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Letter to the Editor

Transverse vibration of an Euler–Bernoulli uniform beam on up to five resilient supports including ends

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1. Introduction

Uniform beams on several resilient supports (usually approximated with sets of linear translational and rotational springs) occur in engineering applications. Several investigators have derived the frequency equations of special and degenerate cases, a few of which are briefly described here: Chun [1] — one end spring-hinged and the other end free, Goel [2] — both ends spring-hinged, Maurizi et al [3] — one end spring-hinged with a translational spring at the other end. Afolabi [4] — cantilever with resilient root, Lui et al [5] — resilient supports with in-span particles, Maurizi and Belles [6] — cantilever with translational spring-sliding end. Rao [7] derived the frequency equation of beams on resilient supports and tabulated the first five frequencies for several combinations of the four spring stiffness parameters but did not include the frequencies of any special or degenerate cases. Register [8] added rigid bodies at the ends but presented results for symmetrical cases. Kang and Kim [9] considered complex stiffness parameters.

Investigators who have used numerical methods and publications include: Venkateswara Rao and Kanaka Raju [10] — finite element method for spring-hinged at one end, Sundararajan [11] — one term Galerkin solution for a beam spring-hinged at both ends, MacBain and Genin [12] — finite difference, Justine and Krishnan [13] — matrix iteration for resilient support at one end, Karmeswara Rao [14] — Galerkin method, Kim and Dickinson [15] — Rayleigh–Ritz and Abbas and Irretier [16] — cantilever with resilient root by finite element method.

Wang and Lin [17] and Li [18] discussed solutions based on Fourier series and the rates of convergence of the solutions.

Bapat and Bapat [19] used the transfer matrix method to tackle a beam with several translational and rotational springs but the only result listed was the frequencies of a cantilever with one resilient in-span support. The frequency equation of a clamped–clamped beam with one in-span translational support, was derived by Karmeswara Rao [20]. Wu and Chou [21] considered a cantilever carrying oscillators and translational springs.

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The first case considered in the present paper is an uniform Euler–Bernoulli beam on five resilient supports (including ends). Combinations of 'general' and degenerate resilient types of end supports are classified into 81 types (which included the classical clamped, pinned, sliding and free boundary conditions). Degenerate in-span supports are not considered. The frequency equation is expressed as a fourth order determinant equated to zero. A scheme is presented to compute the elements of the determinant of the frequency equation. A 'search' followed by an iterative process based on linear interpolation is used to obtain the roots of the frequency equation. The first three frequency parameters for a selected set of the three in-span support locations, sets of the three in-span spring stiffness parameters and 81 types of end supports are tabulated. Computations were in Fortran77 in double precision, on a VAX under VMS operating system. Numerical problems were not encountered. The method was modified to tackle beams on four, three and two resilient supports and the corresponding first three frequency parameters are tabulated. The method may be used to calculate the frequency parameters of beams on any number of resilient in-span supports.

The tables of frequency parameters may be used to judge the quality of frequencies obtained by numerical methods.

2. Theoretical considerations

2.1. Beam on five resilient supports (including ends)

Fig. 1 shows the uniform Euler–Bernoulli beam $O_{11}O_{21}$ of length L, mass per unit length m, flexural rigidity EI, on resilient supports at the ends O_{11}, O_{21} and at three in-span locations O_{12}, O_{00} and O_{22}. The in-span supports are of the 'general' resilient type shown in Fig. 2;. The 'general' type of resilient support at the ends O_{11} and/or O_{21} is shown in Fig. 2;, the classical clamped (*cl*), pinned (*pn*), sliding (*sl*) or free (*fr*) in Fig. 2a–d and the 'degenerate' resilient supports in Fig. 2e–h. The beam portions are of length $R_{11}L$, $R_{12}L$, $R_{22}L$ and $R_{21}L$, the support location parameters are R_{11}, R_{12}, etc. The dynamics of the portions of the beam between consecutive supports were treated separately. The co-ordinate systems at O_{11} and O_{12} are in the same direction but the systems at O_{22} and O_{21} are in contra-direction to the aforementioned systems.

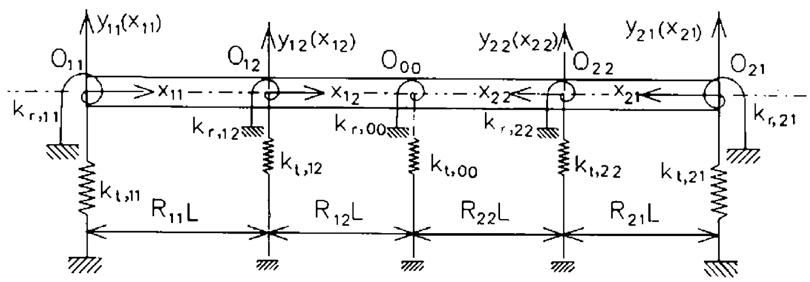

Fig. 1. The uniform beam on five 'general' resilient supports (including ends) and the co-ordinate systems.

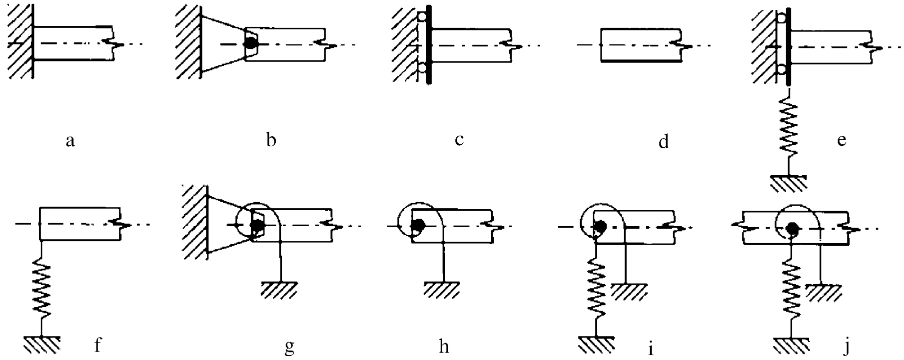

Fig. 2. The classical, degenerate and the 'general' type of resilient end supports.

For free vibration at frequency if $y_j(x_j)$ (j = 11, 12, 22, 21) is the amplitude at abscissa x_j ($0 \leqslant x_j \leqslant R_jL$), then based on the Euler–Bernoulli theory of bending, the bending moment $M_j(x_j)$, shearing force $Q_j(x_j)$ and the mode shape differential equations are

$$M_j(x_j) = EI\,\frac{d^2y_j(x_j)}{dx_j^2},\quad Q_j(x_j) = -EI\,\frac{d^3y_j(x_j)}{dx_j^3},$$
$$EI\,\frac{d^4y_j(x_j)}{dx_j^4} - m\omega^2 y_j(x_j) = 0. \tag{1}$$

To write Eqs. (1) in dimensionless form, introduce the variables X_j ($0 \leqslant X_j \leqslant R_j$) and $Y_j(X_j)$, operators D_j^n and define the dimensionless bending moment $M_j(X_j)$, shearing force $Q_j(X_j)$, frequency Ω, frequency parameter α and the translational and rotational spring stiffness $K_{t,j}$ and $K_{r,j}$ (for use later in the text) as follows:

$$X_{1j} = \frac{x_j}{L},\quad Y_j(X_j) = \frac{y_j(x_j)}{L},\quad D_j = \frac{d}{dX_j},\quad D_j^n = \frac{d^n}{dX_j^n},\quad M_j(X_j) = \frac{M_j(x_j)L}{EI},$$
$$Q_j(X_j) = \frac{Q_j(x_j)L^2}{EI},\quad \boldsymbol{\Omega}^2 = \alpha^4 = \frac{m\omega^2L^4}{EI},\quad K_{t,j} = \frac{k_{t,j}L^3}{EI},\quad K_{r,j} = \frac{k_{r,j}L}{EI}. \tag{2}$$

Eqs. (1) in dimensionless form are

$$M_j(X_j) = D_j^2[Y_j(X_j)],\quad Q_j(X_j) = -D_j^3[Y_j(X_j)],$$
$$D_j^4[Y_j(X_j)] - \boldsymbol{\Omega}^2 Y_j(X_j) = 0. \tag{3}$$

The dimensionless mode shape of the beam portions are

$$Y_j(X_j) = C_{1j}\sin\alpha X_j + C_{2j}\cos\alpha X_j + C_{3j}\sinh\alpha X_j + C_{4j}\cosh\alpha X_j \tag{4}$$

in which $C_{1,j}$ through to $C_{4,j}$ are the constants of integration.

2.1.1. *The mode shape of portion $O_{11}O_{12}$*

Consider the 'general' type of resilient support shown in Fig. 2i. Compatibility of moments and forces acting on the beam element at O_{11} shown in Fig. 3a will lead to the following equations in

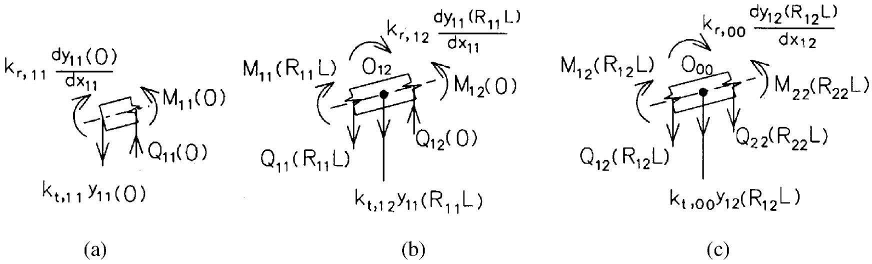

Fig. 3. The forces and moments on beam elements at O_{11}, O_{12} and O_{00}.

dimensionless form:

$$D_{11}^2[Y_{11}(0)] = K_{r,12}D_{11}[Y_{11}(0)], \quad D_{11}^3[Y_{11}(0)] = -K_{t,12}Y_{11}(0). \tag{5}$$

The dimensionless mode shape of the portion $O_{11}O_{12}$ is

$$Y_{11}(X_{11}) = C_{1,11}\sin\alpha X_{11} + C_{2,11}\cos\alpha X_{11} + C_{3,11}\sinh\alpha X_{11} + C_{4,11}\cosh\alpha X_{11}. \tag{6}$$

Eq. (6) must satisfy Eqs. (5) which enables two of the constants of integration in Eq. (6) to be eliminated leading to the mode shape of $O_{11}O_{12}$

$$Y_{11}(X_{11}) = A_1U_{11}(X_{11}) + B_1V_{11}(X_{11}) \tag{7}$$

in which A_1 and B_1 are constants and $U_{11}(X_{11})$ and $V_{11}(X_{11})$, the 'modified' mode shape functions are

$$\begin{aligned} U_{11}(X_{11}) &= \sin\alpha X_{11} + G_1\cos\alpha X_{11} + H_1\cosh\alpha X_{11}, \\ V_{11}(X_{11}) &= \sinh\alpha X_{11} - H_1\cos\alpha X_{11} - G_1\cosh\alpha X_{11} \end{aligned} \tag{8}$$

in which $G_1 = 0.5[\alpha^3/K_{t,11} - K_{r,11}/\alpha]$ and $H_1 = 0.5[\alpha^3/K_{t,11} + K_{r,11}/\alpha]$ or the alternative form

$$\begin{aligned} U_{11}(X_{11}) &= \cos\alpha X_{11} + G_2\sin\alpha X_{11} + H_2\sinh\alpha X_{11}, \\ V_{11}(X_{11}) &= \cosh\alpha X_{11} - H_2\sin\alpha X_{11} - G_2\sinh\alpha X_{11} \end{aligned} \tag{9}$$

in which $G_2 = 0.5[-\alpha/K_{r,11} + K_{t,11}/\alpha^3]$ and $H_2 = 0.5[-\alpha/K_{r,11} - K_{t,11}/\alpha^3]$.

For the classical *cl, pn, sl* and *fr* end supports shown in Fig. 2a–d and the 'degenerate' types of resilient end supports shown in Fig. 2e–i, the functions $U_{11}(X_{11})$ and $V_{11}(X_{11})$ are listed in Appendix A.

2.1.2. The mode shape of portion $O_{12}O_{00}$

The dimensionless mode shape of $O_{12}O_{00}$ is

$$\begin{aligned} Y_{12}(X_{12}) = {} & C_{1,12}\sin\alpha X_{12} + C_{2,12}\cos\alpha X_{12} \\ & + C_{3,12}\sinh\alpha X_{12} + C_{4,12}\cosh\alpha X_{12}. \end{aligned} \tag{10}$$

Continuity of deflection and slope and compatibility of the forces and moments on the beam element at O_{12} shown in Fig. 2b results in

$$
\begin{gathered}
Y_{11}(R_{11}) = Y_{12}(0), \quad D_{11}[Y_{11}(R_{11})] = D_{12}[Y_{12}(0)], \\
D_{1}^{2}[Y_{11}(R_{11})] + K_{r,12} D_{11}[Y_{11}(R_{11})] = D_{12}^{2}[Y_{12}(0)], \\
D_{1}^{3}[Y_{11}(R_{11})] - K_{1,12} Y_{11}(R_{11}) = D_{12}^{3}[Y_{12}(0)].
\end{gathered} \tag{11}
$$

When Eqs. (7) and (10) were substituted into the set of Eqs. (11) and the four constants $C_{1,12}$ through to $C_{4,12}$ were eliminated, the following form of the mode shape of the portion $O_{12}O_{00}$ results:

$$
Y_{12}(X_{12}) = A_1 U_{12}(X_{12}) + B_1 V_{12}(X_{12}), \tag{12}
$$

where A_1 and B_1 are the same constants which appear in Eq. (7) and the functions $U_{12}(X_{12})$ and $V_{12}(X_{12})$ are

$$
\begin{aligned}
U_{12}(X_{12}) &= G_{1,12}\sin\alpha X_{12} + G_{2,12}\cos\alpha X_{12} + G_{3,12}\sinh\alpha X_{12} + G_{4,12}\cosh\alpha X_{12}, \\
V_{12}(X_{12}) &= H_{1,12}\sin\alpha X_{12} + H_{2,12}\cos\alpha X_{12} + H_{3,12}\sinh\alpha X_{12} + H_{4,12}\cosh\alpha X_{12}
\end{aligned} \tag{13}
$$

in which the coefficients $G_{1,12}$ through to $G_{4,12}$ are

$$
\begin{aligned}
G_{1,12} &= \frac{D_{11}[U_{11}(R_{11})]}{2\alpha} - \frac{D_{11}^{3}[U_{11}(R_{11})] - K_{t,12} U_{11}(R_{11})}{2\alpha^3}, \\
G_{2,12} &= \frac{U_{11}(R_{11})}{2} - \frac{D_{11}^{2}[U_{11}(R_{11})] + K_{r,12} D_{11}[U_{11}(R_{11})]}{2\alpha^2}, \\
G_{3,12} &= \frac{D_{11}[U_{11}(R_{11})]}{2\alpha} + \frac{D_{11}^{3}[U_{11}(R_{11})] - K_{t,12} U_{11}(R_{11})}{2\alpha^3}, \\
G_{4,12} &= \frac{U_{11}(R_{11})}{2} + \frac{D_{11}^{2}[U_{11}(R_{11})] + K_{r,12} D_{11}[U_{11}(R_{11})]}{2\alpha^2}.
\end{aligned} \tag{14}
$$

The coefficients $H_{1,12}$—$H_{4,12}$ are obtained by inserting V in place of U in the above equations.

2.1.3. The mode shapes of portions $O_{21}O_{22}$ and $O_{22}O_{00}$

The mode shape of the portion $O_{21}O_{22}$ may be expressed in the form

$$
Y_{21}(X_{21}) = A_2 U_{21}(X_{21}) + B_2 V_{21}(X_{21}) \tag{15}
$$

in which A_2 and B_2 are constants and the functions $U_{21}(X_{21})$ and $V_{21}(X_{21})$ are obtained by replacing the subscript 11 with 21 in Eqs. (8) and (9). Consideration of continuity of deflection and slope and compatibility of bending moment and slope at O_{22} will lead to the mode shape of $O_{22}O_{00}$

$$
Y_{22}(X_{22}) = A_2 U_{22}(X_{22}) + B_2 V_{22}(X_{22}) \tag{16}
$$

in which the functions $U_{22}(X_{22})$ and $V_{22}(X_{22})$ are

$$
\begin{aligned}
U_{22}(X_{22}) &= G_{1,22}\sin\alpha X_{22} + G_{2,22}\cos\alpha X_{22} + G_{3,22}\sinh\alpha X_{22} + G_{4,22}\cosh\alpha X_{22}, \\
V_{22}(X_{22}) &= H_{1,22}\sin\alpha X_{22} + H_{2,22}\cos\alpha X_{22} + H_{3,22}\sinh\alpha X_{22} + H_{4,22}\cosh\alpha X_{2},
\end{aligned} \tag{17}
$$

where the coefficients $G_{1,22}$–$H_{4,22}$ were obtained by replacing the subscripts 11 and 12 with 21 and 22, respectively, on the right side of Eqs. (14).

2.1.4. The frequency equation

The forces and moments acting on the element at O_{00} is shown in Fig. 3c. Continuity of deflection and slope (bearing in mind the contra-directions of X_{12} and X_{22}) and compatibility of moment and shearing forces lead to the following equations in dimensionless form:

$$\begin{gathered} Y_{12}(R_{12}) = Y_{22}(R_{22}), \quad D_{12}[Y_{12}(R_{12})] = -D_{22}[Y_{22}(R_{22})], \\ D_{12}^2[Y_{12}(R_{12})] + K_{r,00}D_{12}[Y_{12}(R_{12})] = D_{22}^2[Y_{22}(R_{22})], \\ D_{12}^3[Y_{12}(R_{12})] - K_{t,00}[Y_{12}(R_{12})] = -D_{22}^3[Y_{22}(R_{22})]. \end{gathered} \tag{18}$$

When Eqs. (12) and (16) are inserted into Eqs. (18), the coefficient matrix of the four equations which results must be singular. This leads to the frequency equation

$$\begin{vmatrix} U_{12}(R_{12}) & V_{12}(R_{12}) & -U_{22}(R_{22}) & -V_{22}(R_{22}) \\ D_{12}[U_{12}(R_{12})] & D_{12}[V_{12}(R_{12})] & D_{22}[U_{22}(R_{22})] & D_{22}[V_{22}(R_{22})] \\ \begin{matrix} D_{12}^2[U_{12}(R_{12})] \\ +K_{r,00}D_{12}[U_{12}(R_{12})] \end{matrix} & \begin{matrix} D_{12}^2[V_{12}(R_{12})] \\ +K_{r,00}D_{12}[V_{12}(R_{12})] \end{matrix} & -D_{22}^2[U_{22}(R_{22})] & -D_{22}^2[V_{22}(R_{22})] \\ \begin{matrix} D_{12}^3[U_{12}(R_{12})] \\ -K_{t,00}U_{12}(R_{12}) \end{matrix} & \begin{matrix} D_{12}^3[V_{12}(R_{12})] \\ -K_{t,00}V_{12}(R_{12}) \end{matrix} & D_{22}^3[U_{22}(R_{22})] & D_{22}^3[V_{22}(R_{22})] \end{vmatrix} = 0. \tag{19}$$

Without loss of generality one may choose

$$R_{11} + R_{12} + R_{22} + R_{21} = 1. \tag{20}$$

2.1.5. The system parameters for five support beam

For sample calculations the following system parameters were chosen: support location parameters: $[R_{11}\ R_{12}\ R_{22}\ R_{21}]=[0.2\ 0.2\ 0.3\ 0.3]$, the end support spring stiffness parameters: $[K_{t,11}\ K_{r,11}]=[10\ 20]$ and $[K_{t,21}\ K_{r,21}]=[100\ 200]$ and the in-span spring stiffness parameters: $[K_{t,12}\ K_{r,12}]=[K_{t,00}\ K_{r,00}]=[K_{t,22}\ K_{r,22}]=[10\ 2]$. In Tables 1–4, the boundary conditions are represented by (B, C) where B and/or C=1, 2, 3,…,9 denote the types of supports shown in Fig. 2a–i, respectively. For the 'general' type of resilient end supports, i.e., (9, 9) all the system parameters will be used. For the other types of supports, the stiffness parameters are selected from this set as required. For example, in the case of (5, 7) type, $K_{r,11}$ and $K_{t,21}$ are not required, (1, 6) type will not require $K_{t,11}$, $K_{r,11}$ and $K_{r,21}$ and so on.

2.1.6. Natural frequency calculations

Account is taken of the type of support at O_{11} and O_{21} in the choice of the functions $U_{11}(X_{11})$, $V_{11}(X_{11})$, $U_{21}(X_{21})$ and $V_{21}(X_{21})$ from Eqs. (8) or (9) for 'general' resilient support or for classical or 'degenerate' resilient support from Appendix A. The functions are all transcendental and their derivatives are obtained by straightforward differentiation. For a trial $\alpha = 0.1$ (say), $U_{11}(R_{11})$, $V_{11}(R_{11})$, $D_{11}[U_{11}(R_{11})]$, $D_{11}[V_{11}(R_{11})]$, etc., were calculated and inserted into Eqs. (14) to get the coefficients $G_{1,12}$ through to $H_{4,12}$, followed by $U_{12}(R_{12})$, $V_{12}(R_{12})$, $D_{12}[U_{12}(R_{12})]$, $D_{12}[V_{12}(R_{12})]$,

Table 1
The first three non-zero frequency parameters α_1, α_2 and α_3 of beam on three in-span 'general' resilient supports and 81 types of end supports

B, C	α_1	α_2	α_3	B, C	α_1	α_2	α_3	B, C	α_1	α_2	α_3
1, 1	5.01200	7.97783	11.13113	2, 1	4.29081	7.22150	10.31448	3, 1	3.05481	5.81451	8.77832
1, 2	4.25497	7.21931	10.40102	2, 2	3.63753	6.44643	9.59628	3, 2	2.68953	5.05489	8.04720
1, 3	2.94690	5.78407	8.79113	2, 3	2.61868	5.03628	8.04225	3, 3	2.33137	3.71783	6.60324
1, 4	2.69299	5.04012	7.97423	2, 4	2.47493	4.37542	7.22180	3, 4	2.31489	3.25030	5.83631
1, 5	4.07118	6.05921	8.87340	2, 5	3.69403	5.43898	8.14985	3, 5	2.90693	4.50848	6.78995
1, 6	3.93106	5.83477	8.19829	2, 6	3.45944	5.35971	7.53438	3, 6	2.65806	4.49182	6.40356
1, 7	4.98908	7.93988	11.08353	2, 7	4.27328	7.18646	10.27005	3, 7	3.04689	5.78723	8.73833
1, 8	2.94400	5.77116	8.76851	2, 8	2.61719	5.02603	8.02182	3, 8	2.33124	3.71198	6.58794
1, 9	4.06837	6.05421	8.85465	2, 9	3.68930	5.43721	8.13415	3, 9	2.90214	4.50812	6.78120
4, 1	2.91220	5.16864	8.01248	5, 1	3.24505	5.84019	8.78592	6, 1	3.19788	5.24898	8.03303
4, 2	2.63078	4.50438	7.27361	5, 2	2.89951	5.09410	8.05735	6, 2	2.89516	4.62010	7.30235
4, 3	2.33129	3.39094	5.90193	5, 3	2.47090	3.81760	6.62110	6, 3	2.44190	3.61467	5.95643
4, 4	2.31305	3.03055	5.22226	5, 4	2.42748	3.37421	5.86144	6, 4	2.38805	3.27344	5.29746
4, 5	2.82256	4.20343	6.15029	5, 5	3.08498	4.54161	6.80504	6, 5	3.07136	4.29012	6.19255
4, 6	2.61068	4.13821	5.91992	5, 6	2.86074	4.52962	6.41790	6, 6	2.85936	4.24784	5.95702
4, 7	2.90663	5.14702	7.97522	5, 7	3.23743	5.81315	8.74602	6, 7	3.19177	5.22795	7.99602
4, 8	2.33115	3.38677	5.88965	5, 8	2.47053	3.81192	6.60586	6, 8	2.44144	3.61059	5.94434
4, 9	2.81865	4.20200	6.14520	5, 9	3.08056	4.54135	6.79627	6, 9	3.06733	4.28918	6.18735
7, 1	4.83844	7.71498	10.77696	8, 1	3.03806	5.71484	8.61036	9, 1	3.23904	5.74776	8.62101
7, 2	4.11987	6.96923	10.06895	8, 2	2.68303	4.97680	7.88990	9, 2	2.89898	5.02507	7.90389
7, 3	2.88623	5.58054	8.51418	8, 3	2.33136	3.67763	6.48330	9, 3	2.46752	3.79057	6.50687
7, 4	2.65598	4.87449	7.71272	8, 4	2.31472	3.22488	5.74012	9, 4	2.42288	3.36121	5.77215
7, 5	3.99738	5.87724	8.60306	8, 5	2.89775	4.46921	6.67605	9, 5	3.08335	4.50846	6.69592
7, 6	3.83798	5.70073	7.95885	8, 6	2.65294	4.44780	6.32063	9, 6	2.86058	4.49325	6.33901
7, 7	4.81740	7.67883	10.73181	8, 7	3.03044	5.68879	8.57158	9, 7	3.23162	5.72196	8.58234
7, 8	2.88366	5.56868	8.49279	8, 8	2.33123	3.67202	6.46875	9, 8	2.46714	3.78512	6.49240
7, 9	3.99417	5.87333	8.58575	8, 9	2.89305	4.46875	6.66807	9, 9	3.07898	4.50812	6.68791

System parameters in Section 2.1.5.

Table 2
Same as Table 1 for beam on two in-span 'general' resilient supports

B, C	α_1	α_2	α_3	B, C	α_1	α_2	α_3	B, C	α_1	α_2	α_3
1, 1	4.91038	7.93581	11.12642	2, 1	4.14786	7.20642	10.28985	3, 1	2.81593	5.68415	8.67423
1, 2	4.15648	7.15979	10.39257	2, 2	3.46394	6.41690	9.57964	3, 2	2.40710	4.92986	7.92741
1, 3	2.88150	5.69540	8.76430	2, 3	2.45637	4.95216	8.03287	3, 3	2.10747	3.58013	6.47333
1, 4	2.64654	4.94840	7.93202	2, 4	2.34288	4.26600	7.20662	3, 4	2.10686	3.10933	5.71223
1, 5	4.01231	5.98822	8.84862	2, 5	3.57184	5.39087	8.14105	3, 5	2.67797	4.43300	6.67222
1, 6	3.85563	5.77919	8.16422	2, 6	3.29931	5.31983	7.52405	3, 6	2.38097	4.40550	6.31068
1, 7	4.88783	7.89725	11.07869	2, 7	4.12984	7.17102	10.24604	3, 7	2.80750	5.65765	8.63378
1, 8	2.87886	5.68256	8.74138	2, 8	2.45521	4.94165	8.01242	3, 8	2.10747	3.57442	6.45840
1, 9	4.00917	5.98354	8.82968	2, 9	3.56640	5.38926	8.12536	3, 9	2.67251	4.43240	6.66405
4, 1	2.49029	4.88585	7.92690	5, 1	3.06260	5.71220	8.68170	6, 1	2.96151	4.97604	7.94574
4, 2	2.21948	4.18432	7.14721	5, 2	2.70091	4.97296	7.93766	6, 2	2.68173	4.32650	7.17392
4, 3	2.09054	3.01144	5.68457	5, 3	2.32582	3.68957	6.49236	6, 3	2.29693	3.33984	5.73982
4, 4	2.08954	2.66723	4.97039	5, 4	2.30646	3.24070	5.73945	6, 4	2.26421	3.03085	5.05147
4, 5	2.44001	3.99147	5.98069	5, 5	2.91693	4.46762	6.68798	6, 5	2.87264	4.09204	6.02069
4, 6	2.21231	3.87043	5.78098	5, 6	2.66855	4.44666	6.32551	6, 6	2.65712	4.01086	5.81532
4, 7	2.48539	4.86403	7.88825	5, 7	3.05487	5.68596	8.64134	6, 7	2.95629	4.95505	7.90732
4, 8	2.09054	3.00771	5.67207	5, 8	2.32566	3.68402	6.47751	6, 8	2.29667	3.33633	5.72755
4, 9	2.43614	3.98886	5.97621	5, 9	2.91217	4.46716	6.67980	6, 9	2.86876	4.09022	6.01610
7, 1	4.72872	7.68523	10.76947	8, 1	2.78490	5.56261	8.50617	9, 1	3.05137	5.59903	8.51649
7, 2	4.00814	6.92073	10.06325	8, 2	2.39108	4.82898	7.76420	9, 2	2.69895	4.88323	7.77817
7, 3	2.80732	5.49271	8.49823	8, 3	2.10638	3.51996	6.33621	9, 3	2.32321	3.64800	6.36138
7, 4	2.59867	4.77772	7.68277	8, 4	2.10592	3.06738	5.59608	9, 4	2.30275	3.21724	5.63114
7, 5	3.92720	5.81145	8.58875	8, 5	2.65768	4.37944	6.54486	9, 5	2.91249	4.42154	6.56553
7, 6	3.74799	5.64810	7.93652	8, 6	2.36711	4.34371	6.21606	9, 6	2.66742	4.39476	6.23500
7, 7	4.70787	7.64845	10.72458	8, 7	2.77690	5.53734	8.46666	9, 7	3.04394	5.57406	8.47710
7, 8	2.80503	5.48082	8.47660	8, 8	2.10638	3.51455	6.32199	9, 8	2.32304	3.64274	6.34724
7, 9	3.92359	5.80781	8.57129	8, 9	2.65237	4.37867	6.53749	9, 9	2.90782	4.42095	6.55812

System parameters in Section 2.2.

Table 3
Same as Table 1 for beam on one in-span 'general' resilient supports

B, C	α_1	α_2	α_3	B, C	α_1	α_2	α_3	B, C	α_1	α_2	α_3
1, 1	4.84006	7.86544	11.05613	2, 1	4.06555	7.08742	10.25539	3, 1	2.53414	5.56610	8.65620
1, 2	4.04193	7.11414	10.30176	2, 2	3.32985	6.31481	9.51454	3, 2	2.03096	4.76367	7.91698
1, 3	2.70705	5.66408	8.67515	2, 3	2.19991	4.89301	7.91981	3, 3	1.74874	3.35973	6.41762
1, 4	2.48439	4.89758	7.86421	2, 4	2.09181	4.20799	7.09346	3, 4	1.74008	2.89233	5.61377
1, 5	3.92457	5.96495	8.76086	2, 5	3.47291	5.33725	8.03483	3, 5	2.40829	4.30673	6.63259
1, 6	3.74062	5.76085	8.09702	2, 6	3.15756	5.27619	7.42633	3, 6	2.00790	4.25958	6.26782
1, 7	4.81649	7.82837	11.00705	2, 7	4.04678	7.05292	10.20998	3, 7	2.52496	5.53823	8.61662
1, 8	2.70458	5.65111	8.65291	2, 8	2.19881	4.88276	7.89942	3, 8	1.74867	3.35423	6.40203
1, 9	3.92090	5.96039	8.74258	2, 9	3.46673	5.33587	8.01937	3, 9	2.40165	4.30571	6.62431
4, 1	2.05555	4.79319	7.86700	5, 1	2.85880	5.59378	8.66404	6, 1	2.74087	4.88261	7.88778
4, 2	1.65846	4.01631	7.11305	5, 2	2.46968	4.80849	7.92734	6, 2	2.44637	4.16375	7.14186
4, 3	1.69461	2.66312	5.65581	5, 3	2.10311	3.48770	6.43590	6, 3	2.05010	3.08682	5.71075
4, 4	1.62340	2.29146	4.90929	5, 4	2.09523	3.04425	5.64016	6, 4	2.02363	2.77196	4.98765
4, 5	2.02110	3.85606	5.96316	5, 5	2.73137	4.34153	6.64762	6, 5	2.67256	3.95417	6.00334
4, 6	1.65677	3.69200	5.76606	5, 6	2.44212	4.30366	6.28158	6, 6	2.42735	3.83994	5.79989
4, 7	2.04957	4.76962	7.82989	5, 7	2.85105	5.56619	8.62454	6, 7	2.73579	4.85987	7.85090
4, 8	1.69420	2.65915	5.64300	5, 8	2.10305	3.48239	6.42039	6, 8	2.04990	3.08334	5.69813
4, 9	2.01588	3.85259	5.95874	5, 9	2.72609	4.34070	6.63930	6, 9	2.66841	3.95163	5.99879
7, 1	4.66091	7.59768	10.71427	8, 1	2.49288	5.45000	8.48128	9, 1	2.84581	5.48584	8.49223
7, 2	3.89543	6.85887	9.98073	8, 2	2.00409	4.66379	7.75220	9, 2	2.46735	4.71998	7.76649
7, 3	2.62515	5.45837	8.39770	8, 3	1.74600	3.28963	6.28748	9, 3	2.09879	3.44047	6.31172
7, 4	2.42714	4.72950	7.59866	8, 4	1.73581	2.83801	5.50437	9, 4	2.08977	3.01398	5.53824
7, 5	3.84006	5.78312	8.49110	8, 5	2.37848	4.25135	6.51190	9, 5	2.72557	4.29354	6.53170
7, 6	3.63198	5.62666	7.85607	8, 6	1.98363	4.19340	6.18027	9, 6	2.44069	4.24791	6.19792
7, 7	4.63912	7.56233	10.66772	8, 7	2.48413	5.42324	8.44289	9, 7	2.83839	5.45939	8.45395
7, 8	2.62300	5.44647	8.37655	8, 8	1.74592	3.28440	6.27262	9, 8	2.09872	3.43543	6.29695
7, 9	3.83591	5.77963	8.47415	8, 9	2.37200	4.25010	6.50442	9, 9	2.72041	4.29254	6.52418

System parameters in Section 2.3.

Table 4
Same as Table 1 for beam without any in-span supports

B, C	α_1	α_2	α_3	B, C	α_1	α_2	α_3	B, C	α_1	α_2	α_3
1, 1	4.73004	7.85320	10.99561	2, 1	3.92660	7.06858	10.21018	3, 1	2.36502	5.49780	8.63938
1, 2	3.92660	7.06858	10.21018	2, 2	3.14159	6.28319	9.42478	3, 2	1.57080	4.71239	7.85398
1, 3	2.36502	5.49780	8.63938	2, 3	1.57080	4.71239	7.85398	3, 3	3.14159	6.28319	9.42478
1, 4	1.87510	4.69409	7.85476	2, 4	3.92660	7.06858	10.21018	3, 4	2.36502	5.49780	8.63938
1, 5	3.84832	5.82222	8.72077	2, 5	3.36984	5.21207	7.96343	3, 5	2.24363	4.23440	6.50197
1, 6	3.64054	5.61600	8.08409	2, 6	2.98864	5.14821	7.38723	3, 6	1.55179	4.18704	6.14569
1, 7	4.70704	7.81562	10.94377	2, 7	3.90744	7.03462	10.16186	3, 7	2.35339	5.47118	8.59818
1, 8	2.36069	5.48412	8.61830	2, 8	1.56689	4.70077	7.83477	3, 8	3.13382	6.26776	9.40182
1, 9	3.84424	5.81751	8.70349	2, 9	3.36261	5.21060	7.94887	3, 9	2.23440	4.23338	6.49379
4, 1	1.87510	4.69409	7.85476	5, 1	2.73320	5.52795	8.64718	6, 1	2.63892	4.79377	7.87565
4, 2	3.92660	7.06858	10.21018	5, 2	2.23892	4.76093	7.86437	6, 2	2.23133	4.09539	7.09735
4, 3	2.36502	5.49780	8.63938	5, 3	1.68490	3.30162	6.30355	6, 3	1.41149	2.93837	5.55952
4, 4	4.73004	7.85320	10.99561	5, 4	1.57707	2.64824	5.52863	6, 4	2.45476	4.82716	7.87411
4, 5	1.83965	3.78182	5.82665	5, 5	2.61448	4.26981	6.51842	6, 5	2.57037	3.88845	5.86973
4, 6	3.60613	5.62875	8.08274	5, 6	2.21906	4.23263	6.16131	6, 6	2.21533	3.77231	5.66560
4, 7	1.86584	4.67128	7.81716	5, 7	2.72408	5.50164	8.60606	6, 7	2.63245	4.77198	7.83826
4, 8	2.35762	5.48444	8.61828	5, 8	1.68433	3.29446	6.28823	6, 8	1.40931	2.93340	5.54646
4, 9	1.83110	3.77809	5.82213	5, 9	2.60776	4.26899	6.51022	6, 9	2.56480	3.88584	5.86508
7, 1	4.54624	7.58494	10.66097	8, 1	2.32500	5.37830	8.46654	9, 1	2.72293	5.41732	8.47745
7, 2	3.76935	6.81958	9.89062	8, 2	1.53376	4.60942	7.69404	9, 2	2.23816	4.67048	7.70831
7, 3	2.26351	5.29156	8.35288	8, 3	3.07045	6.15069	9.23929	9, 3	1.66567	3.25873	6.17757
7, 4	1.79117	4.51272	7.58632	8, 4	2.29558	5.38131	8.46634	9, 4	1.54068	2.62486	5.42095
7, 5	3.76102	5.64569	8.44132	8, 5	2.21395	4.18076	6.38067	9, 5	2.61015	4.22389	6.40219
7, 6	3.52336	5.48553	7.83672	8, 6	1.51696	4.12213	6.05495	9, 6	2.21870	4.17886	6.07480
7, 7	4.52481	7.54945	10.61157	8, 7	2.31374	5.35273	8.42679	9, 7	2.71413	5.39211	8.43779
7, 8	2.25951	5.27877	8.33294	8, 8	3.06291	6.13588	9.21713	9, 8	1.66503	3.25186	6.16288
7, 9	3.75636	5.64204	8.42536	8, 9	2.20486	4.17949	6.37323	9, 9	2.60354	4.22290	6.39472

System parameters: $[K_{t,11} K_{r,11}] = [10\ 20]$, $[K_{t,21} K_{r,21}] = [100\ 200]$.

etc. from Eq. (13). Similarly $U_{22}(R_{22})$, $V_{22}(R_{22})$, $D_{22}[U_{22}(R_{22})]$, $D_{22}[V_{22}(R_{22})]$ were calculated and hence the elements of the determinant of the frequency equation (19). The determinant was expanded by inductive development [22]. The trial α was changed in steps of 0.1 and calculations were repeated till a sign change in the value of the determinant was observed. This gives a 'range' in which a root lies. Calculations were repeated in this 'range' in steps of 0.01 to narrow the 'range'. At this stage an iterative procedure based on linear interpolation was invoked to find a root to a pre-set accuracy. The procedure was repeated from this root to locate the second root and so on. The first three frequency parameters for the selected set of system parameters are tabulated in Table 1.

2.2. Beams on four resilient supports (including ends)

For sample calculations the following system parameters were chosen for the beam on two in-span supports: $[R_{11}\ R_{12}\ R_{21}] = [0.4\ 0.3\ 0.3]$ and $[K_{t,11}\ K_{r,11}]$, $[K_{t,12}\ K_{r,12}]$, $[K_{t,00}\ K_{r,00}]$ and $[K_{t,21}\ K_{r,21}]$ were the same as in Section 2.1.5. The frequency equation was obtained by replacing the subscript 22 with 21 in Eq. (19). The first three frequency parameters are tabulated in Table 2.

2.3. Beam on three resilient supports (including ends)

For sample calculations the following system parameters were chosen for the beam on one in-span support: $[R_{11}\ R_{21}] = [0.7\ 0.3]$ and $[K_{t,11}\ K_{r,11}]$, $[K_{t,00}\ K_{r,00}]$ and $[K_{t,21}\ K_{r,21}]$ were the same as in Section 2.1.5. The frequency equation was obtained by replacing the subscript 12 with 11 and 22 with 21 in Eq. (19). The first three frequency parameters are tabulated in Table 3.

2.4. Beam on resilient end supports

Rao [7] tackled a beam on the 'general' type of resilient end supports but did not present any results for the degenerate types of resilient supports. To fill this shortcoming, the first three frequency parameters of the beam on end supports (spring stiffness parameters $[K_{t,11}\ K_{r,11}]$ and $[K_{t,21}\ K_{r,21}]$ are the same as in Section 2.1.5), are tabulated in Table 4. The boundary conditions (9, 9) were considered in Ref. [7], (7, 4) in Ref. [1], (7, 7) in Ref. [2], (7, 6) in Ref. [3], (9, 4) in Ref. [4] and (5, 4) in Ref. [6].

3. Concluding remarks

Transverse vibrations of uniform Euler–Bernoulli beams on up to five resilient supports (including ends) were considered in this paper. A total of 81 combinations of classical, degenerate and 'general' types of resilient end supports were considered. Degenerate in-span supports were not considered. The frequency equation was expressed as a fourth order determinant equated to zero Schemes to express and calculate the elements of the determinant and then to compute the roots of the frequency equation are presented. Tables of the first three frequencies for one set of the system parameters are presented for three, two, one and nil in-span supports.

The method may be extended to tackle any number of resilient supports provided the in-span supports are not degenerate. The frequencies tabulated may be used to judge the frequencies obtained by numerical methods.

Appendix A

The functions $U_{11}(X_{11})$ and $V_{11}(X_{11})$ are for classical and 'degenerate' resilient supports at O_{11}.For the classical *cl, pn, sl* or *fr* supports at O_{11} shown in:

Fig. 2a: $K_{t,11} \to \infty, K_{r,11} \to \infty$ (*cl*)

$$U_{11}(X_{11}) = \sin \alpha X_{11} - \sinh \alpha X_{11}, V_{11}(X_{11}) = \cos \alpha X_{11} - \cosh \alpha X_{11}. \tag{A.1}$$

Fig. 2b: $K_{t,11} \to \infty, K_{r,11} = 0$ (*pn*)

$$U_{11}(X_{11}) = \sin \alpha X_{11}, V_{11}(X_{11}) = \sinh \alpha X_{11}. \tag{A.2}$$

Fig. 2c: $K_{t,11} = 0, K_{r,11} \to \infty$ (*sl*)

$$U_{11}(X_{11}) = \cos \alpha X_{11}, V_{11}(X_{11}) = \cosh \alpha X_{11}. \tag{A.3}$$

Fig. 2d: $K_{t,11} = 0, K_{r,11} = 0$ (*fr*)

$$U_{11}(X_{11}) = \sin \alpha X_{11} + \sinh \alpha X_{11}, V_{11}(X_{11}) = \cos \alpha X_{11} + \cosh \alpha X_{11}. \tag{A.4}$$

For the 'degenerate' resilient supports shown in:

Fig. 2e: $K_{t,11} \neq 0, K_{r,11} \to \infty$

$$\begin{aligned} U_{11}(X_{11}) &= \cos \alpha X_{11} + \frac{K_{t,11}(\sin \alpha X_{11} - \sinh \alpha X_{11})}{2\alpha^3}, \\ V_{11}(X_{11}) &= \cosh \alpha X_{11} + \frac{K_{t,11}(\sin \alpha X_{11} - \sinh \alpha X_{11})}{2\alpha^3}. \end{aligned} \tag{A.5}$$

Fig. 2f: $K_{t,11} \neq 0, K_{r,11} = 0$

$$\begin{aligned} U_{11}(X_{11}) &= \sin \alpha X_{11} + \frac{\alpha^3(\cos \alpha X_{11} + \cosh \alpha X_{11})}{2K_{t,11}}, \\ V_{11}(X_{11}) &= \sinh \alpha X_{11} - \frac{\alpha^3(\cos \alpha X_{11} + \cosh \alpha X_{11})}{2K_{t,11}}. \end{aligned} \tag{A.6}$$

Fig. 2g: $K_{t,11} \to \infty, K_{r,11} \neq 0$

$$\begin{aligned} U_{11}(X_{11}) &= \mathit{sin}\, \alpha X_{11} - \frac{K_{r,11}(\cos \alpha X_{11} - \cosh \alpha X_{11})}{2\alpha}, \\ V_{11}(X_{11}) &= \sinh \alpha X_{11} - \frac{K_{r,11}(\cos \alpha X_{11} - \cosh \alpha X_{11})}{2\alpha}. \end{aligned} \tag{A.7}$$

Fig. 2h: $K_{t,11} = 0, K_{r,11} \neq 0$

$$\begin{aligned} U_{11}(X_{11}) &= \cos \alpha X_{11} - \frac{\alpha(\sin \alpha X_{11} + \sinh \alpha X_{11})}{2K_{r,11}}, \\ V_{11}(X_{11}) &= \cosh \alpha X_{11} + \frac{\alpha(\sin \alpha X_{11} + \sinh \alpha X_{11})}{2K_{r,11}}. \end{aligned} \tag{A.8}$$

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