Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 261 (2003) 365–371
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

Spectral density of an oscillator with power law damping excited by white noise

F. Rüdinger*, S. Krenk

Department of Mechanical Engineering, Technical University of Denmark, Building 101E, DK-2800 Kgs. Lyngby, Denmark

Received 3 April 2002; accepted 12 August 2002

1. Introduction

Much of the theory developed in non-linear random vibration addresses the joint probability density of the state space vector. Another important function describing the dynamics of the system is the power spectral density, which is the Fourier transform of the displacement covariance function and gives the distribution of energy on the frequency components. Solutions to non-linear random vibration problems in terms of the power spectral density have generally received less attention.

A method based on local similarity with the free undamped response was proposed by Krenk and Roberts [1]. The response is here described by a set of modified phase plane variables, and the free undamped response at a given energy level is expanded in a Fourier series. The power spectral density at a given energy level is then obtained as an infinite series, where the first term corresponds to the fundamental frequency, the second term to the first higher harmonic, and so forth. The method was demonstrated considering the Duffing oscillator with linear–quadratic–cubic damping and both white and coloured noise excitation, and has been extended to include systems with parametric excitation by Krenk et al. [2].

In the present case a system with linear stiffness, power law viscous damping and white noise excitation is considered. The approach proposed by Krenk and Roberts [1] can in this case be considered as an extension of the method of equivalent non-linearization [3]. Since the stiffness of the system is linear no higher harmonics appear in the spectrum. The method is compared to results obtained by statistical linearization and Monte Carlo simulation both with respect to the probability density of the mechanical energy and the power spectral density of the response.

*Corresponding author. Tel.: +45-45-25-19-67; fax: +45-45-88-43-25.
E-mail addresses: fr@mek.dtu.dk (F. Rüdinger), sk@mek.dtu.dk (S. Krenk).

doi:10.1016/S0022-460X(02)01183-5

2. Power law viscous damper

An oscillator with linear stiffness and power law viscous damping excited by external white noise is considered. The normalized equation of motion for a system of this type can be expressed as

$$\ddot{X} + h(\dot{X})\dot{X} + \omega_0^2 X = W(t), \quad h(\dot{X})\dot{X} = c\,\mathrm{sign}(\dot{X})\,|\dot{X}|^{\alpha}, \tag{1a, b}$$

where X is the displacement and a dot indicates the derivative with respect to time. ω_0 is the natural angular frequency and $h(\dot{X})$ is termed the damping function. When given in the form (1b), the damping function describes the behaviour of the Jarret Elastomeric Spring Dampers reasonably well. c is the damping coefficient and has the dimension length$^{(1-\alpha)} \cdot$ time$^{(\alpha-2)}$. The damping law exponent α is a number in the range $\alpha \in [0, 1]$. For $\alpha = 1$ the case of linear viscous damping is retrieved. Dry friction corresponds to $\alpha = 0$. $W(t)$ is assumed to be a broadband process, which is approximated by an ideal white noise,

$$\mathrm{E}[W(t)W(t + \Delta t)] = 2\pi S_W \delta(\Delta t), \tag{2}$$

where S_W is the intensity of the white noise. E[ ] is the expectation operator and $\delta(\Delta t)$ is the Dirac delta function.

2.1. Non-dimensional formulation

Initially the problem is rewritten in non-dimensional form. A length scale and a time scale are introduced as

$$X_0 = \sqrt{\frac{2\pi S_W}{\omega_0^3}}, \quad t_0 = \frac{1}{\omega_0}. \tag{3}$$

The equation of motion is then multiplied by t_0^2/X_0, whereby the following non-dimensional equation of motion is obtained:

$$\ddot{Y} + d(\dot{Y})\dot{Y} + Y = U(\tau). \tag{4}$$

The excitation process $U(\tau)$ is a unit white noise, i.e., a white noise with intensity $1/2\pi$. The non-dimensional time τ, displacement Y and velocity $\dot{Y}$ are given by

$$\tau = \frac{t}{t_0}, \quad Y = \frac{X}{X_0}, \quad \dot{Y} = \frac{\mathrm{d}Y}{\mathrm{d}\tau}. \tag{5}$$

A dot used in connection with Y thus indicates the derivative with respect to τ. The non-dimensional damping function $d(\dot{Y})$ is given by

$$d(\dot{Y})\dot{Y} = \beta\,\mathrm{sign}(\dot{Y})\,|\dot{Y}|^{\alpha}, \quad \beta = \frac{c}{\omega_0}\left(\frac{2\pi S_W}{\omega_0}\right)^{(\alpha-1)/2}, \tag{6}$$

where β is the non-dimensional damping coefficient. The equation of motion is seen only to depend on the parameters α and β in this form.

2.2. Equivalent non-linearization

In the method of equivalent non-linearization [3], an equivalent non-linear system is introduced as

$$\ddot{Y} + d_{eq}(\Lambda)\dot{Y} + Y = U(\tau), \quad \Lambda = \tfrac{1}{2}\dot{Y}^2 + \tfrac{1}{2}Y^2, \tag{7a, b}$$

where Λ is the mechanical energy non-dimensionalized by multiplication with t_0^2/X_0^2 from Eq. (3). $d_{eq}(\Lambda)$ is a non-dimensional equivalent damping function, which is assumed to be a function of the mechanical energy only. The equivalent damping function is evaluated by

$$d_{eq}(\lambda) = \frac{\langle\, d(\dot{Y})\dot{Y}^2 \,|\, \lambda \,\rangle}{\langle\, \dot{Y}^2 \,|\, \lambda \,\rangle}, \tag{8}$$

where $\langle \;|\, \lambda \rangle$ is the mean value for a given energy level, which can be evaluated by considering free undamped vibration at energy level λ as discussed by Krenk et al. [2]. Free undamped vibration at energy level λ is given by

$$y = \sqrt{2\lambda}\sin\tau, \quad \dot{y} = \sqrt{2\lambda}\cos\tau. \tag{9}$$

The equivalent damping function is now evaluated from Eq. (8) considering the harmonic motion given by Eq. (9),

$$d_{eq}(\lambda) = \frac{(1/2\pi)\int_0^{2\pi}(\beta\,|\,\dot{y}\,|^{\,\alpha+1}\,|\lambda)\,\mathrm{d}\tau}{(1/2\pi)\int_0^{2\pi}(\dot{y}^2\,|\,\lambda)\,\mathrm{d}\tau} = a\lambda^{(\alpha-1)/2}, \quad a = \frac{\beta 2^{(\alpha+1)/2}}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}\alpha+1)}{\Gamma(\frac{1}{2}\alpha+\frac{3}{2})}. \tag{10a, b}$$

The probability density of the mechanical energy of the equivalent system (7) is given by

$$p_\lambda(\lambda) = A\exp(-2D_{eq}(\lambda)), \quad D_{eq}(\lambda) = \int_0^{\lambda} d_{eq}(e)\,\mathrm{d}e, \tag{11}$$

where A is a normalizing constant and $D_{eq}(\lambda)$ is the damping potential. With the equivalent damping function $d_{eq}(\lambda)$ evaluated in Eq. (10), the probability density of the energy reduces to

$$p_\lambda(\lambda) = A\exp\left(-\frac{4a}{(\alpha+1)}\lambda^{(\alpha+1)/2}\right), \quad A = \frac{\alpha+1}{2\Gamma(\frac{2}{\alpha+1})}\left(\frac{4a}{\alpha+1}\right)^{2/(\alpha+1)}, \tag{12}$$

where a is as given in Eq. (10b). The power spectral density is obtained by a weighted average of spectral densities at various energy levels [1]. According to this method the two-sided spectral density at a given energy level is introduced as

$$S_y(r|\lambda) = \frac{d_{eq}(\lambda)\lambda}{\pi}\frac{1}{[(1-r^2)^2 + d_{eq}(\lambda)^2 r^2]}, \quad r = \frac{\omega}{\omega_0}, \tag{13a, b}$$

where r is a non-dimensional frequency and $r = 1$ corresponds to resonance. The total spectrum is obtained by integrating over all energy levels each weighted by the probability density of that particular energy level,

$$S_y(r) = \int_0^{\infty} S_y(r|\lambda)p_\lambda(\lambda)\,\mathrm{d}\lambda = \int_0^{\infty}\frac{a\lambda^{(\alpha+1)/2}}{\pi}\frac{p_\lambda(\lambda)}{[(1-r^2)^2 + a^2\lambda^{\alpha-1}r^2]}\,\mathrm{d}\lambda. \tag{14}$$

The integral in the expression for the power spectral density is evaluated numerically in the following.

2.3. Statistical linearization

In statistical linearization (also known as equivalent linearization or stochastic linearization) the non-linear equation of motion is replaced by an equivalent linear one, which is easily solved. Based on the non-dimensional formulation (4) the equivalent linear system is expressed as

$$\ddot{Y} + 2\zeta_{eq}\dot{Y} + Y = U(\tau), \tag{15}$$

where ζ_{eq} is the equivalent damping ratio. The probability density of the non-dimensional velocity $\dot{Y}$ and the non-dimensional energy Λ introduced in Eq. (7b) for the linear system (15) are given by

$$p_{\dot{y}}(\dot{y}) = \sqrt{\frac{2\zeta_{eq}}{\pi}}\exp(-2\zeta_{eq}\dot{y}^2), \quad p_{\lambda}(\lambda) = 4\zeta_{eq}\exp(-4\zeta_{eq}\lambda) \tag{16}$$

which are seen to be a zero mean normal distribution and an exponential distribution. ζ_{eq} is obtained by

$$2\zeta_{eq} = \mathrm{E}\left[\frac{\partial}{\partial\dot{Y}}(d(\dot{Y})\dot{Y})\right] = \int_{-\infty}^{\infty} p_{\dot{y}}(\dot{y})\frac{\partial}{\partial\dot{y}}(d(\dot{y})\dot{y})\,\mathrm{d}\dot{y}. \tag{17}$$

Solving this equation for ζ_{eq} the following value is obtained:

$$\zeta_{eq} = \frac{1}{2}\left(\frac{\alpha\beta\Gamma(\frac{1}{2}\alpha)}{\sqrt{\pi}}\right)^{2/(\alpha+1)}. \tag{18}$$

The power spectral density of the non-dimensional linear system (15) is given by

$$2\pi S_y(r) = \frac{1}{[(1-r^2)^2 + (2\zeta_{eq}r)^2]}, \tag{19}$$

where r is the non-dimensional frequency introduced in Eq. (13b). Both the probability density (16) and the power spectral density (19) are seen to depend only on the parameters α and β in the combination ζ_{eq} in Eq. (18). For a discussion of statistical linearization, see e.g., Roberts and Spanos [4].

3. Numerical examples

In order to investigate the results derived in the previous section, the analytical expressions for the probability density of the energy and for the spectral density of the response are compared to results obtained from stochastic records. The simulation of the stochastic records follows the procedure used in Ref. [5] and will not be discussed here. It should however be mentioned, that 50 time steps are taken per period.

In Figs. 1–4 the probability density of the energy and the power spectral density of the response are given for various combinations of the parameter α governing the magnitude of non-linearity, and the damping level of the system as defined by ζ_{eq}. The dashed line corresponds to the

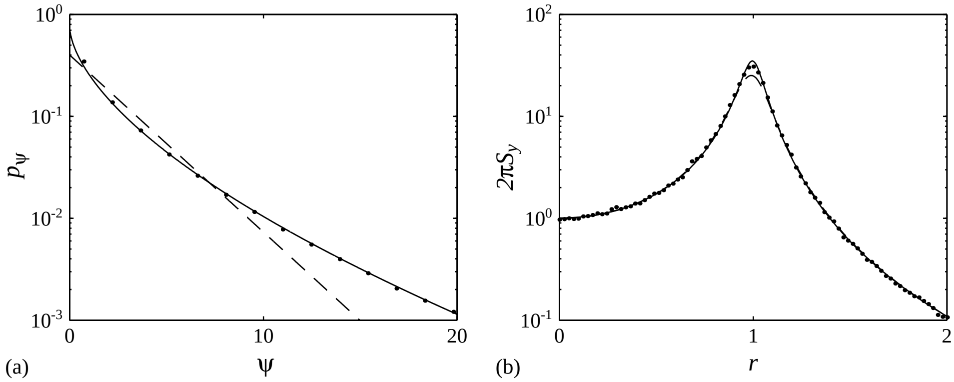

Fig. 1. Probability density and spectral density for $\alpha = 0.2$ and $\zeta_{eq} = 0.1$ ($\beta = 0.355$): –, equivalent non-linearization; - -, statistical linearization; ·, stochastic simulation.

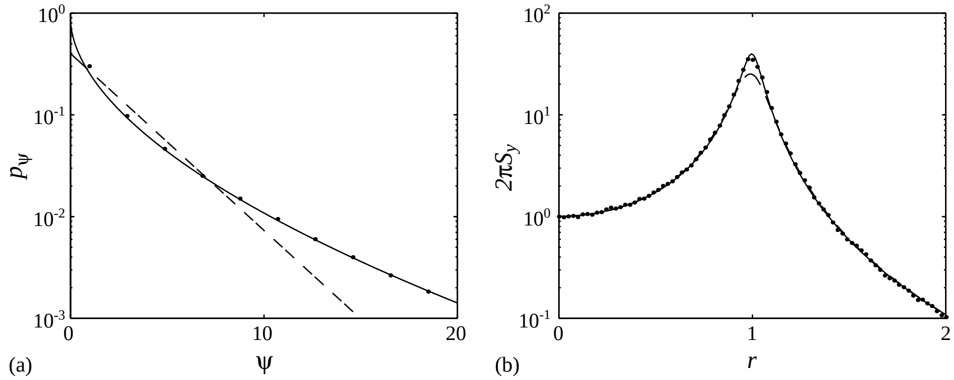

Fig. 2. Probability density and spectral density for $\alpha = 0.1$ and $\zeta_{eq} = 0.1$ ($\beta = 0.376$): –, equivalent non-linearization; - -, statistical linearization; ·, stochastic simulation.

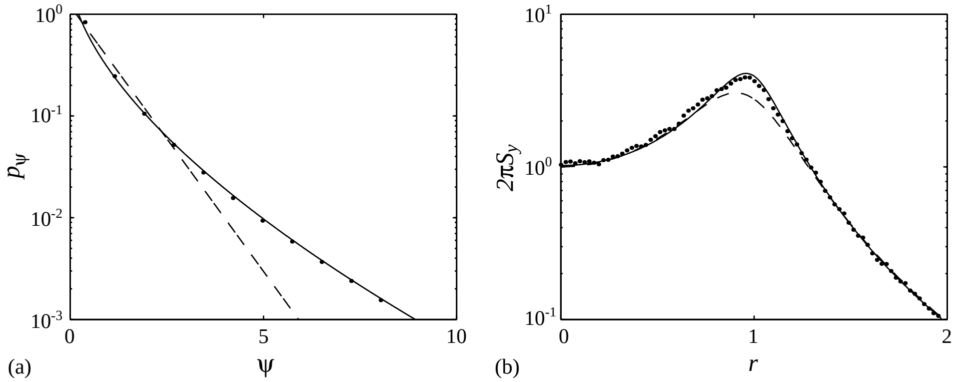

Fig. 3. Probability density and spectral density for $\alpha = 0.2$ and $\zeta_{eq} = 0.3$ ($\beta = 0.686$): –, equivalent non-linearization; - -, statistical linearization; ·, stochastic simulation.

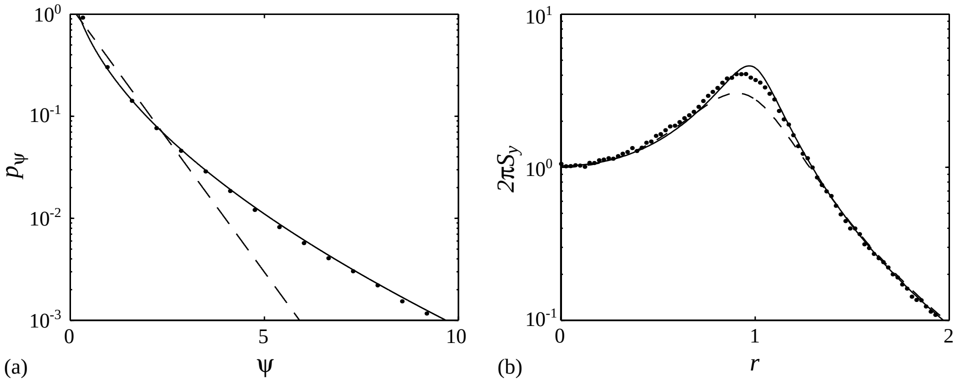

Fig. 4. Probability density and spectral density for $\alpha = 0.1$ and $\zeta_{eq} = 0.3$ ($\beta = 0.687$): –, equivalent non-linearization; - -, statistical linearization; ·, stochastic simulation.

analytical solution obtained by statistical linearization and the solid line corresponds to the method proposed by Krenk and Roberts [1], which can be considered an extension of the method of equivalent non-linearization when oscillators with linear stiffness are considered. The dots correspond to stochastic simulation of records with a length of 50,000 natural periods.

Figs. 1 and 2 show the probability density and the spectral density for $\zeta_{eq} = 0.1$. In the first case $\alpha = 0.2$ and in the second case $\alpha = 0.1$. In both cases the probability density is approximated very accurately by the method of equivalent non-linearization. Statistical linearization on the other hand gives a relatively poor estimate of the probability density, especially the tail. This is an effect of the Gaussian nature of the response of the equivalent linear system leading to an exponential distribution of the energy, i.e., a straight line in the semi-logarithmic plots in Figs. 1–4a. As to the spectral density of the response, both methods seem to give accurate results, except at the resonance peak, where the method of statistical linearization slightly underestimates the peak.

In Figs. 3 and 4 the equivalent damping is increased to $\zeta_{eq} = 0.3$, which can be seen by the significant broadening of the peak, when comparing with the spectral densities in Figs. 1b and 2b. The non-linearity parameter is again chosen as $\alpha = 0.2$ and 0.1, respectively. The probability density is seen to be accurately evaluated by the method of equivalent non-linearization, while the statistical linearization procedure displays the same shortcomings as observed in Figs. 1a and 2a. The accuracy of the statistical linearization with respect to the power spectral density is seen to be decreasing with increasing damping. The peaks in the spectra shown in Figs. 3b and 4b are thus less accurately represented by the statistical linearization than in the previous case in Figs. 1b and 2b. This is partly due to the broadening of the peak and thereby broadening of the range containing the inaccuracy. The method of equivalent non-linearization gives more accurate results. However, in the last case for $\zeta_{eq} = 0.3$ and $\alpha = 0.1$ this method seems to overestimate the peak slightly.

4. Conclusions

Approximate solutions for the probability density of the energy and the power spectral density of the response of an oscillator with power law viscous damping have been considered. As to the probability density of the energy, the method of statistical linearization gives very inaccurate results, since the probability density of the energy of the equivalent linear system is exponential. Especially the tail of the distribution is poorly represented. The method of equivalent non-linearization gives very accurate results.

When the power spectral density of the response is considered, both methods give very accurate results when the system is lightly damped and the non-linearity is moderate. Statistical linearization does however tend to underestimate the peak slightly. For strongly non-linear systems with high levels of damping, the method of statistical linearization fails to give an accurate representation of the resonance peak. The method of equivalent non-linearization still gives reasonable results in this case except for a slight overestimation of the peak.

Acknowledgements

The project is supported by the Danish Technical Research Council and a Marie Curie grant at the Joint Research Centre of the European Commission at Ispra, Italy.

References

[1] S. Krenk, J.B. Roberts, Local similarity in non-linear random vibration, Journal of Applied Mechanics 66 (1999) 225–235.

[2] S. Krenk, Y.K. Lin, F. Rüdinger, Effective system properties and spectral density in random vibration with parametric excitation, Journal of Applied Mechanics 69 (2002) 161–170.

[3] T.K. Caughey, On the response of non-linear oscillators to stochastic excitation, Probabilistic Engineering Mechanics 1 (1986) 2–4.

[4] J.B. Roberts, P.D. Spanos, Random Vibration and Statistical Linearization, Wiley, Chichester, 1990.

[5] F. Rüdinger, S. Krenk, Non-parametric system identification from non-linear stochastic response, Probabilistic Engineering Mechanics 16 (2001) 233–243.