_1^2 - r^2)/(2r) \geqslant 0)$ is the part of the plate placed outside of the vertical strip $|x_1| \geqslant r$, $F_{10} = (\sigma_{10} \equiv (x_2^2 - r^2)/(2r) \geqslant 0)$ is the part of the plane placed outside of the horizontal strip $|x_2| \geqslant r$. The normalized equation of the whole boundary $\partial\Omega$ is given by the equation $\omega \equiv F_2 \wedge_0 F_3 = 0$.

The numerical results presented below were obtained using the solution structure (26). Table 1 presents the first four natural frequency parameters for the isotropic plates shown in Figs. 1(a) and (b). To examine the accuracy of the results obtained by RFM, they are compared with those obtained in Refs. [1–3].

Tables 2 and 3 present the first four natural frequency parameters for orthotropic plates shown in Figs. 1(a) and (b) for the three different combinations of the orthotropic parameters η and μ and for $\gamma = (a_1/a) = 0.5$ and $\gamma = 1$. These numerical results are also compared with those obtained in Refs. [1,2]. It can be seen from the comparisons given in the Tables 1–3 that that all the numerical results obtained by RFM agree well with those obtained by other analytical and numerical methods for the plates shown in Figs. 1(a) and (b). The percentage error did not exceed 2.5%.

4.3. Example 3

Let us consider a vibration problem for isotropic and orthotropic plate of a complex geometry as shown in Fig. 2. It is assumed that the straight parts of the boundary (parts MGA, LDE, FCP, and RBK) are clamped and curvilinear ones are simply supported. The solution structure for this problem is given by Eq. (26). The normalized equation of the domain boundary $\partial\Omega$ may be

Table 1
Dimensionless frequencies $\Lambda = 4\lambda a^2\sqrt{\rho h/D}$ of isotropic square plates shown in Figs. 1(a) and (b) (Examples 1 and 2, $\vartheta = 0.3, \gamma = a_1/a$)

Mode	γ					
	0	1/3	1/2	2/3	0.98	1
Example 1						
Present (1,1)	19.74	27.52	28.45	28.85	28.95	28.95
Ref. [1]	19.74	28.10	28.44	28.80		28.94
Ref. [2]	19.74	27.30	28.37	28.82		28.96
Ref. [3]	19.74	27.85	28.62	28.90		28.96
Present (1,2)	49.35	65.18	67.86	69.035	69.32	69.30
Ref. [1]	49.35	66.86	67.85	68.90		69.29
Present (2,1)	49.35	51.92	53.49	654.44	54.74	54.74
Ref. [1]	49.35	52.85	53.49	54.33		54.74
Present (2,2)	78.96	85.68	90.46	93.58	94.58	94.58
Ref. [1]	78.96	88.44	90.50	93.20		94.59
Example 2						
Present (1,1)	19.74	24.24	25.79	30.18	33.62	35.96
Ref. [1]	19.74	22.52	26.18	29.45		35.96
Present (1,2)	49.35	54.72	58.178	64.49	69.75	73.33
Ref. [1]	49.35	53.51	58.70	63.46		73.35
Present (2,1)	49.35	54.72	58.178	64.49	69.75	73.33
Ref. [1]	49.35	53.51	58.70	63.46		73.35
Present (2.2)		90.41	97.96	105.43	107.85	108.2
Ref. [1]	78.96	88.75	98.58	104.6		108.2

written in the form

$$\omega = (F_3 \wedge_0 F_2) \wedge_0 (-(F_4 \vee_0 F_5) \wedge_0 (-(F_6 \vee_0 F_7))) = 0, \tag{28}$$

where functions $F_i (i = 2, 3)$ and signs R-operations $\wedge_0$, $\vee_0$ have the same sense as in Example 2. The functions $F_k (k = 4, 5, 6, 7)$ describe subdomains placed inside the ellipses. The normalized equations for boundaries of these subdomains have the forms

$$F_i = \frac{f_i}{\sqrt{f_i^2 + |\nabla f_i|^2}} = 0, \quad F_4 = \left(1 - \frac{(x_1 - a)^2}{b^2} - \frac{x_2^2}{a_2^2}\right), \quad F_5 = \left(1 - \frac{(x_1 + a)^2}{b^2} - \frac{x_2^2}{a_2^2}\right),$$

$$F_6 = \left(1 - \frac{x_1^2}{a_2^2} - \frac{(x_2 - a)^2}{b^2}\right), \quad F_7 = \left(1 - \frac{(x_2 + a)^2}{b^2} - \frac{x_1^2}{a_2^2}\right).$$

The function $\omega(x)$ satisfies conditions (18) and the function $\omega_1(x)$ can be taken the same as in Example 2 (formula (27)). It is equal to zero over the parts MGA, KBR, PCF, and EDL.

The next tables present the lowest four frequency parameters for isotropic and orthotropic plates shown in Fig. 2. A comprehensive study of the variations of the frequency parameters depending on the change of the elliptical cut-out depth is presented for the isotropic plate in

Table 2
Dimensionless frequencies $\Lambda = 4\lambda a^2\sqrt{\rho h/H}$ of orthotropic plates shown in Fig. 1a (Example 1), $\eta = D_1/H$ and $\mu = D_2/H$

Mode	γ	$(\alpha;\beta)$		
		(0.5;0.5)	(0.5;1)	(1;2)
Present(1,1)	0.5	22.88	27.52	35.67
Ref. [1]		23.09	27.80	36.0
	1.0	23.17	28.10	36.61
Ref. [1]		23.17	28.10	36.61
Present (1,2)	0.5	52.57	67.32	89.99
Ref. [1]		53.15	68.03	90.78
	1.0	53.41	68.97	92.80
Ref. [1]		53.41	68.97	92.80
Present (2,1)	0.5	43.22	45.51	57.01
Ref. [1]		43.33	45.60	57.04
	1	44.13	47.09	59.25
Ref. [1]		44.13	47.09	59.26
Present (2,2)	0.5	76.78	85.68	105.65
Ref. [1]		76.59	85.90	105.7
	1	78.92	90.37	113.02
Ref. [1]		78.92	90.37	113.0

Table 3
Dimensionless frequencies $\Lambda = 4\lambda a^2\sqrt{\rho h/H}$ of orthotropic square plates shown in Fig. 1b (Example 2), $\eta = D_1/H$, $\mu = D_2/H$

Mode	γ	$(\eta;\mu)$		
		(0.5;0.5)	(0.5;1)	(1;2)
Present (1,1)	0.5	21.59	23.83	29.74
Ref. [1]		22.58	25.09	28.64
	1.0	28.07	32.27	42.396
Ref.[1]		28.07	32.27	42.70
Ref. [2]		28.1	32.3	42.4
Present (1,2)	0.5	47.08	56.67	73.20
Ref. [1]		47.79	60.55	71.82
	1.0	56.62	71.50	98.87
Ref. [1]		56.63	71.50	96.58
Present (2,1)	0.5	47.08	49.19	61.88
Ref. [1]		47.79	50.57	61.31
	1	56.62	58.99	76.84
Ref. [1]		56.63	58.99	77.63
Present (2,2)	0.5	81.99	90.45	111.88
Ref. [1]		85.45	92.17	110.1
	1	88.41	98.81	124.67
Ref. [1]		88.42	98.81	126.4

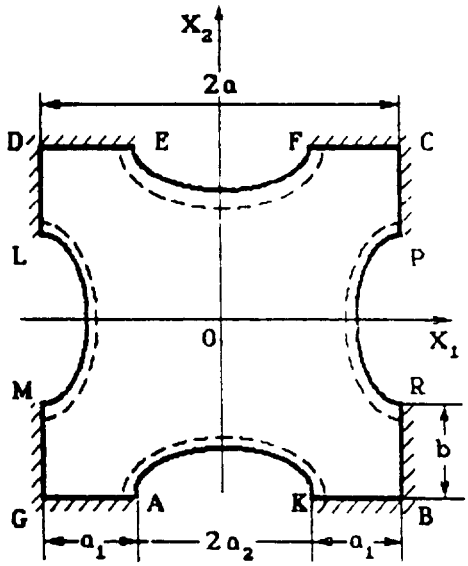

Fig. 2. Numerical Example 3.

Table 4
Dimensionless frequencies $\Lambda = 4\lambda a^2\sqrt{\rho h/D}$ of isotropic plates with mixed boundary conditions (Fig. 2, Example 3, $\vartheta = 0.3$)

b/a	Mode			
	(1,1)	(1,2)	(2,1)	(2,2)
Present 0	25.79	64.49	58.18	97.96
Ref. [1]	26.18	63.46	58.70	98.58
Present 0.05	28.42	64.25	64.25	98.85
Present 0.1	32.23	73.60	73.60	106.7
Present 0.15	41.40	92.10	92.10	125.2
Present 0.2	56.05	123.1	123.1	161.5
Present 0.25	80.82	176.5	176.5	222.9

Table 4 and for the orthotropic plate in Table 5. In both cases it is assumed that $\gamma = (a_1/a) = 0.25$; (a) $\eta = D_1/H = 0.5$, $\mu = D_2/H$; (b) $\eta = D_1/H = 0.5, \mu = D_2/H = 1$; (c) $\eta = D_1/H = 1$, $\mu = D_2/H = 2$.

Tables 4 and 5 present the four first frequency parameters for isotropic and orthotropic plates shown in Fig. 2.

For all of the examples above, the indefinite components Φ_i were approximated by a complete system of polynomials of degree N in expansion (12). A comprehensive numerical study has shown a sufficiently rapid convergence of the method. All of the numerical data given in the Tables 1–5 has been obtained by retaining the first 21 co-ordinate functions in expansion (12).

Table 5
Dimensionless frequencies $\Lambda = 4\lambda a^2\sqrt{\rho h/H}$ of orthotropic square plates with mixed boundary conditions (Fig. 2, Example 3)

$(\eta;\mu)$	b/a	Mode			
		(1,1)	(1,2)	(2,1)	(2,2)
(0.5;0.5)	0.005	23.85	51.53	51.36	89.90
(0.5;1)		26.37	63.10	53.17	91.35
(1;2)		32.52	82.21	66.87	112.6
(0.5;0.5)	0.125	31.86	66.59	66.45	97.12
(0.5;1)		34.53	80.89	68.32	106.5
(1;1)		40.74	104.1	84.67	129.2
(0.5;0.5)	0.25	71.27	144.9	145.0	191.0
(0.5;1)		77.04	174.5	149.2	208.9
(1;1)		90.19	222.1	183.7	252.0

Further increases in the number of co-ordinate functions did not affect the accuracy of the solution.

5. Conclusion

In this paper, the R-function method for solving free vibration problems for thin, orthotropic plates of an arbitrary plan form with mixed boundary conditions has been presented. RFM enables one to seek the solution in the form of some analytical expressions and to construct approximation sequences with the use of variational, projection, and, in principle, any other methods. In this respect, RFM may be coupled with many the network-type methods, such as BEM and FIE, etc. In addition, RFM holds the general advantages of classical analytical methods but, at the same time, enables one to treat vibration eigen value problems for domains of complex geometry.

In the RFM, all the geometric information from the mathematical model of the boundary value problem being analyzed is transformed in the solution structure in analytical form. This enables one to accomplish a total computerization of RFM formulations and to develop the highly intelligent software technology for treating the above-mentioned dynamic problems for isotropic and anisotropic plates of complex geometry with arbitrary boundary conditions.

It has been shown that the method yields rapid and convergent numerical solutions and these results are in an excellent agreement with other sources of numerical or approximate solutions. The accuracy and applicability of the RFM for the class of problems considered in this paper has been verified successfully. Therefore, we can conclude that the method is very useful in addition to the existing numerical and analytical methods available for the solution of free plate vibration problems.

The further extension of the RFM for dynamic analysis by considering the forced flexural vibrations of anisotropic plates and shallow shells of an arbitrary geometry with mixed boundary conditions forms the key subject of the authors continuing research. The results and new findings from this further research will be reported as a paper in due course.

References

[1] G. Irie, G. Yamada, Y. Narita, Free vibration of a rectangular plate supported on the sides and some segments, Bulletin of the Japan Society of Mechanical Engineers 20 (1977) 1085–1092.

[2] L.M. Keer, B. Stahl, Eigenvalue problem of rectangular plates with mixed edge conditions, Journal of Applied Mechanics 39 (1972) 513–520.

[3] A.W. Leissa, Vibrations of plates, NASA SP-160, NASA, Washington, DC, 1969.

[4] G.V. Narayanau, D.E. Beskos, Numerical operational methods for time dependent linear problems, International Journal of Numerical Methods in Engineering 18 (1982) 1829–1854.

[5] P.A. Laura, L.E. Luisoni, Vibrations of orthotropic rectangular plates with edges possesing different rotational flexibility and subjected to in-plane forces, Computers and Structures 9 (1978) 527–532.

[6] K. Nagaya, Vibration of a plate of arbitrary shape with free and simply supported mixed edges, Journal of Acoustical Society of America 73 (3) (1983) 256–261.

[7] Y. Narita, Application of a series-type method to vibration of orthotropic rectangular plates with mixed boundary conditions, Journal of Sound and Vibration 77 (1981) 345–355.

[8] G. Venkateswara Rao, I.S. Raj, T.V. Murthy, Vibration of rectangular plates with mixed boundary conditions, Journal of Sound and Vibration 30 (1973) 257–260.

[9] E.S. Ventsel, On one method for eigenvalue problems of thin plates of an arbitrary geometry, Dynamics of Aerospace Structures 1 (1985) 66–79 (in Russian).

[10] D.E. Beskos, Dynamic analysis of plates, in: D.E. Beskos (Ed.), Boundary Element Analysis of Plates and Shells, Springer, New York, 1991, pp. 35–93.

[11] J. Mackerle, Static and dynamic analysis of plates using finite element and boundary element techniques–A bibliography (1992–1994), Finite Elements in Analysis and Design 20 (1995) 139–154.

[12] H.D. Conway, The bending, buckling, and flexural vibrations of simply supported polygonal plates by point-matching, Journal of Applied Mechanics 28 (1961) 288–291.

[13] M.J.A. O'Callaghan, R.P. Studdert, The edge-function method for the free vibrations of thin orthotropic plates, in: C.A. Brebbia, G. Maier (Eds.), Boundary Elements VII, Springer, Berlin, 1985, pp. 6.37–6.52.

[14] H.C. Huang, Static and Dynamic Analysis of Plates and Shells, Springer, New York, 1985.

[15] J.A. Talbert, A.R. Parkinson, Development of an automatic, two-dimensional finite-element mesh generator using quadrilateral elements and bezier curve boundary definition, International Journal of Numerical Methods in Engineering 29 (1990) 1551–1567.

[16] V.L. Rvachev, Theory of R-Functions and Some of its Applications, Nauka Dumka, Kiev, Ukraine, 1982 (in Russian).

[17] V.L. Rvachev, T.I. Sheiko, R-functions in boundary value problems in mechanics, Applied Mechanics Reviews 48 (4) (1995) 151–187.

[18] V.L. Rvachev, L.V. Kurpa, R-Functions in Problems of the Plate Theory, Nauka Dumka, Kiev, Ukraine, 1987 (in Russian).

[19] S.G. Lekhnitski, Anisotropic Plates, Gordon and Breach, New York, 1968.

[20] S.P. Timoshenko, S. Woinowsky-Krieger, Theory of Plates and Shells, McGraw-Hill, New York, 1959.

[21] S.G. Mikhlin, Variational Methods in Mathematical Physics, Macmillan, New York, 1964.