Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Sound and Vibration 261 (2003) 123–151
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

A perturbation method for computing the simplest normal forms of dynamical systems

P. Yu[a,*,1], A.Y.T. Leung[b]

[a] Department of Applied Mathematics, University of Western Ontario, London, Ont., Canada N6A 5B7
[b] Department of Building and Construction, City University of Hong Kong, Kowloon, Hong Kong

Received 11 October 2001; accepted 3 May 2002

Abstract

A previously developed perturbation method is generalized for computing the simplest normal form (at each level of computation, the minimum number of terms are retained) of general n-dimensional differential equations. This "direct" approach, combining the normal form theory with center manifold theory in one unified procedure, can be used to systematically compute the simplest (or unique) normal form. Two particular singularities of the Jacobian of the system are considered in this paper: the first one is associated with one pair of purely imaginary eigenvalues (Hopf-type singularity), and the other corresponds to a simple zero and a pair of purely imaginary eigenvalues (Hopf-zero-type singularity). The approach can be easily formulated and implemented using a computer algebra system. Maple programs have been developed in this paper which can be "automatically" executed by a user without the knowledge of computer algebra. A physical oscillator model is studied in detail to show the computational efficiency of the "direct" method, and the advantage of using the simplest normal form, which greatly simplifies the analysis on dynamical systems, in particular, for bifurcations and stability.

1. Introduction

Normal form theory has been widely used in the analyses of vibrations and bifurcations for non-linear dynamic systems [1–5]. The basic idea of the method of normal forms is applying a series of near-identity non-linear transformations (NTs) to systematically construct a simple form of the original system. In general, the normal form is not uniquely defined and computing the

*Corresponding author. Tel.: +1-519-679-2111, ext. 88783; fax: +1-519-661-3523.
E-mail address: pyu@pyu1.apmaths.uwo.ca (P. Yu).
[1] On sabbatical leave, as a Visiting Professor in the Department of Building and Construction, City University of Hong Kong, August 2001–July 2002.

0022-460X/03/$ - see front matter
PII: S0022-460X(02)00954-9

explicit formula of a normal form in terms of the coefficients of the original system is not easy. In the past few years, symbolic computation of normal forms using computer algebra systems has received considerable attention (e.g., see Refs. [6–12]). The method of normal form is usually employed together with center manifold theory [13] which uses the same idea of successive NTs. In general, given a non-linear system, center manifold theory is applied before using normal form theory. However, there exist approaches which combine the two steps into one unified procedure (e.g., see Refs. [7,9,11,12]).

This paper generalizes a previously developed perturbation technique to compute the simplest normal form (SNF) of dynamical systems. This technique, based on the methods of multiple time scales (MTSs) [14] and harmonic balance [15], has been widely used to study non-linear vibration and bifurcation problems. Huseyin and Lin applied this approach to obtain the explicit formulae of governing equations up to first order approximation [16]. Later, this method was extended to compute the normal forms of Hopf and generalized Hopf bifurcations up to any high order for general n-dimensional systems [7]. Moreover, user-friendly symbolic programs written in Maple were developed, which can be "automatically" executed on computer systems. This technique combines center manifold theory and normal form theory in one unified procedure through a perturbation procedure. Two singularities will be considered in this paper: one is characterized by a pair of purely imaginary eigenvalues (Hopf-type singularity), and the other by a simple zero and a pair of purely imaginary eigenvalues (Hopf-zero-type singularity). It will be shown that the perturbation technique systematically leads to a "unique" normal form for a given set of differential equations up to an arbitrary order. The procedure is straightforward and does not require solving large matrix systems. With the aid of Maple, this approach can be easily implemented on a computer system to automatically compute the explicit expressions of the simplest normal form in terms of the coefficients of the original system.

The above-mentioned "unique" normal form is different from that discussed by Ushiki [17], Baider and Sanders [18], Chua and Kokubu [19,20]. They used Lie algebra to define the first, the second, …, infinite order normal forms. Those unique normal forms are also called the "minimum" or the "simplest" normal forms. In the conventional normal form (CNF) or classical normal form theory, by saying that "normal forms are not unique" it usually implies that: (1) for a same system, its normal form may have different "forms"; or (2) even for a same "form", the CNF may not be the same, with different coefficients. Thus, in this paper we shall adopt the "simplest normal form" rather than the "unique normal form", emphasizing the meaning of the "simplest", i.e., at each order of computation the number of the terms retained in the normal form reaches the "minimum". Computing the SNF of a system is much more involved than calculating the CNF, and in fact computer algebra systems such as Maple, Mathematica have been introduced in the calculations (for example, see Refs. [21–25]). However, all the computation procedures presented in the above-mentioned papers are based on a known (or pre-calculated) CNF, and thus needs one more NT from the original system. In this paper, it will be shown that the perturbation method can be extended to obtain the SNF directly from the original differential equations, which may be called "direct" method. A comparison between the "direct" method and the approach via the CNF ("indirect" method) is given to show the advantage of the "direct" method developed in this paper.

After an outline of the classical normal form theory, the perturbation method and its procedure to find the CNF of a dynamical system are described in the next section. A comparison is also

given in this section to show the advantages of the perturbation technique. Section 3 considers the SNF for Hopf and generalized Hopf bifurcations, while the SNF for Hopf-zero bifurcation is presented in Section 4. A physical oscillator model is studied in detail in Section 5 to demonstrate the applicability of the technique and the advantage of using the SNF for the study of dynamical systems. The SNF can greatly simplify the dynamical analysis, in particular, for bifurcations and stability. Finally, concluding remarks are drawn in Section 6.

2. Normal form theory and perturbation technique

Although normal form theory can be traced back to researchers 150 years ago, most credit should be given to Poincaré. Later many researchers made contributions in developing the theory and methodology for dynamical systems. Among them are Birkhoff and Takens, and this is why the normal form theory is also called Poincaré normal form theory, Birkhoff normal form theory or Takens normal form theory. Takens normal form theory, based on the linear structure of a system, applies Lie algebra to give a very delicate formulation. To make it easy for a comparison between the standard (conventional) normal form theory and the perturbation method, we first outline Takens normal form theory as follows.

Consider the dynamical system described by the following differential equation:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) = J\,\boldsymbol{x} + \sum_{k\geqslant 2} \boldsymbol{f}_k(\boldsymbol{x}), \quad \boldsymbol{x}\in\boldsymbol{R}^n, \tag{1}$$

where $J\,\boldsymbol{x}\in\boldsymbol{R}$ represents the linear part of the system and J is in Jordan canonical form. $\boldsymbol{f}_k(\boldsymbol{x})$ $(k\geqslant 2)$ is a vector of homogeneous polynomials of degree k. Suppose all the eigenvalues of J have zero real parts, implying that system (1) is described on n-dimensional center manifold.

The purpose of Takens normal form theory is to find an efficient transformation such that the transformed system will be in a simpler form, while the topological structure of the original system near the origin is retained. To achieve this, we introduce a near-identity transformation in a neighborhood of the origin, given by

$$\boldsymbol{x} = \boldsymbol{h}(\boldsymbol{y}) = \boldsymbol{y} + \boldsymbol{h}_2(\boldsymbol{y}) + \boldsymbol{h}_3(\boldsymbol{y}) + \cdots + \boldsymbol{h}_k(\boldsymbol{y}) + \cdots, \tag{2}$$

where $\boldsymbol{y}\in\boldsymbol{R}^n$ denotes the new co-ordinates, and $\boldsymbol{h}_k(\boldsymbol{y})$ is a vector of homogeneous polynomials of degree k in $\boldsymbol{y}$. Upon substitution of Eq. (2) into Eq. (1), we have $\dot{\boldsymbol{h}}(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{h}(\boldsymbol{y}))$. Since $\dot{\boldsymbol{h}}(\boldsymbol{y}) = (\partial\boldsymbol{h}/\partial\boldsymbol{y})\dot{\boldsymbol{y}}$, the equation of the new system can be written as

$$\dot{\boldsymbol{y}} = \boldsymbol{h}_{\boldsymbol{y}}^{-1}(\boldsymbol{y})\boldsymbol{f}(\boldsymbol{h}(\boldsymbol{y})) = \boldsymbol{h}_{\boldsymbol{y}}^{-1}\,J\,\boldsymbol{h}(\boldsymbol{y}) + \boldsymbol{h}_{\boldsymbol{y}}^{-1}(\boldsymbol{f}_2\,(\boldsymbol{h}(\boldsymbol{y})) + \cdots), \tag{3}$$

where $\boldsymbol{h}_{\boldsymbol{y}} \equiv \partial\boldsymbol{h}/\partial\boldsymbol{y}$ is the first order derivative of $\boldsymbol{h}$ with respect to $\boldsymbol{y}$, and $\boldsymbol{h}_{\boldsymbol{y}}^{-1}$ denotes the inverse of $\boldsymbol{h}_{\boldsymbol{y}}$ in the neighborhood of the origin.

Computation of the normal form is a recursive procedure order by order. Suppose the computation from the second order to $k-1$ order has been done, we now process the kth order calculation. Thus, the near-identity transformation can be assumed as $\boldsymbol{h}(\boldsymbol{y}) = \boldsymbol{y} + \boldsymbol{h}_k(\boldsymbol{y})$, and then the first order derivative of $\boldsymbol{h}(\boldsymbol{y})$ with respect to $\boldsymbol{y}$ is given by

$$\boldsymbol{h}_{\boldsymbol{y}}(\boldsymbol{y}) = I + \boldsymbol{h}_{k\boldsymbol{y}}(\boldsymbol{y}), \tag{4}$$

and then the inverse of $\boldsymbol{h}_y(\boldsymbol{y})$ can be approximated by the Taylor expansion

$$(\boldsymbol{h}_y(\boldsymbol{y}))^{-1} = I - \boldsymbol{h}_{ky}(\boldsymbol{y}) + \tfrac{1}{2}\boldsymbol{h}_{2ky} - \cdots = I - \boldsymbol{h}_{ky}(\boldsymbol{y}) + O(|\boldsymbol{y}|^{2k-2}) \quad \text{as } |\boldsymbol{y}| \to 0. \tag{5}$$

Substituting Eqs. (4) and (5) into Eq. (3) yields

$$\dot{\boldsymbol{y}} = J\,\boldsymbol{y} + \sum_{i=2}^{k-1} \boldsymbol{f}^i(\boldsymbol{y}) + \{\boldsymbol{f}^k(\boldsymbol{y}) - [\boldsymbol{h}_{ky}(\boldsymbol{y})\,J\,\boldsymbol{y} - J\boldsymbol{h}_k(\boldsymbol{y})]\} + O(|\boldsymbol{y}|^{k+1}), \tag{6}$$

which implies that a suitable choice of $\boldsymbol{h}(\boldsymbol{y})$ can result in the simplification of $\boldsymbol{f}^k(\boldsymbol{y})$.

Next introduce a linear operator

$$\begin{aligned} &L_J^k : H_n^k \to H_n^k, \\ &(L_J^k\,\boldsymbol{h}^k)(\boldsymbol{y}) = \boldsymbol{h}_{ky}(\boldsymbol{y})\,J\,\boldsymbol{y} - J\,\boldsymbol{h}_k(\boldsymbol{y}) \quad \text{for } \boldsymbol{h}_k \in H_n^k, \end{aligned} \tag{7}$$

where H_n^k denotes the vector space of homogeneous polynomials of order k in n variables with values in $\boldsymbol{R}^n$. The operator L_J^k is also called Lie bracket. Further, let P^k be the range of L_J^k in H_n^k and Q^k be the complementary subspace to P^k in H_n^k, then we have

$$H_n^k = P^k \oplus Q^k \quad (k \geqslant 2). \tag{8}$$

Now Takens normal form theorem can be described as follows [2]: given the dynamical system (1), let the decomposition (8) of homogeneous space H_n^k be given for $k = 2, \ldots, r$. Then there exists a sequence of near-identity transformations, $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}_k(\boldsymbol{y})$, in a neighborhood of the origin, where $\boldsymbol{h}_k \in H_n^k$ $(k = 2, \ldots, r)$, such that Eq. (1) is transformed into

$$\dot{\boldsymbol{y}} = J\,\boldsymbol{y} + \sum_{k=1}^{r} \boldsymbol{g}_k(\boldsymbol{y}) + O(|\boldsymbol{y}|^{r+1}), \tag{9}$$

where $\boldsymbol{g}_k \in Q^k$ $(k = 2, 3, \ldots, r)$. It can be seen from the above discussion that the normal form is determined on the basis chosen for the complementary subspaces Q^k, $k = 2, 3, \ldots, r$. These subspaces are determined by the matrix J, but in general are not unique. Therefore, the normal form is in general not uniquely determined.

However, Takens normal form theory only gives the "form" of the normal form, not telling you how to find the explicit expression of the normal form. Thus, many computation methods have been developed, such as matrix representation method (e.g., see Refs. [3,5]) and the method of adjoint [4]. The matrix method is not efficient since the dimension of the matrices increases very rapidly as the order of the normal form increases, which is, in particular, not suitable for computing higher order normal forms. The method of adjoint, on the other hand, introduces an additional adjoint operator so that the basis for the complementary space is uniquely determined, which may be difficult for those who (e.g., from engineering society) may just want to apply the method of normal forms to solve a particular problem.

There are many other approaches often used in engineering society which may also lead to normal forms, though they are not usually called normal forms. We list some of such well known approaches: Lyapunov constants, succession function, Lindstedt–Poincaré method, Lyapunov–Schmidt reduction, time averaging, MTSs, intrinsic harmonic balancing, etc. The method of MTSs has been used by many researchers for analyzing vibration and stability of oscillator systems. In a paper published in 1998 [7], this method was first applied together with a perturbation technique to develop "automatic" symbolic computation for the normal forms of

Hopf and generalized Hopf bifurcations. Later, this method is generalized to consider other singularities [11,12].

Before giving a comparison between the perturbation technique and the classical normal form theory (e.g., Takens normal form theory), we present the perturbation technique below, showing how to use this approach to compute the CNF of a general non-linear system. The computation of the SNF will be discussed in the next two sections.

The perturbation method is based on the approach of multiple scales or MTSs, which is frequently used for vibration analysis of a dynamical system governed by a second order non-linear differential equation (e.g., single pendulum) [14]:

$$\ddot{x} + x = \varepsilon f(x, \dot{x}), \tag{10}$$

where the dot indicates the differentiation with respect to time t, and ε is a small perturbation parameter ($0 < \varepsilon \ll 1$). f is a non-linear analytic function and can thus be expressed in a Taylor expansion. Recently, the perturbation approach has been extended to consider general n-dimensional systems [7,9,11,12], described by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \boldsymbol{R}^n, \tag{11}$$

where $\boldsymbol{f}$ is assumed to be analytic; and $\boldsymbol{x} = \boldsymbol{0}$ is an equilibrium of the system, i.e., $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. In Ref. [7] a user-friendly ("automatic") symbolic computer program written in Maple has been developed for computing the CNF associated with the Hopf-type singularity (i.e., the Jacobian of the system has a pair of purely imaginary eigenvalues). Maple programs have also been developed for other singularities, including those associated with the Jordan canonical matrix

$$J = \begin{bmatrix} J_0 & 0 \\ 0 & J_1 \end{bmatrix}, \tag{12}$$

where J_0 is given in one of the following forms:

$$\text{(A)} \quad J_0 = \begin{bmatrix} 0 & \omega_c \\ -\omega_c & 0 \end{bmatrix}, \quad \text{(B)} \quad J_0 = \begin{bmatrix} 0 & \omega_c & 0 \\ -\omega_c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\text{(C)} \quad J_0 = \begin{bmatrix} 0 & \omega_{1c} & 0 & 0 \\ -\omega_{1c} & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_{2c} \\ 0 & 0 & -\omega_{2c} & 0 \end{bmatrix}, \quad \text{(D)} \quad J_0 = \begin{bmatrix} 0 & \omega_c & 1 & 0 \\ -\omega_c & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega_c \\ 0 & 0 & -\omega_c & 0 \end{bmatrix}, \tag{13}$$

and J_1, given in Jordan canonical form, involves eigenvalues which have negative real parts. In other words, system (11) does not contain unstable manifold in the vicinity of the origin, which is reasonable from the application point of view. Note that here the frequencies ω, ω_1 and ω_2 are positive, and the ratio ω_1/ω_2 may be an irrational number (non-resonant case) or a form of a fraction number, m/n, in which m and n are positive integers (resonant cases). Also, note that without loss of generality, we may assume $\omega_c = 1$ (otherwise, one may use a transformation $t' = \omega_c t$ to change frequency ω_c or ω_1 to 1). It should be pointed out that although the

perturbation approach has been applied only to the above-mentioned cases, it can be easily extended to consider other cases in which the eigenvalues of J_0 have more zero real parts.

Since the procedure involved in the perturbation approach is similar for all the cases mentioned above, and cases (A), (C) and (D) have been studied in detail in Refs. [7,11,12] so in the following we shall use case (B) to illustrate how the perturbation approach is applied to derive the explicit formulas of the CNF and the associated NT. To begin with we may write Eq. (11) in a more convenient component form:

$$\dot{x}_1 = x_2 + f_1(\boldsymbol{x}), \tag{14}$$

$$\dot{x}_2 = -x_1 + f_2(\boldsymbol{x}), \tag{15}$$

$$\dot{x}_3 = f_3(\boldsymbol{x}), \tag{16}$$

$$\dot{x}_p = -\alpha_p x_p + f_p(\boldsymbol{x}) \quad (p = 4, 5, \ldots, m_1 + 3), \tag{17}$$

$$\begin{aligned} \dot{x}_q &= -\alpha_q x_q + \omega_q x_{q+1} + f_q(\boldsymbol{x}), \\ \dot{x}_{q+1} &= -\omega_q x_q - \alpha_q x_{q+1} + f_{q+1}(\boldsymbol{x}) \quad (q = m_1 + 4, m_1 + 6, \ldots, n-1), \end{aligned} \tag{18}$$

where $\alpha_p,\ \alpha_q,\ \omega_q > 0$ and $3 + m_1 + 2m_2 = n$. The functions $f_i(\boldsymbol{x})$ satisfy $f_i(\boldsymbol{x}) = 0$ and $\partial f_i(\boldsymbol{x})/\partial x_j = 0,\ i,j = 1, 2, \ldots, n$.

The underlying idea of the method of MTS is combining the time scaling with spatial scaling via the same perturbation parameter ε. Let us first consider the expansion which represents the response as a function of multiple independent variables, or scales, instead of a single time variable. Thus, one begins by introducing new independent variables according to

$$T_k = \varepsilon^k t \quad \text{for } k = 0, 1, 2, \ldots\ . \tag{19}$$

It follows that the derivatives with respect to t now become expansions in terms of the partial derivatives with respect to T_k, given by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}T_0}{\mathrm{d}t}\frac{\partial}{\partial T_0} + \frac{\mathrm{d}T_1}{\mathrm{d}t}\frac{\partial}{\partial T_1} + \frac{\mathrm{d}T_2}{\mathrm{d}t}\frac{\partial}{\partial T_2} + \cdots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \tag{20}$$

where the differentiation operator $D_k = \partial/\partial T_k,\ k = 0, 1, 2, \ldots$

Next, assume that the solution of Eq. (11) (or equivalently, Eqs. (14)–(18)) in the neighborhood of $\boldsymbol{x} = \boldsymbol{0}$ is given in the form of

$$x_i(t;\varepsilon) = \varepsilon x_{i1}(T_0, T_1, \ldots) + \varepsilon^2 x_{i2}(T_0, T_1, \ldots) + \cdots \quad (i = 1, 2, \ldots, n). \tag{21}$$

Note that the perturbation parameter ε used in the above solution is identical to that used in Eq. (19) for time scaling. The number of independent time scales needed in Eq. (21) depends on the order of the normal form to be computed. For instance, if the expansion is given to order ε^2, then T_0, T_1 and T_2 are needed in the expansion. In general, if one wants to derive a normal form up to order k, then time scales $T_0, T_1, \ldots, T_k$ should be included in solution (21).

Now substituting solution (21) into Eqs. (14)–(18) with the aid of Eq. (20) and balancing the like powers of ε in the resulting equations yields the following ordered perturbation equations:

$$\varepsilon^1\text{:} \quad D_0 x_{11} = x_{21}, \tag{22}$$

$$D_0 x_{21} = -x_{11}, \tag{23}$$

$$D_0\, x_{31} = 0, \tag{24}$$

$$D_0\, x_{p1} = -\alpha_p\, x_{p1} \quad (p = 4, 5, \ldots, m_1 + 3), \tag{25}$$

$$\begin{aligned} &D_0\, x_{q1} = -\alpha_q\, x_{q1} + \omega_q\, x_{(q+1)1}, \\ &D_0\, x_{(q+1)1} = -\omega_q\, x_{q1} - \alpha_q\, x_{(q+1)1} \quad (q = m_1 + 4, m_1 + 6, \ldots, n - 1); \end{aligned} \tag{26}$$

$$\varepsilon^2\text{:} \quad D_0\, x_{12} = x_{22} - D_1\, x_{11} + f_{12}(\boldsymbol{x}_1), \tag{27}$$

$$D_0\, x_{22} = -x_{12} - D_1\, x_{21} + f_{22}(\boldsymbol{x}_1), \tag{28}$$

$$D_0\, x_{32} = -D_1\, x_{31} + f_{32}(\boldsymbol{x}_1), \tag{29}$$

$$D_0\, x_{p2} = -\alpha_p\, x_{p2} - D_1\, x_{p1} + f_{p2}(\boldsymbol{x}_1) \quad (p = 4, 5, \ldots, m_1 + 3), \tag{30}$$

$$\begin{aligned} &D_0\, x_{q1} = -\alpha_q\, x_{q1} + \omega_q\, x_{(q+1)\,1} - D_1\, x_{q1} + f_{q2}(\boldsymbol{x}_1), \\ &D_0\, x_{(q+1)\,1} = -\omega_q\, x_{q1} - \alpha_q\, x_{(q+1)\,1} - D_1\, x_{(q+1)\,1} + f_{(q+1)\,2}(\boldsymbol{x}_1) \\ &\quad (q = m_1 + 4, m_1 + 6, \ldots, n - 1), \end{aligned} \tag{31}$$

etc., where $f_{i2} = (\mathrm{d}^2/\mathrm{d}\varepsilon^2) f_i(\boldsymbol{x}_1)|_{\varepsilon=0}$. Note that f_{i2} are functions of x_{i1} $(i = 1, 2, \ldots, n)$ only, which have been solved from the ε^1 order perturbation Eqs. (22)–(26). In general, functions f_{ik} only involve the variables which have been obtained from the previous $(k-1)$ step perturbation equations. It should be also noted that the power series (21) starts from $O(\varepsilon)$ order term rather than $O(1)$ order term in order to separate the original equations into different order terms in ε. Usually, one should first use a forward scaling $x_i \to \varepsilon x_i$ in Eqs. (14)–(18), which separates the terms according to the powers of ε, and then the series (21) can be expanded from $O(1)$ order term. Here, we have combined the forward scaling into the solution form (21).

To find the solutions to Eqs. (22)–(25), first note that these equations can be divided into two groups, one consists of Eqs. (22)–(24), and the other includes the remaining equations. Secondly, since we are interested in the asymptotic behavior of the system, so the solutions of the second group are contributed from the first three variables x_1, x_2 and x_3 only. This is in fact the idea of center manifold theory.

The solutions to Eqs. (22) and (23) can be found by differentiating Eq. (21) and then substituting Eq. (22) into the resulting equation, which produces a simple second order, free vibration equation:

$$D_0^2 x_{11} + x_{11} = 0, \tag{32}$$

from which one can readily obtain the solution, written in a general form:

$$x_{11} = r(T_1, T_2, \ldots)\cos[T_0 + \phi(T_1, T_2, \ldots)] \equiv r\cos(T_0 + \phi) \equiv r\cos\theta, \tag{33}$$

where r and ϕ represent, respectively, the amplitude and phase of motion, and $\theta = \omega_c T_0 + \phi = T_0 + \phi$. Once the solution x_{11} is obtained, x_{21} can be directly determined from Eq. (22). The

solution to Eq. (24) is simply given by

$$x_{31} = z(T_1, T_2, \ldots) \equiv z. \tag{34}$$

It can be seen from solutions (33) and (34) that

$$D_0 r = D_0 \phi = D_0 z = 0, \tag{35}$$

since the variables r and z do not contain T_0. The asymptotic ε^1 order solutions of the second group, given by

$$x_{i1} = 0, \quad i = 4, 5, \ldots, n, \tag{36}$$

actually represent the first order steady state solutions for the second group.

The above procedure can be carried out to ε^2 order perturbation equations (27) and (28) to find the following equation:

$$D_0^2 x_{12} + x_{12} = -D_1 D_0 x_{11} - D_1 x_{21} + D_0 f_{12} + f_{22}, \tag{37}$$

which is a non-linear homogeneous ordinary differential equation. Substituting the solutions x_{11} and x_{21} obtained from the first order perturbation equations into the right-hand side of Eq. (37) results in an expression given in terms of trigonometric functions $\cos k(T_0 + \phi)$ and $\sin k(T_0 + \phi)$, $k = 0, 1, 2$. To eliminate possible secular terms which may appear in the solution of x_{12}, it is required that the coefficients of the two terms $\cos(T_0 + \phi)$ and $\sin(T_0 + \phi)$ equal zero, which in turn determines the expressions for $D_1 r$ and $D_1 \phi$. Then the solution to the second order perturbation equation can be determined, and thus x_{12} involves a particular solution only. Having found x_{12}, one can solve x_{22} from Eq. (27). Similarly, Eq. (29) can be used to determine $D_1 z$ and to find the solution for x_{31} by simply balancing the harmonics. The solutions for other components of x_{i2}, $i = 4, 5, \ldots$ can be easily obtained from Eqs. (30) and (31) by harmonic balancing.

The procedure can be applied to any high order perturbation equations, and finally, the normal form is obtained, given in polar co-ordinates:

$$\dot{r} = \frac{\partial r}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial r}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial r}{\partial T_2}\frac{\partial T_2}{\partial t} + \cdots = D_0 r + D_1 r + D_2 r + \cdots, \tag{38}$$

$$\dot{\theta} = \omega_c + \frac{\partial \phi}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial \phi}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial \phi}{\partial T_2}\frac{\partial T_2}{\partial t} + \cdots = 1 + D_0 \phi + D_1 \phi + D_2 \phi + \cdots, \tag{39}$$

$$\dot{z} = \frac{\partial z}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial z}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial z}{\partial T_2}\frac{\partial T_2}{\partial t} + \cdots = D_0 z + D_1 z + D_2 z + \cdots, \tag{40}$$

where the back scaling $\varepsilon r \to r$ (i.e., $\varepsilon x_i \to x_i$) has been used. It has been shown through the above procedure that $D_i r$, $D_i \phi$ and $D_i z$ are monomials of r and z. It should be pointed out that the particular solution to the differential equation (37), etc. can be found using the intrinsic harmonic balancing [15] so that the solution is uniquely determined from three algebraic equations. Thus, $D_i r$, $D_i \phi$ and $D_i z$ are also uniquely defined, which implies that the normal form given in Eqs. (38)–(40) is actually uniquely determined.

The NT between system (11) and the CNF described by Eqs. (38)–(40) can be easily found from the solutions x_i if the transformation:

$$y_1 = r\cos\theta, \quad y_2 = -r\sin\theta, \quad y_3 = z, \tag{41}$$

is applied. Note that the resulting NT not only includes the transformation on the 3-D center manifold spanned by the critical variables x_1, x_2 and x_3, but also contains the projection of the subspace spanned by the non-critical variables $x_4, x_5, \ldots, x_n$ to the center manifold. This is why the perturbation method can be used to find normal forms from the original n-dimensional system without employing center manifold theory.

The above discussion as well as the results obtained in Refs. [7,11,12] suggest that the perturbation technique, compared with the classical normal form theory (e.g., Takens normal form theory) has the following advantages:

(1) The perturbation technique combines center manifold theory and normal form theory in one unified procedure and can be used to treat a general non-linear system whose dimension is greater than its center manifold's dimension.
(2) The technique determines a unique normal form.
(3) The perturbation procedure, unlike the matrix method, does not increase the number of the equations (whose number is equal to the dimension of the system) to be solved at each perturbation order.
(4) The approach, unlike other methods which need to solve differential equations, needs to solve only algebraic equations.
(5) The method generates explicit solutions for both the normal form and associated NT.
(6) The technique uses one consistent NT through all order equations, which is convenient in applications.
(7) The approach is computationally efficient. The results given in Ref. [7] show that the perturbation technique can be used to verify a center of a system up to 60th order, which certainly cannot be handled by the matrix method.

The weakness of the perturbation technique is that it can only be applied to the systems whose Jacobian contains at least one pair of purely imaginary eigenvalues. It is not applicable to purely zero (e.g., double zero) singularity.

The CNF given by Eqs. (38)–(40) for Hopf-zero singularity can be further simplified. In fact, Ushiki [17] applied Lie algebra to consider this case and obtained a simpler form than the CNF. However, the simpler form given in Ref. [17] is only up to fifth order. In the following two sections, we will extend the perturbation technique described above to compute the SNF of system (11) up to any order. In particular, we focus on two singularities: one is associated with Hopf and generalized Hopf bifurcations, and the other with Hopf-zero bifurcation.

3. The SNF for Hopf and generalized Hopf bifurcations

For a system associated with Hopf singularity, i.e., the Jacobian of the system contains a pair of purely imaginary eigenvalues; Baider and Churchill [26] developed grading functions to obtain the simplest "form". In general, they defined the first, the second, …, infinite order normal forms, and

the first order normal form is actually the CNF. The CNFs of Hopf and generalized Hopf bifurcations obtained using the perturbation method can be found in Ref. [7], where it is shown that the perturbation technique is computationally efficient. It was used to confirm the results of a planar system up to the 60th order. Recently, explicit computation of the SNF for Hopf and generalized Hopf bifurcations have been developed. Based on a general CNF, the explicit SNF and associated NT are obtained [22].

To be more specific, consider the system

$$\dot{\boldsymbol{x}} = J\boldsymbol{x} + \boldsymbol{F}(\boldsymbol{x}), \quad \boldsymbol{x} \in \boldsymbol{R}^n, \tag{42}$$

where function $\boldsymbol{F}$ and its first derivative vanish at the origin $\boldsymbol{0}$, and J is given by

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & A \end{bmatrix}, \quad A \in R^{(n-2)\times(n-2)}, \tag{43}$$

in which A is "stable" (i.e., all eigenvalues of A have negative real parts).

By using the perturbation technique described in the previous section (the Maple program is available in Ref. [3]) one can find the CNF of system (42), given in polar co-ordinates:

$$\begin{aligned} \dot{r} &= D_2 r + D_4 r + D_6 r + \cdots + D_{2n} r + \cdots \\ &\equiv a_{13} r^3 + a_{15} r^5 + a_{17} r^7 + \cdots + a_{1(2n+1)} r^{2n+1} + \cdots, \\ \dot{\theta} &= 1 + D_2\phi + D_4\phi + D_6\phi + \cdots + D_{2n}\phi + \cdots \\ &\equiv 1 + a_{23} r^2 + a_{25} r^4 + a_{27} r^6 + \cdots + a_{2(2n+1)} r^{2n} + \cdots, \end{aligned} \tag{44}$$

where the notations $D_i r$ and $D_i \phi$ have similar meaning as that given in Eqs. (38) and (39). The coefficients a_{ij}'s are expressed explicitly in terms of the coefficients of the original function $\boldsymbol{F}$.

Based on the CNF given in Eq. (44), the explicit formulas of the SNF of Hopf and generalized Hopf bifurcations have been obtained by Yu [22] as

$$\text{(Hopf)} \quad \begin{cases} \dot{R} = a_{13} R^3 + a_{15} R^5 \\ \dot{\Theta} = 1 + a_{23} R^2 \end{cases} \quad \text{if } a_{13} \neq 0; \tag{45}$$

$$\text{(GH-I)} \begin{cases} \dot{R} = a_{1(2k+1)} R^{2k+1} + b_{1(4k+1)} R^{4k+1}, \\ \dot{\Theta} = 1 + a_{2\,(2k+1)} R^{2k}, \end{cases} \quad \text{if} \quad \begin{cases} a_{13} = a_{15} = \cdots = a_{1(2k-1)} = 0, \ a_{1(2k+1)} \neq 0, \\ a_{23} = a_{25} = \cdots = a_{2(2k-1)} = 0; \end{cases} \tag{46}$$

$$\text{(GH-II)} \begin{cases} \dot{R} = a_{1(2k+1)} R^{2k+1} + b_{1(4k+1)} R^{4k+1}, \\ \dot{\Theta} = 1 + b_{2(2j-1)} R^{2(j-1)} + b_{2(2(j+1)-1)} R^{2j} + \cdots + b_{2(2k+1)} R^{2k}, \end{cases} \quad \text{if} \quad \begin{cases} a_{13} = a_{15} = \cdots = a_{1(2k-1)} = 0, \ a_{1(2k+1)} \neq 0, \\ a_{23} = a_{25} = \cdots = a_{2(2j-3)} = 0, \ a_{1(2j-1)} \neq 0 \quad (2 \leqslant j \leqslant k), \end{cases} \tag{47}$$

where GH denotes the generalized Hopf bifurcation. The coefficients b_{ij}'s are explicitly expressed in terms of the a_{ij}'s. The proof and the detailed computation procedure can be found in Ref. [22]. The general idea of the proof is that the two kth order coefficients of the NT can be used to eliminate the two $(k+1)$th order CNF coefficients.

The approach described above for computing the SNF is via the CNF in two steps: the first step is to find the CNF from the original n-dimensional system using a CNF method; and the second step is to further simplify the CNF described on the 2-D center manifold to obtain the SNF. Thus, the NT between the original system and the SNF cannot be obtained directly. One must combine the two NTs, one from the original system to the CNF and the other from the CNF to the SNF, to find the required NT. This is quite time consuming, in particular, for higher order normal forms. Therefore, it is necessary to develop a method to compute the SNF directly from the original system (42). In the following, the perturbation technique, used to find CNF from system (42), will be extended to compute the SNF directly from system (42).

In order to achieve this, i.e., to use the perturbation approach described in the previous section to directly compute the SNF as well as the associated NT, first note that in the previous section when we solve a differential equation (e.g., Eq. (37)) after the secular terms are removed, we only assume a form for the particular solution (as usually used in the method of MTS, e.g., $x_{12} = x_{12}^p$) and then use the method of harmonic balance (e.g., $x_{12}^p = p_0 + p_2 \cos 2T_0 + \sin 2T_0 + p_3 \cos 3T_0 + \sin 3T_0$) to find the unique particular solution. However, in general, the solution to a non-homogeneous differential equation should include two parts: one is the particular solution while the other is the solution to the homogeneous equation (e.g., $D_0^2 x_{12} + x_{12} = 0$). Therefore, for the kth order perturbation equation, in addition to the particular solution, one may have an extra part (homogeneous solution), given in the general form: $A_k\, r^{k+1} \cos T_0 + B_k\, r^{k+1} \sin T_0$. The two extra coefficients A_k and B_k introduced in the homogeneous solution can be taken arbitrarily since this solution automatically satisfies the homogeneous differential equation. Hence, instead of using $D_k a$ and $D_k \phi$ to eliminate the secular terms at the kth order perturbation equation (which is the main step in using the perturbation technique to compute the CNF) one may use the two extra coefficients A_k and B_k to balance the secular terms, and therefore $D_k a$ and $D_k \phi$ can be set zero, resulting in a simpler form—the SNF.

To show the detailed procedure, we can still use Eqs. (14)–(18) to consider the Hopf bifurcation by neglecting Eq. (16) which corresponds to the zero eigenvalue. Let us look again at Eq. (37) which is a second order, non-homogeneous differential equation. The general form of solution to this equation can be written as

$$x_{12} = x_{12}^h + x_{12}^p, \tag{48}$$

where x_{12}^h represents the solution to the homogeneous differential equation $D_0^2\, x_{12} + x_{12} = 0$, whereas x_{12}^p denotes a particular solution to Eq. (37). The part x_{12}^h was not included in the solution (i.e., $x_{12}^h = 0$) when we were deriving the CNF in the previous section. It is easy to see that the solution of x_{12}^h can be written in the form of

$$x_{12}^h = A_{21} \cos(T_0) + B_{21} \sin(T_0), \tag{49}$$

where A_{21} and B_{21} are monomials of r, to be determined by harmonic balance. However, it should be noted that for Hopf bifurcation all the terms appearing on the right-hand side of Eq. (37) can be put in the form of Cr^3, where C is a constant. Thus, in order to explicitly show r in Eq. (49), we

may rewrite Eq. (49) as

$$x_{12}^h = A_{21} r^3 \cos(T_0) + B_{21} r^3 \sin(T_0), \tag{50}$$

where A_{21} and B_{21} are now arbitrary constant coefficients. Also, note that the r appearing in the above solution should be treated as a variable of the MTSs, $T_1, T_2, \ldots$. Thus, it will generate derivatives $D_1 r, D_2 r$, etc. when it is substituted into higher order perturbation equations. In general, similar to Eq. (50), one can write the solution form

$$x_{1\,2k}^h = A_{2k\,1} r^{2k+1} \cos(T_0) + B_{2k\,1} r^{2k+1} \sin(T_0) \tag{51}$$

for the $(2k+1)$th order perturbation equation. However, the coefficients $A_{2k\,1}$ and $B_{2k\,1}$ cannot be used to simplify the $(2k+1)$th order normal form coefficients, but the $(2k+3)$th order normal form coefficients. For example, the third order CNF coefficients a_{13} and a_{23} (see Eq. (44)) of Hopf bifurcation cannot be eliminated since the coefficients A_{01} and B_{01} do not exist. For the fifth order CNF coefficients a_{15} and a_{25}, the coefficient B_{21} can be used to eliminate a_{25}, whereas A_{21} cannot be used, and thus a_{15} has to be retained. In general, the $(2k+3)$th order ($k \geqslant 2$) CNF coefficients a_{12k+3} and a_{22k+3} can be removed using both $A_{2k\,1}$ and $B_{2k\,1}$. The results obtained from the above discussion are summarized below.

3.1. Hopf

If $D_2 r \neq 0$ (i.e., $a_{13} \neq 0$, here we use the notation D rather than the a coefficients since the Maple program uses D notation), then

$$\begin{aligned} &B_{21} \Rightarrow D_4\phi = 0 \quad (A_{21} \text{ can be set zero}),\\ &A_{2k\,1} \Rightarrow D_{2k+2} r = 0 \\ &B_{2k\,1} \Rightarrow D_{2k+2}\phi = 0 \end{aligned} \quad \text{for } k \geqslant 2, \tag{52}$$

where the notation $\Rightarrow$ means an elimination. For example, $B_{21} \Rightarrow D_4\phi = 0$ implies that $D_4\phi$ can be eliminated using the coefficient B_{21}. Thus, the SNF of Hopf bifurcation can be found as

$$\begin{aligned} \dot{R} &= D_2 r(R) + D_4 r(R) \equiv a_{13} R^3 + a_{15} R^5, \\ \dot{\Theta} &= 1 + D_2\phi(R) \equiv 1 + a_{23} R^2, \end{aligned} \tag{53}$$

up to any order. Note that we use the explicit expression $D_2 r(R)$, $D_4 r(R)$ and $D_2\phi(R)$ to indicate that the variable r for the CNF has been replaced by R in the expression of the derivatives. The coefficients a_{13}, a_{15} and a_{23} are identical to those given in Eq. (44) and thus the above equation obtained using the perturbation technique is identical to that obtained in Ref. [22], given by Eq. (45).

When $D_2 r = 0$, we can similarly find the procedure for computing the SNF of the generalized Hopf bifurcations given in Eqs. (46) and (47). The proof is omitted here but the detailed "rules" for choosing $A_{2k\,1}$ and $B_{2k\,1}$ to eliminate $D_{2k+2} r$ and $D_{2k+2}\phi$ are listed below.

3.2. GH-I

If

$$D_2 r = D_4 r = \cdots = D_{2k-2} r = 0, \quad D_{2k} r \neq 0,$$
$$D_2 \phi = D_4 \phi = \cdots = D_{2k-2} \phi = 0;$$

then

first terms: $D_{2k} r \neq 0$ and $D_{2k}\phi$,

$$\begin{aligned}
A_{2\,1} &\Rightarrow D_{2k+2} r = 0,\\
B_{2\,1} &\Rightarrow D_{2k+2}\phi = 0,\\
A_{4\,1} &\Rightarrow D_{2k+4} r = 0,\\
B_{4\,1} &\Rightarrow D_{2k+4}\phi = 0,\\
&\vdots\\
A_{2k-2\,1} &\Rightarrow D_{4k-2} r = 0,\\
B_{2k-2\,1} &\Rightarrow D_{4k-2}\phi = 0,\\
(A_{2k\,1}) &\Rightarrow D_{4k} r \neq 0,\\
B_{2k\,1} &\Rightarrow D_{4k}\phi = 0,\\
A_{2k+2\,1} &\Rightarrow D_{4k+2} r = 0,\\
B_{2k+2\,1} &\Rightarrow D_{4k+2}\phi = 0,\\
&\vdots
\end{aligned} \tag{54}$$

Note that the coefficient $A_{2k\,1}$ does not appear in the corresponding equation and thus $D_{4k} r$ cannot be eliminated. ($A_{2k\,1}$ can be set zero.) Therefore, the SNF for this case is given by

$$\begin{aligned}
\dot{R} &= D_{2k} r(R) + D_{4k} r(R) \equiv a_{1(2k+1)} R^{2k+1} + b_{1(4k+1)} R^{4k+1},\\
\dot{\Theta} &= 1 + D_{2k}\phi(R) \equiv 1 + a_{2(2k+1)} R^{2k+1},
\end{aligned} \tag{55}$$

which is the same as Eq. (46) obtained before [22].

3.3. GH-II

If

$$D_2 r = D_4 r = \cdots = D_{2k-2} r = 0, \quad D_{2k} r \neq 0,$$
$$D_2 \phi = D_4 \phi = \cdots = D_{2j-4}\phi = 0, \quad D_{2j-2}\phi \neq 0; \quad \text{for } 2 \leqslant j \leqslant k,$$

then

$$\begin{aligned}
&B_{2j-2\,1} \neq 0,\\
&B_{2j\,1} \neq 0,\\
&\vdots\\
&B_{2k-2\,1} \neq 0,
\end{aligned}$$

$$
\begin{aligned}
&A_{2k\,1} \neq 0,\\
&B_{2k\,1} \neq 0,\\
&*B_{2\,1} \Rightarrow D_{2k+2}r = 0,\\
&B_{2k+2\,1} \Rightarrow D_{2k+2}\phi = 0,\\
&B_{4\,1} \Rightarrow D_{2k+4}r = 0,\\
&B_{2k+4\,1} \Rightarrow D_{2k+4}\phi = 0,\\
&\vdots\\
&**A_{2\,1} \Rightarrow D_{4k-2j+4}r = 0,\\
&B_{4k\,1} \Rightarrow D_{4k-2j+4}\phi = 0,\\
&A_{4\,1} \Rightarrow D_{4k-2j+6}r = 0,\\
&B_{4k+2\,1} \Rightarrow D_{4k-2j+6}\phi = 0,\\
&\vdots\\
&A_{2j-6\,1} \Rightarrow D_{4k-4}r = 0,\\
&B_{4k-2j\,1} \Rightarrow D_{4k-4}\phi = 0,\\
&A_{2j-4\,1} \Rightarrow D_{4k-2}r = 0,\\
&B_{4k-2j+2\,1} \Rightarrow D_{4k-2}\phi = 0,\\
&***(A_{2j-2\,1}) \Rightarrow D_{4k}r \neq 0,\\
&B_{4k-2j+4\,1} \Rightarrow D_{4k}\phi = 0,\\
&A_{2j\,1} \Rightarrow D_{4k+2}r = 0,\\
&B_{4k-2j+6\,1} \Rightarrow D_{4k+2}\phi = 0,\\
&\vdots
\end{aligned}
\tag{56}
$$

Note that in case GH-I, the coefficients $A_{2i\,1}$ and $B_{2i\,1}$ are consistently used to eliminate $D_{2i+2k}r$ and $D_{2i+2k}\phi$, respectively; while for case GH-II these two coefficients must exchange their rules at the place marked by $**$. Again, $D_{4k}r \neq 0$ is due to the fact that the coefficient $A_{2j-2\,1}$ does not appear in the equation (and can thus be set to zero). The SNF for this case is given by

$$
\begin{aligned}
\dot{R} &= D_{2k}r(R) + D_{4k}r(R)\\
&\equiv a_{1(2k+1)}R^{2k+1} + b_{1(4k+1)}R^{4k+1},\\
\dot{\Theta} &= 1 + D_{2j-2}\phi(R) + \cdots + D_{2k}\phi(R)\\
&\equiv 1 + b_{2(2j-1)}R^{2(j-1)} + b_{2(2(j+1)-1)}R^{2j} + \cdots + b_{2(2k+1)}R^{2k},
\end{aligned}
\tag{57}
$$

which is identical to that given in Eq. (47).

It should be pointed out that without the results and the proof presented in Ref. [22], it is very difficult (if not impossible) to find the "rule" of computing the coefficients $A_{2k\,1}$ and $B_{2k\,1}$, especially for case GH-II.

The perturbation procedure described in the previous section and this section can be easily implemented using a computer algebra system such as Maple or Mathematica. In fact, Maple programs for computing the SNF of Hopf and generalized Hopf bifurcations have been

developed, and the results for the examples presented in Section 5 are obtained by executing the Maple programs. The Maple source code can be found in this journal and Appendix A. The programs have been coded in a user-friendly fashion, which can be "automatically" executed on a main frame, a work station or a PC without users' interaction.

4. The SNF for Hopf-zero bifurcation

In this section, we turn to consider the SNF of Hopf-zero bifurcation using the perturbation approach. The SNF for the generic case of Hopf-zero singularity was first reported in Ref. [17] using the Lie algebra approach, but that form was given only up to fifth order. Recently, this case has been reconsidered Refs. [21,23], and the SNF for the generic case has been obtained explicitly up to any order [23]. The proof given in Ref. [23] for the Hopf-zero singularity, based on the CNF, follows the similar idea for Hopf and generalized Hopf bifurcations [22].

Let us consider the system

$$\dot{\boldsymbol{x}} = J\,\boldsymbol{x} + \boldsymbol{F}(\boldsymbol{x}), \quad \boldsymbol{x} \in \boldsymbol{R}^n, \tag{58}$$

with $F_i(\mathbf{0}) = \partial F_i(\mathbf{0})/\partial x_j = 0$, $i,j = 1,2,\ldots,n$, but the Jacobian J now is given by

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A \end{bmatrix}, \quad A \in R^{(n-3)\times(n-3)}, \tag{59}$$

where A is a stable matrix. By the normal form theory, the CNF of system (58) can be found in the cylindrical co-ordinates up to an arbitrary order n in the form of (e.g., see Refs. [21,23])

$$\begin{aligned}
\dot{r} &= r\left(a_{101}\,z + \sum_{i=1}^{m_1}\sum_{j=0}^{i} a_{1\,2(i-j)\,2j}\,r^{2\,(i-j)}\,z^{2\,j} + \sum_{i=1}^{m_2}\sum_{j=0}^{i} a_{1\,2(i-j)\,2j+1}\,r^{2\,(i-j)}\,z^{2\,j+1}\right),\\
r\,\dot{\theta} &= r\left(1 + a_{201}\,z + \sum_{i=1}^{m_1}\sum_{j=0}^{i} a_{2\,2(i-j)\,2j}\,r^{2\,(i-j)}\,z^{2j} + \sum_{i=1}^{m_2}\sum_{j=0}^{i} a_{2\,2(i-j)\,2j+1}\,r^{2\,(i-j)}\,z^{2\,j+1}\right),\\
\dot{z} &= \sum_{i=1}^{m_3}\sum_{j=0}^{i} a_{3\,2(i-j)\,2j}\,r^{2\,(i-j)}\,z^{2\,j} + \sum_{i=1}^{m_1}\sum_{j=0}^{i} a_{3\,2(i-j)\,2j+1}\,r^{2\,(i-j)}\,z^{2\,j+1},
\end{aligned} \tag{60}$$

where $m_1 = m_2 + 1 = m_3 = \frac{1}{2}(n-1)$ when n is an odd integer; and $m_1 = m_2 = m_3 - 1 = n/2 - 1$ when n is an even integer. The coefficients a_{ijk}'s are given explicitly in terms of the original

coefficients of function $\boldsymbol{F}$. Then, based on Eq. (60), it has been proved that the SNF of system (58) up to an arbitrary order n is given by Ref. [23]

$$\begin{aligned}
\dot{R} &= R(b_{101}Z + b_{120}R^2 + b_{140}R^4),\\
\dot{\Theta} &= 1 + b_{201}Z + b_{220}R^2 + \sum_{i=2}^{m_3} b_{202i}Z^{2i},\\
\dot{Z} &= b_{302}Z^2 + b_{303}Z^3 + \sum_{i=1}^{m_3} b_{3\,2i\,0}R^{2i} + \sum_{i=1}^{m_1} b_{3\,2i\,1}\,R^{2\,i}\,Z,
\end{aligned} \tag{61}$$

if a_{101}/a_{302} is not an algebraic number. In particular, a_{101} and a_{302} should satisfy the following conditions:

$$\begin{aligned}
&2(m_1 - i)a_{101} + (2\,i - 1)\,a_{302} \neq 0 \quad \text{for } i = 1, 2, \ldots, k;\ k = 1, 2, \ldots, m_1,\\
&(m_1 - i)\,a_{101} + (i - 1)\,a_{302} \neq 0 \quad \text{for } i = 0, 1, \ldots, k;\ k = 0, 1, \ldots, m_1,
\end{aligned} \tag{62}$$

when n is odd; and

$$\begin{aligned}
&2\,(m_1 + 1 - i)\,a_{101} + (2\,i - 3)\,a_{302} \neq 0 \quad \text{for } i = 1, 2, \ldots, k;\ k = 1, 2, \ldots, m_1,\\
&(m_1 - i)\,a_{101} + i\,a_{302} \neq 0 \quad \text{for } i = 0, 1, \ldots, k;\ k = 0, 1, \ldots, m_1,
\end{aligned} \tag{63}$$

when n is even. Here, note in Eq. (61) that $b_{101} = a_{101}$, $b_{201} = a_{201}$, $b_{320} = a_{320}$, $b_{302} = a_{302}$, and other coefficients are expressed explicitly in terms of a_{ijk}'s.

The similar idea and procedure used in the previous section for computing the SNF of Hopf bifurcation can be applied here to directly compute the SNF of the Hopf-zero singularity using the perturbation approach. However, the procedure is more involved for this case. Again consider the differential equation (37) for which we can still use the general solution form given by Eq. (48), but we must change Eq. (49) to the form

$$x_{12}^h = r[A_{2\,1\,1}\,z\cos(T_0) + B_{2\,1\,1}\,z\sin(T_0)], \tag{64}$$

where the coefficient $r\,z$ follows the pattern of the second terms in the first two equations of Eq. (60). In addition, following the pattern of the third equation of Eq. (60), one can have another equation, given by

$$x_{32}^h = C_{2\,2\,0}\,r^2 + C_{2\,0\,2}\,z^2. \tag{65}$$

Thus, we have a total of four arbitrary coefficients $A_{2\,1\,1}$, $B_{2\,1\,1}$, $C_{2\,2\,0}$ and $C_{2\,0\,2}$ which can be used in the third-order perturbation equations to possibly eliminate $D_2\,r$, $D_2\,\phi$ and $D_2\,z$. It is clear that the second-order CNF given in Eq. (60) cannot be simplified since no coefficients can be used at this step.

According to the pattern of the CNF given in Eq. (60), it is easy to find the general formulas for the homogeneous solution, written in the form of

$$\begin{aligned}
x_{1n}^h &= (A_{n\,n-1\,1}\,r^{n-1}z + A_{n\,n-3\,3}\,r^{n-2}\,z^3 + \cdots + A_{n\,1\,n-1}\,r\,z^{n-1})\cos(T_0)\\
&\quad + (B_{n\,n-1\,1}\,r^{n-1}\,z + B_{n\,n-3\,3}\,r^{n-2}\,z^3 + \cdots + B_{n\,1\,n-1}\,r\,z^{n-1})\sin(T_0),\\
x_{3n}^h &= C_{n\,n\,0}\,r^n + C_{n\,n-2\,2}\,r^{n-2}\,z^2 + \cdots + C_{n\,0\,n}\,z^n
\end{aligned} \tag{66}$$

when n is an even number; and

$$x_{1n}^{h} = (A_{n\,n\,0} r^{n} + A_{n\,n-2\,2} r^{n-2} z^{2} + \cdots + A_{n\,1\,n-1} r z^{n-1}) \cos(T_0) + (B_{n\,n\,0} r^{n} + B_{n\,n-2\,2} r^{n-2} z^{2} + \cdots + B_{n\,1\,n-1} r z^{n-1}) \sin(T_0),$$
$$x_{3n}^{h} = C_{n\,n-1\,1} r^{n-1} z + C_{n\,n-3\,3} r^{n-3} z^{3} + \cdots + C_{n\,0\,n} z^{n}, \tag{67}$$

when n is an odd number.

Next, from the pattern of the SNF described by Eq. (61) we may find the procedure to solve the coefficients from the ordered perturbation equations. The procedure is similar to that for the Hopf bifurcation; thus, we only show the computation procedure below for brevity.

Second order:

$$\left.\begin{array}{ll} \text{1st Eqn.:} & r z \\ \text{2nd Eqn.:} & r z \\ \text{3rd Eqn.:} & r^2 \quad z^2 \end{array}\right\} \quad \text{(cannot be simplified).}$$

Third order:

$$\begin{array}{llll} \text{1st Eqn.:} & r^3 & r z^2 & \text{term } r^3 \text{ retained,} \\ & & A_{2\,1\,1} & \\ \text{2nd Eqn.:} & r^3 & r z^2 & \text{term } r^3 \text{ retained,} \\ & & B_{2\,1\,1} & \\ \text{3rd Eqn.:} & r^2 z & z^3 & \text{term } z^3 \text{ retained.} \\ & C_{2\,2\,0} & & \end{array}$$

Fourth order:

$$\begin{array}{llll} \text{1st Eqn.:} & r^3 z & r z^3 & \\ & A_{3\,3\,0} & A_{3\,1\,2} & \\ \text{2nd Eqn.:} & r^3 z & r z^3 & \\ & B_{3\,3\,0} & B_{3\,1\,2} & \\ \text{3rd Eqn.:} & r^4 & r^2 z^2 \quad z^4 & \text{term } r^4 \text{ retained,} \\ & & C_{3\,2\,1} \quad C_{3\,0\,3}, & \end{array}$$

Fifth order:

$$\begin{array}{llll} \text{1st Eqn.:} & r^5 & r^3 z^2 \quad r z^4 & \text{term } r^5 \text{ retained,} \\ & & A_{4\,3\,1} \quad A_{4\,1\,3} & \\ \text{2nd Eqn.:} & r^5 & r^3 z^2 \quad r z^4 & \text{term } r z^4 \text{ retained,} \\ & B_{4\,3\,1} & B_{4\,1\,3} & \\ \text{3rd Eqn.:} & r^4 z & r^2 z^3 \quad z^5 & \\ & C_{4\,4\,0} & C_{4\,2\,2} \quad C_{4\,0\,4}. & \end{array}$$

kth order:

$k \geqslant 6$ (even)

1st Eqn. :	$r^{k-1}\, z$	$r^{k-3}\, z^3$	$\cdots$	$r\, z^{k-1}$	
	$A_{k-1\,k-1\,0}$	$A_{k-1\,k-3\,2}$	$\cdots$	$A_{k-1\,1\,k-2}$	
2nd Eqn. :	$r^{k-1}\, z$	$r^{k-3}\, z^3$	$\cdots$	$r\, z^{k-1}$	
	$B_{k-1\,k-1\,0}$	$B_{k-1\,k-3\,2}$	$\cdots$	$B_{k-1\,1\,k-2}$	
3rd Eqn. :	r^k	$r^{k-2}\, z^2$	$\cdots$	z^k	
		$C_{k-1\,k-2\,0}$	$\cdots$	$C_{k-1\,0\,k-2}$	term r^k retained.

kth order:

$k \geqslant 6$ (odd)

1st Eqn. :	r^k	$r^{k-2}\, z^2$	$\cdots$	$r\, z^{k-1}$	
	$*\, C_{k-1\,k-1\,0}$	$A_{k-1\,k-2\,1}$	$\cdots$	$A_{k-1\,1\,k-2}$	
2nd Eqn. :	r^k	$r^{k-2}\, z^2$	$\cdots$	$r\, z^{k-1}$	term $r\, z^{k-1}$ retained,
	$B_{k-1\,k-2\,1}$	$B_{k-1\,k-4\,3}$	$\cdots$		
3rd Eqn. :	$r^{k-1}\, z$	$r^{k-3}\, z^3$	$\cdots$	z^k	
		$C_{k-1\,k-3\,2}$	$\cdots$	$C_{k-1\,0\,k-1}$	term $r^{k-1}\, z$ retained.

4.1. Notes

(1) Although the CNF coefficients are referred in the above notations, the computations including the algorithm and Maple program are not restricted to the CNF, but for the general system (58).

(2) The notations of the 1st, 2nd, and 3rd Eqns. denote the 1st, 2nd and 3rd equations of Eq. (60), respectively.

(3) The variables like $r\, z$, $r\, z^2$, etc. represent the pattern of the terms appearing in the corresponding order perturbation equation.

(4) A coefficient given under a pattern variable means that the pattern variable can be eliminated by using the coefficient. For example, in the fifth order perturbation equation, the term $r^3\, z^2$ can be removed using the coefficient $A_{4\,3\,1}$.

(5) If a variable does not have a corresponding coefficient, then the variable must be retained in the SNF. For example, the variable r^5 in the fifth order perturbation equation must be retained in the SNF.

(6) Unlike Hopf and generalized Hopf bifurcations, the "form" of the SNF for Hopf-zero bifurcation is not unique. This can be seen from the above list that we could use the coefficients to eliminate the pattern variable other than those assigned in the list. For example, it is seen from the fifth order perturbation equation that one may use the coefficients $A_{4\,3\,1}$ and $A_{4\,1\,3}$ to eliminate r^5 and $r^3\, z^2$, or r^5 and $r\, z^4$. However, once a "form" like the one listed above is selected, the SNF is unique.

(7) In the third order perturbation equation, the coefficient $C_{2\,0\,2}$ is not used, and has been set zero. In fact, it can been shown that at this order perturbation (and only at this order

perturbation) only three of the four linear algebraic equations, which involve the four coefficients $A_{2\,1\,1}$, $B_{2\,1\,1}$, $C_{2\,2\,0}$ and $C_{2\,0\,2}$, are independent. Further, one may prove that one such remaining coefficient from this order perturbation equation cannot be used for simplifying higher order normal forms.

5. An oscillator model

In this section, we use the double pendulum system, shown in Fig. 1, to demonstrate the application of the results obtained in the previous sections and the Maple programs developed in this paper. This double pendulum model has been considered by many authors (e.g., see Refs. [8,15,27]) for a number of singularities including Hopf, double zero, Hopf-zero, and double Hopf bifurcation. However, the equations (presented in all these papers) describing the motion of the system are expanded up to only third order terms. In other words, all higher order terms are neglected. Here, in order to obtain the SNF of Hopf bifurcation, one needs to expand the equations up to fifth order. For generalized Hopf bifurcations, the equations must be expanded at least up to ninth order. Such high order expansions result in enormous large expressions (with more than 700 lines computer output), and developing efficient computation methods is essential.

The double pendulum system (Fig. 1) consists of two rigid weightless links of equal length l which carry two concentrated masses $2\,m$ and m, respectively. A follower force P is applied to this system.

The system energy for the three linear springs k_1, k_2 and k_3 is given in the form of

$$V = \tfrac{1}{2}k_1\,\theta_1^2 + \tfrac{1}{2}k_2\,\theta_2^2 + \tfrac{1}{2}k_3\,l^2(\sin\theta_1 + \sin\theta_2)^2, \tag{68}$$

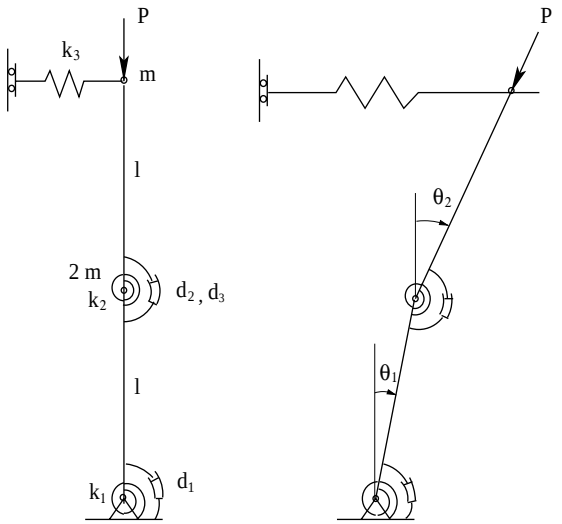

Fig. 1. A double pendulum system.

where θ_1 and θ_2 are generalized co-ordinates which specify the configuration of the system completely. The kinetic energy T of the system is expressed by

$$T = \frac{m l^2}{2 \Omega^2}[3 \theta_1'^2 + \theta_2'^2 + 2 \theta_1' \theta_2' \cos(\theta_1 - \theta_2)], \tag{69}$$

where Ω is an arbitrary value rendering the time variable non-dimensional [15], and the prime denotes differentiation with respect to the non-dimensional time variable τ with $\tau = \Omega t$.

The generalized force corresponding to the generalized co-ordinates θ_1 and θ_2 may be written as

$$Q = P l \sin(\theta_1 - \theta_2), \tag{70}$$

and the damping is assumed to be

$$D = \tfrac{1}{2}[d_1 \theta_1'^2 + d_2(\theta_1' - \theta_2')^2] - \tfrac{1}{4} d_3(\theta_1' - \theta_2')^4, \tag{71}$$

where d_1, d_2 represents the linear parts and d_3 describes the non-linear parts, respectively. In general, one may assume that $d_1, d_2, d_3 \geqslant 0$, indicating that the system has positive linear damping, but may have a negative non-linear damping term.

With the aid of the Lagrangian equations, in addition, choosing the state variables

$$z_1 = \theta_1, \quad z_2 = \theta_1', \quad z_3 = \theta_2 \quad \text{and} \quad z_4 = \theta_2', \tag{72}$$

one can find a set of first order differential equations as follows:

$$\begin{aligned}
z_1' &= z_2, \\
z_2' &= \tfrac{1}{2}\cos(z_1 - z_3)\{-f_1 z_1 - f_2 (z_1 - z_3) - \eta_1 z_2 - \eta_2 (z_2 - z_4) - 2 z_4^2 \sin(z_1 - z_3) \\
&\quad - f_3 \cos(z_1)(\sin z_1 + \sin z_3) + f_4 \sin(z_1 - z_5) + f_5 (z_2 - z_4)^3 \\
&\quad - \cos(z_1 - z_3)[f_2 (z_1 - z_3) + \eta_2 (z_2 - z_4) + 2 z_2^2 \sin(z_1 - z_3) \\
&\quad - f_3 \cos(z_3)(\sin z_1 + \sin z_3) - f_5 (z_2 - z_4)^3]\}, \\
z_3' &= z_4, \\
z_4' &= \tfrac{1}{2}\cos(z_1 - z_3)\{3[f_2 (z_1 - z_3) + \eta_2 (z_2 - z_4) + 2 z_2^2 \sin(z_1 - z_3) \\
&\quad - f_3 \cos(z_3)(\sin z_1 + \sin z_3) - f_5 (z_2 - z_4)^3] \\
&\quad + \cos(z_1 - z_3)[f_1 z_1 + f_2 (z_1 - z_3) + \eta_1 z_2 + \eta_2 (z_2 - z_4) \\
&\quad + 2 z_4^2 \sin(z_1 - z_3) + f_3 \cos(z_1)(\sin z_1 + \sin z_3) \\
&\quad - f_4 \sin(z_1 - z_5) - f_5 (z_2 - z_4)^3]\},
\end{aligned} \tag{73}$$

where f_i's and η_j's are dimensionless coefficients, defined as

$$\begin{aligned}
f_1 &= \frac{k_1 \Omega^2}{m l^2}, \quad f_2 = \frac{k_2 \Omega^2}{m l^2}, \quad f_3 = \frac{k_3 \Omega^2}{m}, \quad f_4 = \frac{P \Omega^2}{m l}, \\
f_5 &= \frac{d_3 \Omega^4}{m l^2}, \quad \eta_1 = \frac{d_1 \Omega^2}{m l^2}, \quad \eta_2 = \frac{d_2 \Omega^2}{m l^2},
\end{aligned} \tag{74}$$

and $f_1, f_2, f_3 \geqslant 0$ due to physical restrictions, and $f_5, \eta_1, \eta_2 \geqslant 0$.

The Jacobian matrix of Eq. (73) evaluated at an arbitrary point at the initial equilibrium solution $z_i = 0$ takes the form

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{2}f_1 - f_2 + \frac{1}{2}f_4 & -\eta_2 - \frac{1}{2}\eta_1 & f_2 - \frac{1}{2}f_4 & \eta_2 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2}f_1 + 2f_2 - f_3 - \frac{1}{2}f_4 & 2\eta_2 + \frac{1}{2}\eta_1 & -2f_2 - f_3 + \frac{1}{2}f_4 & -2\eta_2 \end{bmatrix} \tag{75}$$

from which one may obtain the characteristic polynomial

$$P(\lambda) = \lambda^4 + a_1\,\lambda^3 + a_2\,\lambda^2 + a_3\,\lambda + a_4, \tag{76}$$

where

$$\begin{aligned} a_1 &= \tfrac{1}{2}\eta_1 + 3\,\eta_2, \\ a_2 &= 3f_2 + \tfrac{1}{2}\eta_1\,\eta_2 + \tfrac{1}{2}f_1 - f_4 + f_3, \\ a_3 &= 2\,\eta_2 f_3 + \tfrac{1}{2}f_1\,\eta_2 + \tfrac{1}{2}\eta_1 f_3 + \tfrac{1}{2}\eta_1 f_2, \\ a_4 &= 2\,f_2 f_3 + \tfrac{1}{2}f_1 f_2 + \tfrac{1}{2}f_1 f_3 - f_3 f_4. \end{aligned} \tag{77}$$

Applying the Hurwitz criterion shows that when

$$a_1 > 0,\ \ a_2 > 0,\ \ a_4 > 0 \quad \text{and} \quad a_3\,(a_1 a_2 - a_3) - a_4\,a_1^2 > 0, \tag{78}$$

the initial equilibrium solution $z_i = 0$ is stable. It should be noted that the conditions given in Eq. (78) implies $a_3 > 0$, which is of course as expected. It is easy to show that Hopf bifurcation occurs when $a_3\,(a_1\,a_2 - a_3) - a_4\,a_1^2 = 0$, at which the Jacobian has a pair of purely imaginary eigenvalues while the other two eigenvalues still have negative real parts. When $a_1\,a_2 - a_3 = 0$ and $a_4 = 0$, the Jacobian has a Hopf-zero singularity.

In this paper, we focus on the computation of the SNF without perturbation parameters (unfolding). The computation of the SNFs for two cases are given below. Simple bifurcation and stability analysis are presented to show the advantage of using the SNF.

5.1. Hopf and generalized Hopf bifurcations

We shall compute the SNFs for Hopf and two generalized Hopf bifurcations.

5.1.1. Hopf: $a_{13} \neq 0$

For this case, one can find a critical point, defined by

$$f_1 = 5, \quad f_2 = \tfrac{3}{4}, \quad f_3 = \tfrac{1}{4}, \quad f_4 = \tfrac{5}{2}, \quad f_5 = 3, \quad \eta_1 = \tfrac{1}{2}, \quad \eta_2 = 3, \tag{79}$$

at which the eigenvalues of the Jacobian are

$$\lambda_{1,2} = \pm \mathrm{i}, \quad \lambda_3 = -\tfrac{1}{4}, \quad \lambda_4 = -9. \tag{80}$$

With the linear transformation

$$\begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & -9 \\ 1 & -\frac{1}{2} & -4 & -\frac{43}{22} \\ \frac{1}{2} & 1 & 1 & \frac{387}{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}, \tag{81}$$

one can obtain the Jordan canonical form for the linear part:

$$J_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & -9 \end{bmatrix}. \tag{82}$$

Substituting the transformation (81) into Eq. (73) and expanding the resulting equations in Taylor series up to, say, ninth order yields 780 lines computer output for the four equations, and thus are not presented here. (The computer input files are available from the website: http://www.pyu1.apmaths.uwo.ca/~pyu/pub/software.)

For this example, we have used two procedures to find the SNF: the first procedure is an indirect approach, combining the computations of the CNF and then the SNF obtained on the basis of the CNF using the method given in Ref. [22]. The second procedure is the direct method developed in this paper to compute the SNF from the four transformed state equations directly. A comparison for the two approaches will be given to show the advantage of the direct method.

The first procedure is described as follows: executing the Maple program given in Ref. [7] yields the CNF up to, say, ninth order:

$$\begin{aligned} \dot{r} &= \tfrac{1373}{356864} r^3 - \tfrac{20503743297091}{6435410619269120} r^5 - \tfrac{617766602346825150499326839}{3008048150041913956585439232 00} r^7 \\ &\quad - \tfrac{208541212477078660527047077255149233854109}{230573050515681687613954117620295727579136000} r^9 + \cdots, \\ \dot{\theta} &= 1 - \tfrac{7651}{356864} r^2 + \tfrac{2856512037384017}{231674782293688320} r^4 + \tfrac{230321514983502022819543333}{3008048150041913956585439232 00} r^6 \\ &\quad + \tfrac{160216194991408050498557554591818063303998 3}{138343830309409012568372470572177436547481600 0} r^8 + \cdots. \end{aligned} \tag{83}$$

Since $a_{13} = \frac{1373}{356864} \neq 0$, so according to formula (45), the SNF is

$$\begin{aligned} \dot{R} &= \tfrac{1373}{356864} R^3 - \tfrac{20503743297091}{6435410619269120} R^5, \\ \dot{\Theta} &= 1 - \tfrac{7651}{356864} R^2, \end{aligned} \tag{84}$$

up to *infinite* order. The NT between the original system (with its Jacobian in canonical form) and the SNF (84) via the transformation

$$y_1 = R\cos\Theta, \quad y_2 = -R\sin\Theta, \tag{85}$$

has also been obtained by combining the two NTs: one between the original system (given in the transformed state equation) and system (83), and the other between systems (83) and (84). It has been noticed that computing the combination of the two NTs is very time consuming. In fact, it took about 65 min to obtain the NT from a PC machine (PENTINUM III-700MMX 1024 K system).

The second procedure is to execute the Maple program developed in this paper, resulting in the same equation (84) as well as the explicit NT up to ninth order. The computation time is only about 20 s on the same PC, which is roughly the same as that for computing the CNF (83). This suggests that the direct method for computing the SNF is indeed computationally efficient. The explicit 9th order NT is not listed here due to its extremely long length.

Now it is very clear to see the advantage of using the SNF from Eqs. (83) and (84). If we use the CNF (83) to study the bifurcation and stability for the double pendulum system near a Hopf critical point, one must neglect the seventh and ninth order terms from the first equation of Eq. (83). However, with the SNF given in Eq. (84), one can easily find the steady state solutions:

$$\text{(a) } R=0 \quad \text{and} \quad \text{(b) } R=\frac{11152}{20503743297091}\sqrt{4081987734301361735}. \tag{86}$$

Solution (a) is actually the original equilibrium $x_i=0$ (or $z_i=0$) while solution (b) represents the motion of a limit cycle. It is not difficulty to use the linearization, based on the first equation of Eq. (84), to prove that the limit cycle is stable. However, the equilibrium $R=0$ is a non-linear center and linearization does not work for determining its stability. However, it is easy to see from this simple equation that the cubic term dominates the equation for sufficiently small R, implying that $\dot{R}>0$ when $R\approx 0$. This indicates that $R=0$ is unstable. For further detailed bifurcation analysis, a perturbation parameter is needed, which will not be discussed here.

5.1.2. Generalized Hopf

$a_{13}=a_{23}=0$, $a_{15}\neq 0$

If one chooses the following parameter values:

$$f_1=\frac{261+3\sqrt{1689}}{70}, \quad f_2=\frac{121+3\sqrt{1689}}{280}, \quad f_2=\frac{159-3\sqrt{1689}}{280},$$
$$f_4=\frac{261+3\sqrt{1689}}{140}, \quad f_5=\frac{-8219+303\sqrt{1689}}{1680}, \quad \eta_1=\frac{1}{2}, \quad \eta_2=\frac{121+3\sqrt{1689}}{70}, \tag{87}$$

then the Jacobian of the system becomes

$$J_c=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & -\frac{363+9\sqrt{1689}}{70} \end{bmatrix}. \tag{88}$$

Similarly, executing the Maple program results in the following SNF:

$$\begin{aligned}\dot{R}=&-\left[\frac{24496973786396176127}{32817625267531776000}-\frac{197651938936617393}{10939208422510592000}\sqrt{1689}\right]R^5\\ &+\left[\frac{3044173547069301319598859274167139190643723418436752377163661822527895348759485881149}{21893010787635772815848483260167462094617332198530341793041672013143110320128000000}\right.\\ &\left.-\frac{24690641843116591009553514618197321847424970841145898548059928751087234219936363681651}{729767026254525760528282775338915403153911073284344726434722400438103677337600000000}\sqrt{1689}\right]R^9,\\ \dot{\Theta}=&1-\left[\frac{16667499335273736857}{65635250535063552000}-\frac{138322651547879863}{21878416845021184000}\sqrt{1689}\right]R^4,\end{aligned} \tag{89}$$

which falls in category GH-I with $k = 2$ (see Eq. (55)). Similarly, we can find the solution of the limit cycle and its stability from the first equation of Eq. (89). It is seen from this example that the formulas and Maple programs developed in this paper can be applied not only to rational numbers, but also to irrational numbers. In general, the programs can be applied to any numerical numbers and/or symbolic notations.

5.1.3. Generalized Hopf

$a_{13} = 0,\ a_{15} \neq 0,\ a_{23} \neq 0$

Keep the parameter values the same as that used in Hopf case, except that f_5 is now taken as

$$f_5 = \tfrac{1291}{888}, \tag{90}$$

then the Jacobian has the same eigenvalues as the Hopf case and the Jordan canonical form is identical to Eq. (82). But the SNF is now given by

$$\begin{aligned}\dot{R} &= -\tfrac{767702969}{133344542720} R^5 + \tfrac{58368205538268968132154081702099744397609}{13081082538944209919354538297562961182457856 0} R^9,\\ \dot{\Theta} &= 1 - \tfrac{13}{592} R^2 + \tfrac{5633653476952156762841}{2819242229707135538462720} R^4,\end{aligned} \tag{91}$$

which belongs to category GH-II with $k = j = 2$, by comparing Eq. (91) with Eq. (57). Again a similar bifurcation and stability analysis can be given for this case.

5.1.4. Remarks

It has been noted that the SNFs of Hopf and generalized Hopf bifurcations are indeed unique regardless the methods used and no matter how different the CNFs may be. Further note that the SNFs are finite. This is different from other singularities. For example, the SNF for Takens–Bogodanov singularity (a double zero eigenvalue) does not have a unique "form". Only if a fixed "form" is chosen, is the SNF then unique [24]. We use the example of generalized Hopf bifurcation given above to illustrate this fact. We have used two different methods to find the CNF for this example up to, say, ninth order, given below:

$$\begin{aligned}\dot{r} &= -\tfrac{767702969}{133344542720} r^5 - \tfrac{307083432712581766 27}{103395761232282648576 00} r^7 - \tfrac{228432103271351602727140253610433}{210361106679274885978423352623104000} r^9,\\ \dot{\theta} &= 1 - \tfrac{13}{592} r^2 + \tfrac{353143783073}{40803430072320} r^4 - \tfrac{3180695772121758372 47}{1757727940948805025792 00} r^6\\ &\quad - \tfrac{1903439059823162190766991787105018 1}{4291366576257207673959836393511321600 0} r^8,\end{aligned} \tag{92}$$

which is obtained using the perturbation technique [7]. The other CNF is given by

$$\begin{aligned}\dot{r} &= -\tfrac{767702969}{133344542720} r^5 - \tfrac{163198706644600701 71}{516978806161413242880 0} r^7 - \tfrac{168905353064869754194974555856983}{131475691674546803736514595389440 00} r^9\\ \dot{\theta} &= 1 - \tfrac{13}{592} r^2 + \tfrac{324072151913}{40803430072320} r^4 - \tfrac{222751322503702316267}{1757727940948805025792 00} r^6\\ &\quad - \tfrac{2644640085908679766430524342288 9}{3134672444307675437516315846246400 0} r^8,\end{aligned} \tag{93}$$

obtained using the CNF theory (e.g., Takens normal form theory). It is observed from Eqs. (92) and (93) that except for the two leading terms $-\tfrac{767702969}{133344542720} r^5$ and $-\tfrac{13}{592} r^2$, all the other coefficients are different. However, when we use the same formulas given in Ref. [22] to compute the coefficients of the SNF from the two different CNFs, we obtain the same SNF, given by Eq. (91).

5.2. Hopf-zero singularity

In order to obtain a critical point at which the system has a simple zero and a pair of purely imaginary eigenvalues, it is required that $a_4 = 0$ and $a_1 a_2 = a_3$. Choosing the parameter values

$$f_1 = 3, \quad f_2 = \tfrac{1}{4}, \quad f_3 = \tfrac{3}{4}, \quad f_4 = \tfrac{5}{2}, \quad f_5 = 1, \quad \eta_1 = \eta_2 = 1, \tag{94}$$

yields the eigenvalues

$$\lambda_{1,2} = \pm \mathrm{i}, \quad \lambda_3 = 0 \quad \text{and} \quad \lambda_4 = -\tfrac{7}{2}. \tag{95}$$

Similar to Hopf bifurcation, introduce the linear transformation

$$\begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -\frac{7}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{3} \\ \frac{1}{2} & 1 & 0 & \frac{35}{6} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}, \tag{96}$$

into Eq. (73) to obtain the Jordan canonical form

$$J_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{7}{2} \end{bmatrix}. \tag{97}$$

The CNF for this case can be found using the perturbation method [28]:

$$\begin{aligned} \dot{r} &= r\left[\tfrac{1817}{27136} r^2 - \tfrac{1047}{6784} z^2 - \tfrac{5836100159}{146352046080} r^4 - \tfrac{3303078383571}{15537708892160} r^2 z^2 - \tfrac{9709115}{140944384} z^4\right], \\ \dot{\theta} &= 1 + \tfrac{1577}{13568} r^2 - \tfrac{195}{3392} z^2 - \tfrac{98827513073}{2341632737280} r^4 + \tfrac{1872907276221}{7768854446080} r^2 z^2 + \tfrac{4784053931}{4510220288} z^4, \\ \dot{z} &= z\left[-\tfrac{207}{896} r^2 - \tfrac{101}{448} z^2 + \tfrac{43162232923}{1026075115520} r^4 + \tfrac{228685241}{1973221376} r^2 z^2 + \tfrac{138661}{3512320} z^4\right]. \end{aligned} \tag{98}$$

It is seen from the above equations that since $a_{101} = a_{201} = a_{320} = a_{302} = 0$ so we cannot apply the results and Maple program developed in this paper for Hopf-zero singularity to find the SNF of the above system. In fact, if we apply the Maple program to this example, we would obtain the normal form up to fourth order which is in the "form" of the CNF, but with different coefficients. Executing the program to higher order terms results in the CNF with undetermined A and B coefficients.

We use another known example to demonstrate the computation of the SNF for Hopf-zero singularity. This example is a simple 3-D system described by

$$\dot{x}_1 = -x_2 - (x_1 - x_3)^2, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = -(x_1 - x_3)^2, \tag{99}$$

which was considered by Chow et al. [6] for computing the CNF. They used Takens normal form theory to find the normal form up to fourth order terms. We applied the perturbation approach

[28] to obtain the CNF up to, say, 10th order, given in cylindrical co-ordinates as follows:

$$\begin{aligned}
\dot{r} &= r\,z\big[1-\tfrac{1}{36}r^2-6\,z^2+\tfrac{5257}{1536}r^4-\tfrac{52}{3}r^2\,z^2+82\,z^4+\tfrac{8380395263}{298598400}r^6-\tfrac{647040649}{746496}r^4\,z^2\\
&\quad-\tfrac{1445}{3}r^2\,z^4-2915\,z^6+\tfrac{2833569591307051}{2149908480000}r^8-\tfrac{422554075263397}{33592320000}r^6\,z^2\\
&\quad+\tfrac{15852923765}{186624}r^4\,z^4+\tfrac{1301908}{9}r^2\,z^6+216682z^8\big],\\
\dot{\theta} &= 1+\tfrac{5}{24}r^2+2\,z^2+\tfrac{5743}{6912}r^4+\tfrac{71}{36}r^2\,z^2-19\,z^4+\tfrac{90259819}{9953280}r^6+\tfrac{12826997}{497664}r^4\,z^2\\
&\quad-\tfrac{9983}{144}r^2\,z^4+401\,z^6+\tfrac{221989021403}{1592524800}r^8-\tfrac{4078616364881}{5971968000}r^6\,z^2-\tfrac{1509761095}{186624}r^4\,z^4\\
&\quad-\tfrac{161231}{18}r^2\,z^6-21608\,z^8+\tfrac{586278398752593881}{180592312320000}r^{10}-\tfrac{1336799327788071511}{42998169600000}r^8\,z^2\\
&\quad-\tfrac{74470597523800247}{716636160000}r^6\,z^4+\tfrac{6974100659453}{8957952}r^4\,z^6+\tfrac{239776423}{144}r^2\,z^8+2075676\,z^{10},\\
\dot{z} &= -\tfrac{1}{2}r^2-z^2-\tfrac{313}{288}r^4+2\,r^2\,z^2+4\,z^4-\tfrac{1325465}{165888}r^6+\tfrac{43663}{1296}r^4\,z^2-40\,r^2\,z^4-100\,z^6\\
&\quad-\tfrac{7357559261}{66355200}r^8+\tfrac{62758273}{62208}r^6\,z^2-\tfrac{57671}{432}r^4\,z^4+2804\,r^2\,z^6+5280\,z^8\\
&\quad-\tfrac{77291316228697}{35831808000}r^{10}+\tfrac{4808794663208693}{134369280000}r^8\,z^2-\tfrac{35063334614201}{859963392}r^6\,z^4\\
&\quad-\tfrac{88740547}{1296}r^4\,z^6-325064\,r^2\,z^8-453440\,z^{10}.
\end{aligned}\tag{100}$$

Note that Eq. (100) does not involve odd order terms because of the special form of Eq. (99). It is seen from Eq. (100) that $a_{101}=a_{302}=1$, which violates the second condition of Eq. (63), indicating that this case is not generic. However the Maple program has been developed to allow for such non-genericness. Executing the Maple program developed in this paper yields the SNF up to 10th order:

$$\begin{aligned}
\dot{R} &= RZ,\\
\dot{\Theta} &= 1-\tfrac{19}{24}R^2+\tfrac{865}{192}Z^4+\tfrac{185253037}{233280}Z^6-\tfrac{143758879489111}{22394880000}Z^8+\tfrac{68158920358549065431}{65840947200000}Z^{10},\\
\dot{Z} &= -\tfrac{1}{2}R^2-Z^2-\tfrac{343}{96}R^4+\tfrac{14020496369}{29859840}R^8.
\end{aligned}\tag{101}$$

It is seen from Eq. (101) that the third equation is even simpler than the generic case since the terms R^6, R^{10}, etc. do not appear in the $\dot{Z}$ equation. This is due to the condition $a_{101}=a_{302}=1$. The SNF given by Eq. (101) is indeed much simpler than the CNF described by Eq. (100).

It is observed from Eqs. (100) and (101) that using the SNF to give a bifurcation analysis is much simpler than using the CNF. In fact, it is not easy to find the steady state solutions (by setting $\dot{r}=\dot{z}=0$) from the CNF since the two polynomials are coupled through r and z. However, it is straightforward to find the steady state solutions and their stabilities from the first and the third equations of the SNF given in Eq. (101). Setting $\dot{R}=\dot{Z}=0$ yields

$$\text{(i) } R=Z=0 \quad \text{and} \quad \text{(ii) } R=0.355778, \qquad Z=0. \tag{102}$$

Solution (i) is the initial equilibrium solution while solution (ii) represents a limit cycle. To find the stability of the limit cycle, one can apply linearization to the two equations $\dot{R}$ and $\dot{Z}$ to find the two eigenvalues evaluated on the solution of the limit cycle: $\lambda_{12}=\pm 0.780203$, indicating that the limit cycle is unstable. Two perturbation parameters (unfolding) are needed for further bifurcation and stability analyses.

6. Conclusions

A previously developed perturbation technique for calculating CNF has been extended to compute the SNF of dynamical systems associated with a purely imaginary pair, and a simple zero and a pair of purely imaginary eigenvalues. It has been shown that the perturbation method has computational advantages over the standard method of normal forms. The technique combines the normal form theory with center manifold theory in one unified procedure to determine a unique CNF. The comparison between the indirect and direct computations of the SNF shows that the direct computation is much more efficient. It has also shown the advantage of using the SNF that greatly simplifies the bifurcation and stability analyses. Moreover, other advantages can be observed: (1) the technique is straightforward and systematic, and can be easily implemented using a computer algebra system such as Maple; (2) the approach can be straightforwardly extended to consider the SNF of systems associated with other singularities. (3) the method can be directly extended to study non-autonomous systems involving forcing functions and/or parametric excitations; and (4) the technique can be generalized to investigate systems which may involve perturbing (ε) terms, or which may involve non-linear terms not necessary given in homogeneous polynomials. However, this perturbation approach is only applicable to the cases in which the Jacobian of the system evaluated at a critical point involves, at least, a pair of purely imaginary eigenvalues.

Acknowledgements

The supports received from the City University of Hong Kong and the Natural Sciences and Engineering Research Council of Canada (NSERC) are greatly acknowledged.

Appendix A. Input files

The complete input files for the system can be found at the website: http://www.pyu1.apmath-s.uwo.ca/~pyu/pub/software.

```
sysin := 1:
if sysin = 1 then
   M1 := 2:
   M2 := 0:
   N := 2+M1+M2*2:
   Order := 8:
   func := table([(1)=x[2]+...,(2)=-x[1]+...,(3)=-1/4*x[3]+...,(4)=-
   9*x[4]+...]);
elif sysin = 2 then
   M1 := 2:
   M2 := 0:
   N := 2+M1+M2*2:
```

```
   Order := 8:
   func := table([(1)=x[2]+...,(2)=-x[1]+...,(3)=-1/4*x[3]+...,
   (4)=-363/70*x[4]-9/70*x[4]*1689^(1/2)+...]);
elif sysin = 3 then
   M1 := 2:
   M2 := 0:
   N := 2+M1+M2*2:
   Order := 8:
   func := table([(1)=x[2]+...,(2)=-x[1]+...,(3)=-1/4*x[3]+...,(4)=-
   9*x[4]+...]);
elif sysin = 4 then
   M1 := 1:
   M2 := 0:
   N := 3+M1+M2*2:
   Order:=8:
   func := table([(1)=x[2]+...,(2)=-x[1]+...,(3)=0+...,(4)=-7/2*x[4]+...]);
elif sysin = 5 then
   M1 := 0:
   M2 := 0:
   N := 3+M1+M2*2:
Order:=10:
   func[1] := -x[2]-(x[1]-x[3])^2:
   func[2] := x[1]:
   func[3] := -(x[1]-x[3])^2:
   for i from 1 to 3 do
     func[i] := subs(x[1]=X1,func[i]):
     func[i] := subs(x[2]=x[1],func[i]):
     func[i] := subs(X1=x[2],func[i]):
   od:
   temp := func[1]:
   func[1] := func[2];
   func[2] := temp;
   func[3] := func[3];
fi:
```

In this appendix, the input files for the examples presented in Section 5 are given. A user can follow the samples to prepare one's own input files. Due to the large size of the input files, we only list the linear parts for the pendulum system.

References

[1] V.I. Arnold, Lectures on bifurcations in versal families, Russian Mathematical Surveys 27 (1972) 54–123.
[2] F. Takens, Unfoldings of certain singularities of vector fields: generalized Hopf bifurcations, Journal of Differential Equations 14 (1973) 476–493.

[3] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer, New York, 1983.
[4] C. Elphich, E. Tirapegui, M.E. Brachet, P. Coullet, G. Iooss, A simple global characterization for normal forms of singular vector fields, Physica D 29 (1987) 95–127.
[5] S.-N. Chow, C.-Z. Li, D. Wang, Normal Forms and Bifurcation of Planar Vector Fields, Cambridge University Press, Cambridge, 1994.
[6] S.-N. Chow, B. Drachman, D. Wang, Computation of normal forms, Journal of Computational and Applied Mathematics 29 (1990) 129–143.
[7] P. Yu, Computation of normal forms via a perturbation technique, Journal of Sound and Vibration 211 (1) (1998) 19–38.
[8] P. Yu, Q. Bi, Analysis of non-linear dynamics and bifurcations of a double pendulum, Journal of Sound and Vibration 217 (4) (1998) 691–736.
[9] P. Yu, W. Zhang, Q. Bi, Vibration analysis on a thin plate with the aid of computation of normal forms, International Journal of Non-Linear Mathematics 36 (2001) 597–627.
[10] Q.C. Zhang, A.Y.T. Leung, J.E. Cooper, Computation of normal forms for higher dimensional semi-simple systems, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis 8a (2001) 559–574.
[11] P. Yu, Symbolic computation of normal forms for resonant double Hopf bifurcations using multiple time scales, Journal of Sound and Vibration 247 (2001) 615–632.
[12] P. Yu, Analysis on double Hopf bifurcation using computer algebra with the aid of multiple scales, Nonlinear Dynamics 27 (2001) 19–53.
[13] J. Carr, Applications of Center Manifold Theory, Springer, New York, 1981.
[14] A.H. Nayfeh, D.T. Mook, Nonlinear Oscillations, Wiley, New York, 1979.
[15] K. Huseyin, Multiple Parameter Stability Theory and its Applications, Oxford University Press, Oxford, 1986.
[16] K. Huseyin, R. Lin, An intrinsic multiple-scale harmonic balance method for non-linear vibration and bifurcation problems, International Journal of Non-linear Mechanics 26 (1991) 727–740.
[17] S. Ushiki, Normal forms for singularities of vector fields, Japan Journal of Applied Mathematics 1 (1984) 1–37.
[18] A. Baider, J.A. Sanders, Further reduction of the Takens–Bogdanov normal forms, Journal of Differential Equations 99 (1992) 205–244.
[19] L.O. Chua, H. Kokubu, Normal forms for nonlinear vector fields—Part I: theory and algorithm, IEEE Transactions on Circuits and Systems 35 (1988) 863–880.
[20] L.O. Chua, H. Kokubu, Normal forms for non-linear vector fields—Part II: Applications, IEEE Transactions on Circuits and Systems 36 (1988) 51–70.
[21] A. Algaba, E. Freire, E. Gamero, Hypernormal form for the Hopf-zero bifurcation, International Journal of Bifurcation and Chaos 8 (1998) 1857–1887.
[22] P. Yu, The simplest normal forms of Hopf and generalized Hopf bifurcations, International Journal of Bifurcation and Chaos 9 (10) (1999) 2279–2319.
[23] P. Yu, Y. Yuan, The simplest normal forms associated with a pure imaginary pair and a simple zero eigenvalue, Dynamics of Continuous, Discrete and Impulsive Systems (DCDIS) Series B: Applications and Algorithms 8b (2001) 219–249.
[24] Y. Yuan, P. Yu, Computation of the simplest normal forms of differential equations associated with a double zero eigenvalues, International Journal of Bifurcation and Chaos 11 (5) (2001) 1307–1330.
[25] P. Yu, Y. Yuan, The simplest normal forms associated with a triple zero eigenvalue of index one and two, Proceedings of the Third World Congress of Nonlinear Analysis, July 19–26, 2000, Catania, Sicily, Italy (Special issue) Nonlinear Analysis: Theory, Methods and Applications 47 (2) (2001) 1105–1116.
[26] A. Baider, R. Churchill, Unique normal forms for planar vector fields, Mathematische Zeitschrift 199 (1988) 303–310.
[27] R. Scheidl, H. Troger, K. Zeman, Coupled flutter and divergence bifurcation of a double pendulum, International Journal of Non-linear Mechanics 19 (1984) 163–176.
[28] P. Yu, Symbolic computation of normal forms using a perturbation method, Proceedings of the 1999 Joint Mechanics and Materials Conference, Blacksburg, VA, June 27–30, 1999.