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Letter to the Editor

Experimental validation of the shear correction factor

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1. Introduction

The resonant beam technique (RBT) is a widely used method to determine the elastic moduli of isotropic materials [1–3]. Both the shear and the rotary inertia perturb the classical one-dimensional Euler–Bernoulli theory for the flexural vibrations of elastic beams and lead to a shift of the resonance frequencies in particular of the higher modes. From the resonance frequencies of the flexural vibrations, both Young's modulus and the shear modulus can be determined simultaneously. The basic equations of the theory have been established by Timoshenko [4,5]. The solutions of the equations for different loading configurations (i.e., supported, clamped or free at the respective end) can be found in a work by Huang [6].

Timoshenko introduced the shear correction factor k via a comparison of the two-dimensional solution with the approximations of his theory, because the shear stress is not uniformly distributed over the cross-section of the beam. There has been an elaborate discussion about the correction factor in the literature for different test geometries (see Kaneko [7] for an overview). Several efforts have been made to improve the shear correction factor: Cowper [8] used an approach based on the integration of the three-dimensional theory of elasticity, which satisfies the boundary conditions for the cross-section. Another approach was the development of the solution into a series by Hutchinson and Zillmer [9], who also noted a possible dependence on the width-to-depth ratio. The shear correction factor published by Stephen [10,11] was the first to incorporate a dependence on the aspect ratio of the cross-section. The same results were obtained by Hutchinson with a simple dynamic beam theory in a recent publication [12]. Hutchinson found the experimental evidence for the new correction factor inconclusive, because very little data for different aspect ratios were available from the literature. It is the goal of this paper to overcome the absence of experimental data and to compare different predictions for the correction factor with experimental measurements and an "ideal" specimen simulated with resonant ultrasound spectroscopy (RUS) [13–16].

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2. Theoretical background

The basis for the RBT is Timoshenko's equation for the flexural motion of bars. It takes into account the effects of shear deformation and rotary inertia, with L being the length of the specimen, I the moment of inertia, y the transverse displacement, ρ the density, A the cross-sectional area, E and G Young's and the shear modulus, respectively, and k the shear correction factor:

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} - \rho I\left(1 + \frac{E}{kG}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho I}{kG}\frac{\partial^4 y}{\partial t^4} = 0. \tag{1}$$

By solving this equation and taking into account the correct boundary conditions (i.e., in our case the free–free solution), the so-called frequency equation can be obtained [6]. The roots of this frequency equation are the resonance frequencies of the flexural vibrations. Since the two main parameters of the equation are E and E/kG, the shear correction factor k is required to calculate the shear modulus G from E/kG.

Various derivations of the shear correction factor have been proposed. One of the simplest approaches is to assume a parabolic distribution over the cross-section. For a rectangular beam, the factor can then be easily calculated and has a numerical value of $\frac{5}{6}$ [17].

Timoshenko obtained this correction factor by comparing his solution with a 2-D solution of the bending problem, and found a dependence on the Poisson's ratio ν [5]:

$$k_T(\nu) = \frac{(5 + 5\nu)}{(6 + 5\nu)}. \tag{2}$$

Cowper derived a solution for the shear correction [8], which differs only slightly from Timoshenko's solution

$$k_C(\nu) = \frac{(10 + 10\nu)}{(12 + 11\nu)}. \tag{3}$$

In a recent paper [12], Hutchinson used an approach somewhat similar to the one by Cowper, but improved it to find the best "guess" for the stress and the displacement field through a variational form. He derived an analytical expression (i.e., an expansion into a series) for the shear correction factor, which shows, contrary to the expression by Cowper, a distinct dependence on the aspect ratio (b and a are width and depth of the specimen, respectively)

$$k_H(\nu) = -\frac{2(1+\nu)}{\left(\frac{9}{4a^5 b}C_4 + \nu\left(1 - \frac{b^2}{a^2}\right)\right)} \tag{4}$$

with

$$C_4 = \frac{4}{45}a^3 b\left(-12a^2 - 15a^2\nu + 5b^2\nu\right) + \sum_{n=1}^{\infty}\frac{16\nu^2 b^5\left(n\pi a - b\tanh\left(\frac{n\pi a}{b}\right)\right)}{n^5\pi^5(1+\nu)} \tag{5}$$

3. Experimental set-up and evaluation

Since the equipment and experimental set-up has been described more precisely elsewhere [3], only a short summary will be given here: The specimen is suspended in two carbon fibre loops, which are attached to two piezoelectric transducers, and excited to flexural vibrations via the loops (see Fig. 1). The resonance frequencies are detected by sweeping the frequency range with a network analyzer. The noise level is at about -140 dBm, thus allowing the detection of even very weak resonances.

As model materials brass with high Poisson's ratio (and thus a low factor E/kG) and aluminium with low Poisson's ratio were chosen. The specimens were machined from the centre of isotropic rods to ensure that the samples were influenced or deformed as little as possible by the internal stresses originating from the manufacturing process. A high precision of the specimens was ensured by a very careful machining. Additionally, "ideal" brass specimens were simulated via RUS.

The method is based on a complete numerical solution fulfilling the frequency equation and the boundary condition of zero surface traction. This is achieved by expanding the basis functions in each of the dimensions into a series. Care has to be taken to include enough elements in the series expansion, so that higher orders do not change the calculated resonance frequencies anymore. There is an extraordinary good correlation between the frequencies obtained from RUS and RBT for vibrating beams, and deviations are much less than in experiments with "real" materials. This can be seen as a good confirmation of the predictions of both the theories. But it should also be noticed that this also indicates the difficulties and the limitations of an experiment with "real" materials.

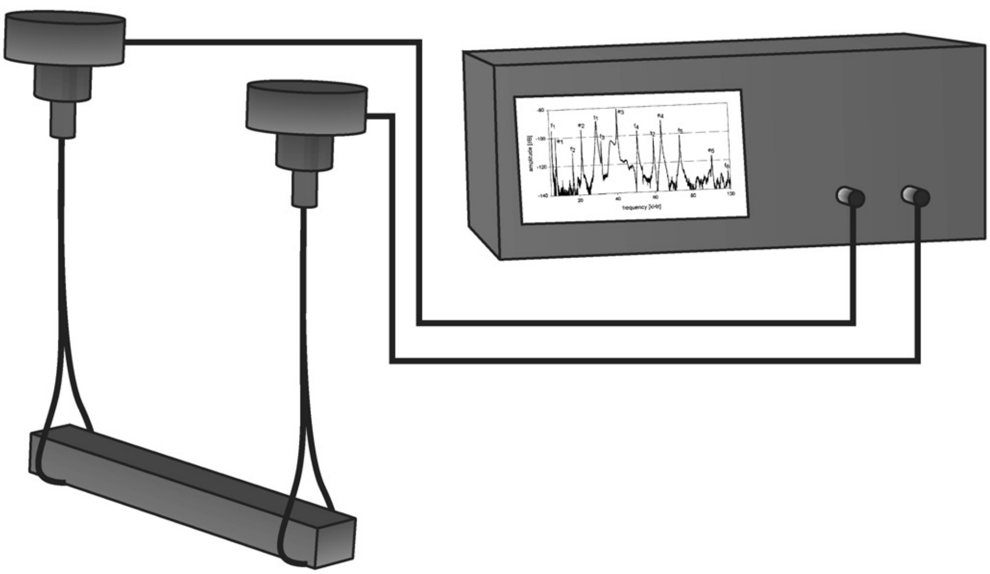

Fig. 1. The experimental set-up. The network analyzer stimulates the sample, which is suspended in two carbon fibre loops, via a piezo electric transducer in a given frequency range and records the answer signal via a piezo electric receiver.

The evaluation of both the experimental and the simulated data was done with software, that numerically solves the frequency equation from the RBT for the first six modes of the transversal vibrations and fits them with the parameters E and E/kG to the measured data. The result of the evaluation of a measurement with RBT is thus always E and E/kG.

4. Results

To calculate the shear correction factor k and its dependency on the aspect ratio of the specimens, the "true" values of E and G are required. This is no problem for the computer experiment: E and G are predetermined and the resonance frequencies are obtained via the RUS procedure. This is an experiment with an "ideal" computer-generated material. The resonance frequencies are now evaluated by RBT, which results in values for E and E/kG. With the knowledge of the predetermined E and G, the k-factor is easily obtained.

The circles in Fig. 2 represent this computer-generated experiment for different aspect ratios. The lines show the predictions from the different theories, i.e., from Timoshenko, Cowper and Hutchinson. It is obvious that there is a distinct dependence on the aspect ratio, which is predicted only by the Hutchinson model. The Poisson's ratio of the computer-generated brass samples is 0.3516.

The same experiment was performed for brass specimens. The material brass was chosen, because of its high Poisson's ratio mentioned before. To obtain the "true" values of E and G for these brass samples, a different procedure had to be chosen: From Fig. 3 it can be seen that for low aspect ratios (less than one) the k-factor tends to be constant and is moreover coincident in the Timoshenko and the Hutchinson model. The values obtained for E and G by the RBT-evaluation procedure for samples with aspect ratios less than one were averaged. They were then assigned as true values and used for the further calculation of the k-factor. Again, the results from

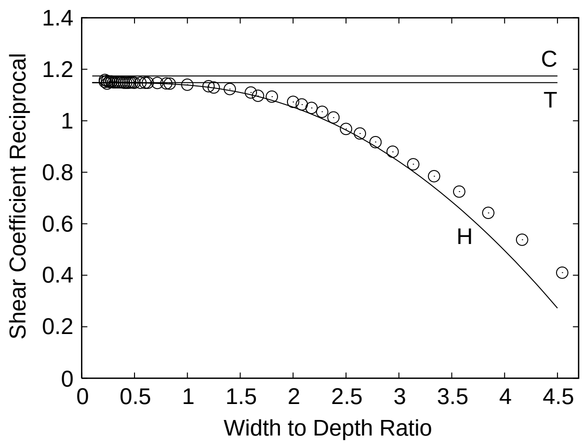

Fig. 2. Shear coefficient reciprocal versus width-to-depth ratio for a rectangular cross-section. The circles depict the shear coefficients from the evaluation of the RUS-simulated brass specimen with Poisson's ratio of 0.3516. The lines indicate the three corresponding shear correction factors: C—Cowper, H—Hutchinson, T—Timoshenko.

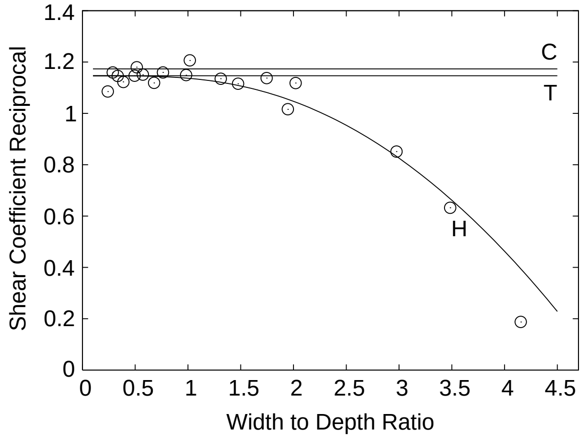

Fig. 3. Shear coefficient reciprocal versus width-to-depth ratio for a rectangular cross section. The circles depict the shear coefficients from the evaluation of the brass specimen. The lines indicate the three corresponding shear correction factors: C—Cowper, H—Hutchinson, T—Timoshenko.

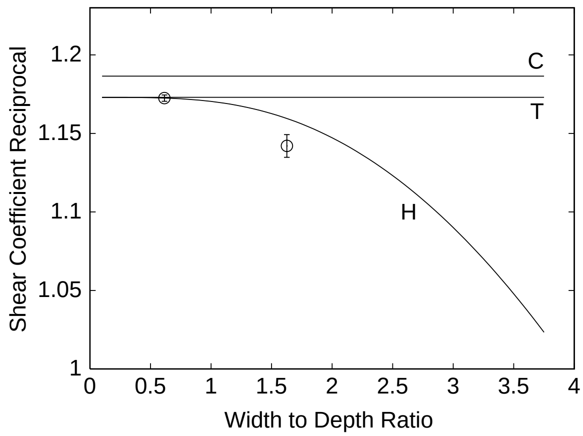

Fig. 4. Shear coefficient reciprocal versus width-to-depth ratio for a rectangular cross-section. The squares depict the shear coefficients for the two width-to-depth ratios of the aluminium specimen with Poisson's ratio of 0.1564. The lines indicate the three corresponding shear correction factors: C—Cowper, H—Hutchinson, T—Timoshenko.

Fig. 3 show a distinct dependence of the k-factor on the aspect ratio as predicted by the Hutchinson model.

Fig. 4 shows an experiment with aluminium, a material with low Poisson's ratio of 0.1564. Here, only two points are drawn, which represent the results from flatwise and edgewise loading of a beam with a rectangular cross-section. Each of the two points represents the mean of 16 single tests, which is the reason for the low standard error. The high number of tests was chosen to show that there is a statistically significant difference between the shear correction factor for the two aspect ratios, even for materials with low Poisson's ratios and aspect ratios close to 1.0.

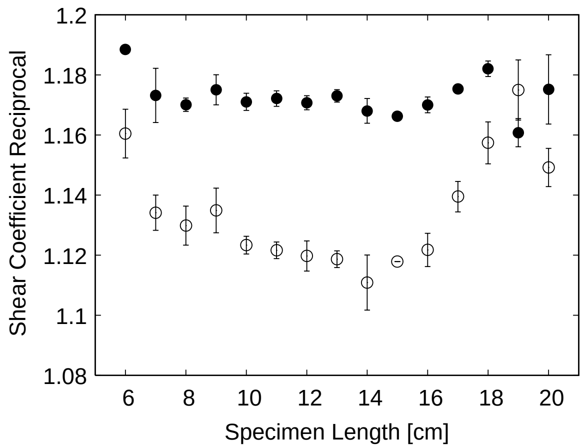

Fig. 5. Shear coefficient reciprocal versus the specimen length of the aluminium samples for two aspect ratios: filled circles—width to depth = 0.615, empty circles—width to depth = 1.626.

The tests in Fig. 4 are the mean of a large number of tests performed with 16 specimens with a constant aspect ratio but with a variation of the length of the specimen. With this experiment the possible influence of testing the specimens at different lengths could be investigated.

The means and the standard errors at each length are shown in Fig. 5, from which it can be deduced that there is no clear influence of the length of the specimen in a wide range. Only for very short and very long specimens, an effect is observed, which will be discussed in the next section.

5. Discussion

The main conclusion from Fig. 2 is that Hutchinson's k-factor is the only one to describe the distinct dependence on the aspect ratio correctly, Timoshenko's k-factor is valid only for small aspect ratios, whereas Cowper's k-factor is too high by about one per cent. The small difference of Hutchinson's k-factor in comparison to the computer-generated experimental values is attributed to two possible reasons: (1) In the model from Hutchinson terms of the eight order are neglected. It could be possible that there is a small influence of these terms. (2) In Hutchinson's model $\sigma_y = \sigma_z = \tau_{yz} = 0$ was assumed. A possible hypothesis is that the last condition could be violated for plate-like specimen, with dimensions $x \gg y$ and that the shear distribution of τ_{yz} in the material is better described by $\int_A \tau_{yz}\, \mathrm{d}A = 0$. This deviation of the experimental values from the theoretical values will be subject to further investigation.

Fig. 3 confirms the results of Fig. 2; however, this "real" experiment features a number of difficulties: First, the "true" values of E and G were not known. Substitutes had to be obtained by averaging the values gained from the RBT evaluation for the range of aspect ratios less than one, where the k-factor is constant. Another major problem was to find isotropic specimens, that are not deformed by internal stresses (which was achieved by machining them from the centre of isotropic brass rods). The third experimental problem is the machining of the specimens, in particular for large aspect ratios: The precision of the measurement is dependent on the uniformity and the precision of their dimensions. In particular for the investigation of small effects, a precision of about 0.01 mm along the length of the specimen is required and trapezoidal cross section have to be avoided. Thus, the scatter in this experiment is significantly larger than in the computer experiment in Fig. 2. The tendency that the k-factor decreases with increasing aspect ratio is, however, obvious even in this case.

The dependence on the aspect ratio increases with increasing Poisson's ratio and is thus very pronounced for brass with Poisson's ratio of about 0.35. It is, however, even visible for aluminium, a material with low Poisson's ratio of 0.1564 and aspect ratios close to 1.0 (see Fig. 4). Again, there is a small difference between the experimental values and Hutchinson's prediction, which is attributed to the uncertainties of the experiment: Firstly, the true values are not known but obtained as for the brass by an averaging procedure. Secondly, the difference is statistically significant, but it varies along the length, as can be observed in Fig. 5. Whereas the k-factor is nearly constant in a wide range of length, it seems to increase in particular for the higher aspect ratio. This effect is very small, i.e., only some per cent. It is very probable that this reflects experimental problems: Hutchinson recommended that the length of the highest mode used for evaluation should be twice the beam depth [9]. For short lengths, one can either use only the lower modes or violate this condition, which both leads to a higher uncertainty of the results. For the long beams, there is an additional effect of the load transfer from the carbon fibre loops: If the specimens are too heavy, in particular the first frequency mode is shifted, as the free–free boundary condition is not fulfilled any more. This frequency shift changes the obtained values for E and G, which makes the precise evaluation of the k-factor more difficult.

Summarizing the results it is obvious that the k-factor proposed by Hutchinson is a significant improvement. This might not affect the measurements of materials with low Poisson's ratio or of specimens with square cross-section. It will, however, significantly influence measurements of specimens with high Poisson's ratio and large aspect ratios.

Acknowledgements

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