ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Sound and Vibration 261 (2003) 169–176

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Letter to the Editor

Vibrations of the system with quadratic non-linearity and a constant excitation force

L. Cveticanin*

Faculty of Technical Sciences, Trg D. Obradovica 6, 21000 Novi Sad, Yugoslavia

Received 17 May 2002; accepted 2 July 2002

1. Introduction

In this paper the mechanical vibration of a one degree-of-freedom mass–spring system under the influence of a constant positive excitation force ($a > 0$) is considered. The elastic property of the spring is non-linear and is a quadratic function of the deflection y. The mathematical model of the system is

$$\ddot{y} + cy + (\pm)b^2 \operatorname{sign}|y|(y^2) = a, \tag{1}$$

where c and b^2 are constant coefficients of the linear and quadratic term, respectively. The sign function is used to elastic force in the spring to satisfy the antisymmetric condition. Namely, if the spring is extended, a force which has the tendency to relax and to put the spring into the previous state appears. The same happens when the spring is pressed. In the first case the deformation is positive, and in the second case it is negative. Due to the fact that the force in the spring is a quadratic function of deformation y the change of the sign of the force is described using the sign function. The signs in the bracket $(\pm)$ indicate the hard and the soft spring, respectively; the plus sign in the bracket is for hard spring and the minus sign in the bracket is for soft spring. This meaning of the bracket $(\pm)$ is applied in the whole paper. The forced vibration of system (1) is subject to the following initial conditions:

$$y(0) = y_0 = 0, \qquad \dot{y}(0) = \dot{y}_0 = 0. \tag{2}$$

For analyzing the vibration properties of the mechanical system, the exact solution of Eq. (1) with respect to the initial conditions (2) is obtained. For the case when the non-linearity is small ($b^2 \equiv |\varepsilon| \ll 1$) a perturbation (series) solution is developed which is usual by more convenient for practical application than the exact solution.

In the paper of Mickens [1] the uniformly valid asymptotic solution for a second order differential equation with the small quadratic non-linearity which does not change its sign due to

*Tel.: +381-21-350-122/815; fax: +381-21-58-133.
E-mail address: cveticanin@uns.ns.ac.yu (L. Cveticanin).

doi:10.1016/S0022-460X(02)01178-1

the fact that the model

$$\ddot{y} + cy = a + \varepsilon y^2, \tag{3}$$

describes another physical phenomena like those in general relativity [2,3] or solid-state physics [4]. The arbitrary conditions are

$$y(0) = y_0 = A, \qquad \dot{y}(0) = \dot{y}_0 = 0. \tag{4}$$

In Refs. [5–7] a list of alternative asymptotic methods for solving Eq. (3) with Eq. (4) is given. The solution is valid only for the case when the non-linearity is small. In Ref. [8] the free vibration of the one mechanical system where the elastic force in the spring is a strong non-linear quadratic function of deflection

$$\ddot{y} + cy + (\pm)b^2 \operatorname{sign}|y|(y^2) = 0, \tag{5}$$

is analyzed. The exact solution of Eq. (5) is in the form of the Jacobi elliptic function. Using the suggested procedures for obtaining the exact and the asymptotic solution of Eq. (5), and also of Eq. (3), in this paper the solution methods for Eq. (1) are developed.

2. Exact solution

Assume an exact solution of Eq. (1) for the initial conditions (2) in the form of the Jacobi elliptic function

$$y = A \operatorname{sn}^2(\omega t, k^2), \tag{6}$$

where ω is the frequency of vibrations, A is the amplitude of vibrations and k^2 is the modulus of the elliptic Jacobi function sn [9]. Namely, it is assumed that the solution of Eq. (1) is of an oscillatory type and one of the most general types of oscillatory functions is the Jacobi elliptic function with a special group of circular functions. In solution (6) there are three unknown values: A, ω and k^2 which have to be determined.

The first and second time derivatives of Eq. (6) are

$$\dot{y} = 2A\omega \operatorname{sn}(\omega t, k^2) \operatorname{cn}(\omega t, k^2) \operatorname{dn}(\omega t, k^2),$$

$$\ddot{y} = 2A\omega^2(3k^2 \operatorname{sn}^4 - 2(k^2+1)\operatorname{sn}^2 + 1). \tag{7}$$

Substituting solution (6) and the corresponding time derivatives (7) into Eq. (1) and equating coefficients by the same order of the function sn, the following algebraic equations are obtained:

$$\operatorname{sn}^4\colon\ (\pm)\operatorname{sign}|A|b^2 A + 6\omega^2 k^2 = 0, \tag{8}$$

$$\operatorname{sn}^2\colon\ c - 4\omega^2(k^2+1) = 0, \tag{9}$$

$$\operatorname{sn}^0\colon\ 2A\omega^2 = a. \tag{10}$$

Solving Eqs. (8)–(10) the values of A, ω and k^2 are obtained, i.e., the exact solution (6) of Eq. (1) with respect to the initial conditions (2).

2.1. Discussion of the parameters

1. The condition of the oscillatory motion of the mass–spring system is $\omega^2 > 0$. From Eq. (10) it is evident that the previous condition is satisfied only for $A > 0$. For the positive amplitude A the vibration y (6) is for all values of time t a non-negative function. It means that the deflection y does not change its sign: it is positive for the whole time interval of motion. This fact tremendously simplifies the solution procedure. Namely, that sign $|x|$ is always positive and the differential equation of motion corresponds to

$$\ddot{y} + cy + (\pm)b^2y^2 = a. \tag{11}$$

2. From Eq. (8) the modulus of the Jacobi function is

$$k^2 = -(\pm)\frac{b^2A}{6\omega^2}. \tag{12}$$

According to Eq. (12) it is evident that the modulus of the Jacobi function is negative ($k^2 < 0$) for the mass–hard spring system, and positive ($k^2 > 0$) for the mass–soft spring system.

3. Analyzing Eq. (9), i.e.,

$$\omega^2 = \frac{c}{4(1 + k^2)}, \tag{13}$$

and using the previous remarks it can be concluded that the modulus of the Jacobi elliptic function is for the mass–hard spring system in the interval

$$-1 < k^2 < 0. \tag{14}$$

Due to the fact that the modulus of the Jacobi function of the mass–hard spring is negative, it is convenient to transform the sn Jacobi function to an sd function with the corresponding positive Jacobi elliptic function [10]. Then solution (6) is

$$y = \frac{A}{(1 + |k^2|)}\,\mathrm{sd}^2(\omega_1 t, k_1^2), \tag{15}$$

where

$$\omega_1 = \omega\sqrt{1 + |k^2|}, \qquad k_1^2 = \frac{|k^2|}{(1 + |k^2|)}. \tag{16}$$

4. Analyzing relation (13) and previous comments on the modulus of Jacobi function it is evident that the frequency of vibration is higher for the mass–hard spring system and lower for the mass–soft spring compares to the frequency of the system with linear spring.

5. For the mass–linear spring system $k^2 = 0$, the frequency of vibration (13) is $\omega = \sqrt{c}/2$ and the amplitude of vibration (10) is $A = 2(a/c)$. The Jacobi elliptic function sn transforms to the circular sin function and solution (1) for $b^2 = 0$ is the well-known result for the mass–linear spring system with constant excitation

$$y = \frac{2a}{c}\sin^2\left(\frac{\sqrt{c}}{2}t\right) = \frac{a}{c}(1 - \cos\sqrt{c}t). \tag{17}$$

6. Solving the system of Eqs. (8)–(10) and applying the previous considerations the amplitude of vibration is

$$A = -\frac{3c}{4b^2}(\pm)\left[1 - \sqrt{1 + (\pm)\frac{16}{3}\frac{ab^2}{c^2}}\right]. \tag{18}$$

The amplitude of vibration depends on the excitation force a, coefficient of linear elasticity c and the coefficient of the non-linearity b^2. There is a limitation for the amplitude of the system with soft spring. Namely, the motion of the mass–soft spring system is oscillatory for $1 - (16/3)(ab^2/c^2) > 0$, i.e.,

$$a < \frac{3}{16}\frac{c^2}{b^2}. \tag{19}$$

7. The motion is periodical and the period of vibration is

$$T = \frac{4K(k^2)}{\omega}, \tag{20}$$

where K is the total elliptic integral of the first kind [9].

8. The forced vibration of the mechanical system described with Eq. (1) and initial conditions (2) is

$$y = A\,\mathrm{sn}^2\left(\sqrt{\frac{a}{2A}}\,t, -(\pm)\frac{b^2A^2}{2a}\right), \tag{21}$$

where A is given by Eq. (18).

2.2. The power series solution

Consider the case when the non-linearity of the spring is small, i.e., $b^2 = \varepsilon$, where ε is a small parameter. The mathematical model is

$$\ddot{y} + cy = a - (\pm)\varepsilon\,\mathrm{sign}\,|y|y^2, \tag{22}$$

where the parameters a, c and ε are positive.

For the linear oscillator with mathematical model

$$\ddot{y} + cy = a, \tag{23}$$

for the initial conditions (2) and $a > 0$ solution (17) is non-negative for all values of time t. For the case when the quadratic non-linearity is small and the solution of Eq. (22) differs from the linear solution only for a small value, it can be assumed that the oscillation remains a non-negative time function. Then the change of the sign of the quadratic term in Eq. (22) is neglected and Eq. (22) transforms to

$$\ddot{y} + cy = a - (\pm)\varepsilon y^2. \tag{24}$$

Introducing the new variable

$$x = y - \frac{a}{c}, \tag{25}$$

into the differential equation (24) gives

$$\ddot{x} + cx = -(\pm)\varepsilon\left(x + \frac{a}{c}\right)^2, \tag{26}$$

with the initial values

$$x(0) = X_0 = -\frac{a}{c}, \qquad \dot{x}(0) = \dot{X}_0 = 0. \tag{27}$$

Now suppose that the solution of Eq. (26) is a series with respect to powers of the small parameter ε

$$x = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots, \tag{28}$$

where $x_0, x_1, x_2, \ldots$ have to be determined to satisfy Eq. (26) for arbitrary values of ε. For the oscillatory solution, the frequency ω is a function of the initial amplitude X_0 and it is assumed as

$$\omega^2 = c + \varepsilon b_1(X_0) + \varepsilon^2 b_2(X_0) + \cdots, \tag{29}$$

i.e.,

$$c = \omega^2 - \varepsilon b_1(X_0) - \varepsilon^2 b_2(X_0) + \cdots, \tag{30}$$

where $b_1(X_0), b_2(X_0), \ldots$ are unknown coefficients for the initial amplitude (27). Substituting Eqs. (28) and (30) into Eq. (26) and equating to zero the terms by the same order of the small parameter ε, the following system of differential equations for $O = O(\varepsilon^3)$ is obtained:

$$\ddot{x}_0 + \omega^2 x_0 = 0,$$

$$\ddot{x}_1 + \omega^2 x_1 = b_1 x_0 - (\pm)\left(x_0 + \frac{a}{c}\right)^2,$$

$$\ddot{x}_2 + \omega^2 x_2 = b_2 x_0 + b_1 x_1 - 2(\pm)x_1\left(x_0 + \frac{a}{c}\right). \tag{31}$$

According to Eqs. (27) and (28) the corresponding initial conditions are

$$x_0(0) = X_0 = -\frac{a}{c}, \qquad x_1(0) = x_2(0) = 0,$$

$$\dot{x}_0(0) = \dot{X}_0 = 0, \qquad \dot{x}_1(0) = \dot{x}_2(0) = 0. \tag{32}$$

For Eq. (32) the solution of the first equation is

$$x_0 = -\frac{a}{c}\cos\omega t, \qquad \omega = \sqrt{c}. \tag{33}$$

Substituting solution (33) into the second equation (31), the secular term in the equation vanishes for

$$b_1 = 2(\pm)\frac{a}{c}, \tag{34}$$

and the solution in the first approximation is

$$x_1 = (\pm)\frac{1}{c}\left(\frac{a}{c}\right)^2\left(\tfrac{4}{3}\cos\omega t - \tfrac{3}{2} + \tfrac{1}{6}\cos 2\omega t\right). \tag{35}$$

Substituting Eqs. (33)–(35) into the third equation (31) the secular term is eliminated if

$$b_2 = -\frac{17}{6c}\left(\frac{a}{c}\right)^2, \tag{36}$$

and the solution is

$$x_2 = \frac{1}{c^2}\left(\frac{a}{c}\right)^3\left(\tfrac{4}{3} - \tfrac{125}{144}\cos\omega t - \tfrac{4}{9}\cos 2\omega t - \tfrac{1}{48}\cos 3\omega t\right). \tag{37}$$

Using the obtained results the approximate solution of order $O(\varepsilon^3)$ for Eq. (22) is

$$\begin{aligned} y = {} & \left(\frac{a}{c}\right)(1-\cos\omega t) \\ & + (\pm)\frac{\varepsilon}{c}\left(\frac{a}{c}\right)^2\left(\tfrac{4}{3}\cos\omega t - \tfrac{3}{2} + \tfrac{1}{6}\cos 2\omega t\right) \\ & + \frac{\varepsilon^2}{c^2}\left(\frac{a}{c}\right)^3\left(\tfrac{4}{3} - \tfrac{125}{144}\cos\omega t - \tfrac{4}{9}\cos 2\omega t - \tfrac{1}{48}\cos 3\omega t\right), \end{aligned} \tag{38}$$

where

$$\omega^2 = c + 2\varepsilon(\pm)\left(\frac{a}{c}\right) - \frac{17}{6}\frac{\varepsilon^2}{c}\left(\frac{a}{c}\right)^2. \tag{39}$$

Analyzing relation (39), it is evident that the frequency of vibration depends on the type of the spring. For the mass–hard spring system the frequency of vibration is higher and for the mass–soft spring system it is lower. From Eq. (38), it is seen that the amplitude of vibration is lower for the mass–hard spring system than for the linear system. The amplitude of vibration of the mass–soft spring system is higher than for the linear system.

Comparing solution (38) and (39) with the uniformly valid approximate solution [1], it can be concluded that they are equal in the first approximation.

3. Comparison of the solutions

Now compare the suggested approximate solution (29), (30) with the exact solution (21), (14). The parameters of the system are $c = 1$ and $\varepsilon = 0.1$, and the excitation force is $a = 0.1$ and 1, respectively. In Fig. 1 the exact y_E and the approximate y_A solution for the excitation parameter $a = 0.1$ (Fig. 1a) and $a = 1$ (Fig. 1b) are plotted. It can be concluded that the accuracy of the approximate solution depends not only on the value of the small parameter ε but also on the value of the parameter a. For $\varepsilon \leqslant 0.1$ and $a \leqslant 0.1$ the approximate solution is on the top of the exact solution. For $\varepsilon \leqslant 0.1$ and $a \geqslant 0.1$ the difference between the approximate and the exact solution is evident.

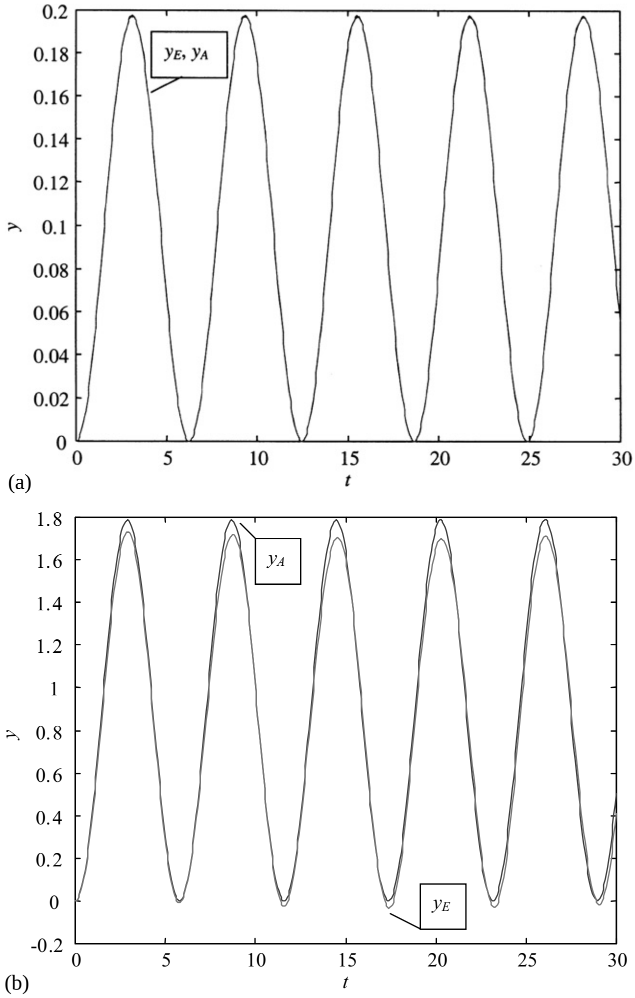

Fig. 1. y_{A-t} and y_{E-t} diagrams for $a = 0.1$ (a) and 1 (b).

Acknowledgements

The present study has been sponsored by Ministry of Science, Technologies and Development, Republic of Serbia (Project No. 1874).

References

[1] R.E. Mickens, A uniformly valid asymptotic solution for $d^2y/dt^2 + y = a + \varepsilon y^2$, Journal of Sound and Vibration 76 (1) (1981) 150–152.

[2] R. Adler, M.Bazin, M. Schiffer, Introduction to General Relativity, McGraw-Hill, New York, 1975.

[3] C. Clark, Elementary General Relativity, Wiley, New York, 1979.

[4] I.G. Main, Vibrations and Waves in Physics, Cambridge University Press, London, 1978.

[5] P.Th. van Woerkom, Comments on "A uniformly valid asymptotic solution for $d^2y/dt^2 + y = a + \varepsilon y^2$", Journal of Sound and Vibration 80 (1982) 157–158.
[6] S.G. Kelly, Comments on "A uniformly valid asymptotic solution for $d^2y/dt^2 + y = a + \varepsilon y^2$", Journal of Sound and Vibration 80 (1982) 155–156.
[7] A.S. Atadan, K. Huseyin, A note on "A uniformly valid asymptotic solution for $d^2y/dt^2 + y = a + \varepsilon y^2$", Journal of Sound and Vibration 85 (1) (1982) 129–131.
[8] L. Cveticanin, Vibrations of the system with quadratic non-linearity, submitted for publication.
[9] P.F. Byrd, Handbook of Elliptic Integrals for Engineers and Physicists, Springer, Berlin, 1954.
[10] M. Abramowitz, J.A. Stegun, Handbook with Formulas, Graphs and Mathematical Tables, Nauka, Moscow, 1979 (in Russian).