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SCIENCE DIRECT
Journal of Sound and Vibration 261 (2003) 697–714
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Elastodynamics of an axisymmetric problem in an anisotropic liquid-saturated porous medium

Rajneesh Kumar[a,*], N.R. Garg[b], Aseem Miglani[c]

[a] *Department of Mathematics, Kurukshetra University, Kurukshetra 136 119, India*
[b] *Department of Mathematics, Maharshi Dayanand University, Rohtak 124 001, India*
[c] *Department of Mathematics, M.M. Engineering College, Mullana 133 203, Ambala, India*

Received 13 September 2001; accepted 28 May 2002

Abstract

The Laplace and Hankel transforms have been employed to find the general solution to the field equations in an anisotropic liquid-saturated porous medium for plain axisymmetric problem, in the transformed form. An application of an infinite space with impulsive force at the origin has been considered to show the utility of the solution obtained. The results of the corresponding problem in isotropic liquid-saturated porous medium can be derived as a special case. To get the solutions in the physical form, a numerical inversion technique has been applied. The results in the form of displacement and stress components have been obtained, numerically and discussed graphically for a particular model.

1. Introduction

In many branches of engineering, e.g., material science, soil mechanics, as well as biomechanics, the response of the material system undergoing external and/or internal loadings are important to study. The field of geomechanics, dealing with the various phenomena occurring in an earthquake, deals with the problems of elastodynamic response of an Earth material due to the presence of certain sources. The liquid-saturated porous materials are often present on and below the surface of the Earth, and are of great importance in the field of geomechanics and engineering.

The theoretical treatment of porous media via continuum theories was based on various methods which fall into two general categories, namely the continuum theory of mixtures and the continuum theory of materials with microstructures. The volume fraction theories—macroscopic

*Corresponding author.
E-mail address: dalalsuman@rediffmail.com (R. Kumar).

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PII: S0022-460X(02)00976-8

theories substituting the microscopic response by means of averaging procedure—are the simplest continuum theories of materials with microstructures. The heterogeneous composition of the porous material can be investigated by using the volume fraction concept. In many papers, either Terzaghi [1–2] or Biot [3–8] has been mentioned as the first investigator to study porous media models by means of the volume fraction concept.

The combination of the mixture theory with the volume fraction concept has been used on micro- as well as macroscopic level. After the development of the mixture theory, it seems that Morland [9] was the first to use the volume fraction concept with elements of the mixture theory to construct "a simple constitutive theory for a fluid-saturated porous solid". Nunziato and Walsh [10] gave a review of known theories and extended the multiphase mixture theory to include chemically reacting materials. The theory of multiphase mixtures was reviewed and extended by Passman et al. [11]. Bowen [12–13] treated incompressible and compressible porous media by use of the theory of mixtures restricted by the volume fraction concept.

Different models have been used for the numerical treatment of initial and boundary value problems ranging from improved classical models proposed by Biot [4–8] to the model based on the mixture theory restricted by the volume fraction concept. Zienkiewicz et al. [14] discussed the static and dynamic behaviour of soils by using an improved Biot model. In recent times, there have again been new attempts to improve the fundamentals of the porous media theory in such a way that the basic equations be mathematically and physically better understood. Li and Li [15] constructed a theory on the thermoelasticity of multicomponent, fluid-saturated, reacting porous media. They pointed out that Bowen's [13] theory had to be expanded through additional constitutive equations. de Boer [16] and de Boer and Kowalski [17] obtained the closure of the theory by assuming α constitutive equations for the interface pressure of a porous medium consisting of α constituents. Recently, de Boer [18] presented a comprehensive review of the porous media theory.

The classical poroelastic model of Biot has been widely used by various authors. His theory—a systematic theory for the acoustic propagation in porous saturated materials—was formulated in such a way that quite general forms of solid and fluid dissipation may be incorporated by using the principle of viscoelastic correspondence. Thus, it represents a potentially powerful tool for studying the behaviour of many kinds of porous media. This theory takes into account the motion of the fluid in the interconnected pores of a porous material, as well as, predicts the effects which are not revealed by the uncoupled theories. Deresiewicz and Skalak [19] derived the conditions sufficient for uniqueness of solution of the field equations of Biot's theory of liquid-filled porous media. Burridge and Vargas [20] gave the fundamental solution in dynamic poroelasticity theory given by Biot. Altay and Dokmeci [21] proved the uniqueness of the solution of Biot poroelasticity equations.

Many researchers have discussed the different types of problems related to liquid-saturated porous medium using different theories, e.g., Armero and Callari [22], Fellah and Depollier [23], Reddy and Tajjuddin [24], etc. Elastodynamic response of the liquid-saturated porous medium due to the presence of certain sources has been discussed by many authors, e.g., Paul [25–26], Pal [27], Philippacopolous [28], Sharma [29], Kumar et al. [30], etc.

There are reasonable grounds for the assumption that anisotropy may exist in the continents. Anisotropy has significant effects on the characteristics of various phenomena occurring in earthquakes, e.g., wave propagation. Therefore, many investigators have studied the problems related to anisotropic liquid-saturated porous medium. Kazi-Aoual et al. [31] discussed the Green function for transversely isotropic poroelastic medium. Sharma and Gogna [32] have studied the wave propagation in anisotropic liquid-saturated porous solids. The wave propagation theory in anisotropic periodically layered fluid-saturated porous media has been discussed by Sun et al. [33]. Propagation of plane waves in transversely isotropic fluid-saturated porous medium was studied by Wang and Zhang [34].

The determination of the state of stress in the materials of the Earth due to the presence of certain sources (axisymmetric source) in the interior of the Earth is of great importance in the fields of geomechanics, geophysics and soil mechanics, etc. Here, in this investigation, we employ the Laplace and the Hankel transforms to study the elastodynamic response of a transversely isotropic liquid-saturated porous medium due to the presence of an instantaneous source through a plain axisymmetric problem. The method of solution used here is an integral transformation method, which is well suited to the initial and boundary conditions of the problem considered here—a time-dependent elastodynamics of a plain axisymmetric problem. It transforms the physical domain problem of space and time variables into a transformed domain problem involving only one space variable, and hence simpler to solve by using the eigenvalue approach.

2. Basic equations

In the absence of body forces, the equations of motion for the liquid-saturated porous medium without dissipation are given by Biot [6–7] as

$$\sigma_{ij,j} = \frac{\partial^2}{\partial t^2}(\rho_{11} u_i + \rho_{12} U_i), \tag{1}$$

$$\sigma_{,i} = \frac{\partial^2}{\partial t^2}(\rho_{12} u_i + \rho_{22} U_i) \quad (i = x, y, z), \tag{2}$$

where σ_{ij} are the stress components in the solid, $\sigma = -\beta f_p$ is the stress in the fluid (f_p is the pressure in the fluid and β is the porosity); u_i, U_i $(i = x, y, z)$ are the components of the displacement vectors in the solid and liquid parts, respectively, of the porous medium; ρ_{11}, ρ_{12} and ρ_{22} are the dynamical coefficients and are related to the mass densities of the solid ρ_s and fluid ρ_f as

$$\rho_{11} + \rho_{12} = (1-\beta)\rho_s, \quad \rho_{12} + \rho_{22} = \beta\rho_f, \tag{3}$$

so that the mass density of the bulk material is

$$\rho = \rho_{11} + 2\rho_{12} + \rho_{22} = \rho_s + \beta(\rho_f - \rho_s). \tag{4}$$

For the transversely isotropic liquid-saturated porous solid with symmetry about the z-axis, the stress strain relations are given by Biot [4] as

$$\begin{aligned}
\sigma_{xx} &= 2Ne_{xx} + A(e_{xx} + e_{yy}) + Fe_{zz} + M\varepsilon, \\
\sigma_{yy} &= 2Ne_{yy} + A(e_{xx} + e_{yy}) + Fe_{zz} + M\varepsilon, \\
\sigma_{zz} &= Ce_{zz} + F(e_{xx} + e_{yy}) + Q\varepsilon,
\end{aligned}$$

$$\sigma_{yz} = Le_{yz}, \quad \sigma_{xz} = Le_{xz}, \quad \sigma_{xy} = Ne_{xy},$$
$$\sigma = M(e_{xx} + e_{yy}) + Qe_{zz} + R\varepsilon, \tag{5}$$

where

$$e_{ij} = \begin{cases} \dfrac{\partial u_i}{\partial x_j}, & i = j, \\ \dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}, & i \neq j, \end{cases}$$

$$\varepsilon = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}. \tag{6}$$

A, N, F, M, C, Q, L and R are the elastic constants for transversely isotropic liquid-saturated porous solid. These elastic constants can be reduced to that of isotropic liquid-saturated porous solid through relations

$$F = A, \quad M = Q, \quad L = N, \quad C = A + 2N. \tag{7}$$

3. Formulation of the problem

The medium considered is transversely isotropic liquid-saturated porous with symmetry about z-axis, taken vertically downwards. Taking the cylindrical polar co-ordinates (r, ϑ, z), the problem considered is plain axisymmetric, i.e., the field component u_ϑ is zero and others (u_r, u_z) are independent of ϑ. An application of a concentrated force acting at some interior point of the infinite medium, taken as origin, along vertical direction is considered, i.e., the force is applied at the origin of the co-ordinate system along z-axis.

4. Solution of the problem

Since the problem considered is plain axisymmetric, we take

$$\vec{u} = (u, 0, w), \quad \vec{U} = (U, 0, W),$$

where (u, w) and (U, W) represent, respectively, the radial and vertical component of the displacement vectors in solid and liquid parts. Thus, the field Eqs. (1) and (2) in the component form can be written as

$$(A + 2N)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}u\right) + L\frac{\partial^2 u}{\partial z^2} + (F + L)\frac{\partial^2 w}{\partial r \partial z}$$
$$+ M\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \frac{1}{r^2}U + \frac{\partial^2 W}{\partial r \partial z}\right) = \frac{\partial^2}{\partial t^2}(\rho_{11}u + \rho_{12}U),$$

$$L\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+C\frac{\partial^2 w}{\partial z^2}+(F+L)\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}\right)$$
$$+Q\left(\frac{\partial^2 U}{\partial r\partial z}+\frac{1}{r}\frac{\partial U}{\partial z}+\frac{\partial^2 W}{\partial z^2}\right)=\frac{\partial^2}{\partial t^2}(\rho_{11}w+\rho_{12}W),$$
$$M\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{1}{r^2}u\right)+Q\frac{\partial^2 w}{\partial r\partial z}$$
$$+R\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}-\frac{1}{r^2}U+\frac{\partial^2 W}{\partial r\partial z}\right)=\frac{\partial^2}{\partial t^2}(\rho_{12}u+\rho_{22}U),$$
$$M\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}\right)+Q\frac{\partial^2 w}{\partial z^2}+R\left(\frac{\partial^2 U}{\partial r\partial z}+\frac{1}{r}\frac{\partial U}{\partial z}+\frac{\partial^2 W}{\partial z^2}\right)=\frac{\partial^2}{\partial t^2}(\rho_{12}w+\rho_{22}W), \tag{8}$$

To represent the field Eq. (8) in the non-dimensional variable form, we define the following non-dimensional quantities:

$$u'=\frac{u}{h},\quad w'=\frac{w}{h},\quad U'=\frac{U}{h},\quad W'=\frac{W}{h},$$
$$r'=\frac{r}{h},\quad z'=\frac{z}{h},\quad t'=\omega t,$$
$$\sigma'_{zz}=\frac{\sigma_{zz}}{C},\quad \sigma'_{rz}=\frac{\sigma_{rz}}{C},\quad \sigma'=\frac{\sigma}{C}, \tag{9}$$

where

$$\omega=\frac{1}{h}\sqrt{\frac{C}{\rho}},$$

and 'h' has the dimension of length, along with

$$a^2=\frac{Q}{C},\quad b^2=\frac{R}{C},\quad c^2=\frac{A+2N}{C},$$
$$d^2=\frac{L}{C},\quad e^2=\frac{F+L}{C},\quad f^2=\frac{M}{C}, \tag{10}$$
$$R_{11}=\frac{\rho_{11}}{\rho},\quad R_{12}=\frac{\rho_{12}}{\rho},\quad R_{22}=\frac{\rho_{22}}{\rho}.$$

Using Eqs. (9) and (10), the field Eq. (8) in the non-dimensional form, after suppressing the dashes, can be written as

$$c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{1}{r^2}u\right)+d^2\frac{\partial^2 u}{\partial z^2}+e^2\frac{\partial^2 w}{\partial r\partial z}$$
$$+f^2\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}-\frac{1}{r^2}U+\frac{\partial^2 W}{\partial r\partial z}\right)=\left(R_{11}\frac{\partial^2 u}{\partial t^2}+R_{12}\frac{\partial^2 U}{\partial t^2}\right),$$

$$d^2\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+\frac{\partial^2 w}{\partial z^2}+e^2\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}\right)$$
$$+a^2\left(\frac{\partial^2 U}{\partial r\partial z}+\frac{1}{r}\frac{\partial U}{\partial z}+\frac{\partial^2 W}{\partial z^2}\right)=\left(R_{11}\frac{\partial^2 w}{\partial t^2}+R_{12}\frac{\partial^2 W}{\partial t^2}\right),$$
$$f^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{1}{r^2}u\right)+a^2\frac{\partial^2 w}{\partial r\partial z}$$
$$+b^2\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}-\frac{1}{r^2}U+\frac{\partial^2 W}{\partial r\partial z}\right)=\left(R_{12}\frac{\partial^2 u}{\partial t^2}+R_{22}\frac{\partial^2 U}{\partial t^2}\right),$$
$$f^2\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}\right)+a^2\frac{\partial^2 w}{\partial z^2}+b^2\left(\frac{\partial^2 U}{\partial r\partial z}+\frac{1}{r}\frac{\partial U}{\partial z}+\frac{\partial^2 W}{\partial z^2}\right)=\left(R_{12}\frac{\partial^2 w}{\partial t^2}+R_{22}\frac{\partial^2 W}{\partial t^2}\right). \tag{11}$$

Applying the Laplace transformation, Sneddon [35], with respect to 't' defined as

$$\{\bar{u}(r,z,p),\bar{w}(r,z,p),\bar{U}(r,z,p),\bar{W}(r,z,p)\}$$
$$=\int_0^\infty \{u(r,z,t),w(r,z,t),U(r,z,t),W(r,z,t)\}\,\mathrm{e}^{-pt}\,\mathrm{d}t, \tag{12}$$

and then the Hankel transformation, Sneddon [35], with respect to 'r' defined as

$$\{\hat{u}(q,z,p),\hat{U}(q,z,p)\}=\int_0^\infty r\{\bar{u}(r,z,p),\bar{U}(r,z,p)\}J_1(qr)\,\mathrm{d}r,$$

$$\{\hat{w}(q,z,p),\hat{W}(q,z,p)\}=\int_0^\infty r\{\bar{w}(r,z,p),\bar{W}(r,z,p)\}J_0(qr)\,\mathrm{d}r, \tag{13}$$

on Eq. (11), we obtain a system of four ordinary differential equations in four unknowns $\hat{u},\hat{w},\hat{U},\hat{W}$, which can be written as

$$A_1\ddot{V}+B_1\dot{V}+C_1=0, \tag{14}$$

where dot represents the differentiation with respect to z,

$$V=[\hat{u},\hat{w},\hat{U},\hat{W}]^{\mathrm{T}},$$

$$A_1=\begin{bmatrix} d^2 & 0 & 0 & 0\\ 0 & 1 & 0 & a^2\\ 0 & 0 & 0 & 0\\ 0 & a^2 & 0 & b^2\end{bmatrix},\quad B_1=\begin{bmatrix} 0 & -qe^2 & 0 & -qf^2\\ qe^2 & 0 & qa^2 & 0\\ 0 & -qa^2 & 0 & -qb^2\\ qf^2 & 0 & qb^2 & 0\end{bmatrix},$$
$$C_1=-\begin{bmatrix} -(q^2c^2+H_{11}) & 0 & -(q^2f^2+H_{12}) & 0\\ 0 & -(q^2d^2+H_{11}) & 0 & -H_{12}\\ -(q^2f^2+H_{12}) & 0 & -(q^2b^2+H_{22}) & 0\\ 0 & -H_{12} & 0 & -H_{22}\end{bmatrix}, \tag{15}$$

and

$$H_{11} = R_{11}p^2, \quad H_{12} = R_{12}p^2, \quad H_{22} = R_{22}p^2. \tag{16}$$

To solve the system of Eq. (14), we assume

$$V(q, z, p) = X(q, p)\mathrm{e}^{mz}, \tag{17}$$

which lead to the characteristic equation

$$\det(m^2 A_1 + mB_1 + C_1) = 0, \tag{18}$$

i.e.,

$$T_0 m^6 + T_1 m^4 + T_2 m^2 + T_3 = 0, \tag{19}$$

where

$$\begin{aligned}
&T_0 = -d^2 H_{22} X, \quad T_1 = T_{11} + T_{12} q^2,\\
&T_2 = T_{21} + T_{22} q^2 + T_{23} q^4, \quad T_3 = T_{31} + T_{32} q^2 + T_{33} q^4 + T_{34} q^6,\\
&T_{11} = ZX + d^2 H_{22} Y, \quad T_{12} = (XX_1' + b^2 X_1 + f^2 X_2) H_{22},\\
&T_{21} = -(Y + d^2 H_{22}) Z,\\
&T_{22} = -\big[(X + X')H_{11}H_{22} + 2(a^2e^2 - f^2 - a^2c^2 + e^2f^2)H_{12}H_{22}\\
&\qquad +2b^2d^2Z + (X_1 + c^2)H_{22}^2 - 2(b^2e^2 - a^2f^2)H_{12}^2\big],\\
&T_{23} = -[(b^2X_1 + c^2X) + d^2X' + f^2X_2]H_{22},\\
&T_{31} = Z^2, \quad T_{32} = (Y' + d^2H_{22})Z,\\
&T_{33} = X'Z + d^2H_{22}Y', \quad T_{34} = d^2H_{22}X',
\end{aligned}$$

and

$$\begin{aligned}
&X = b^2 - a^4, \quad X' = b^2c^2 - f^4,\\
&X_1 = d^4 - e^4, \quad X_1' = c^2 + d^2,\\
&X_2 = 2a^2e^2 - f^2, \quad Z = H_{11}H_{22} - H_{12}^2,\\
&Y = b^2H_{11} + H_{22} - 2H_{12}a^2, \quad Y' = b^2H_{11} + c^2H_{22} - 2H_{12}f^2.
\end{aligned} \tag{20}$$

Using Cardan's method, Eq. (19) is reduced to

$$z^3 + 3Hz + G = 0, \tag{21}$$

where

$$z = m^2 + \frac{T_1}{3T_0},$$
$$H = \frac{1}{3}\left(T_2' - \frac{1}{3}T_1'2\right), \quad G = T_3' - \frac{1}{3}T_1'T_2' + \frac{2}{27}T_1'3,$$
$$T_1' = \frac{T_1}{T_0}, \quad T_2' = \frac{T_2}{T_0}, \quad T_3' = \frac{T_3}{T_0}. \tag{22}$$

The roots of Eq. (21) are given by

$$z = r + s, \tag{23}$$

where r and s are given by

$$r^3 = \frac{-G + \sqrt{G^2 + 4H^3}}{2}, \quad s^3 = \frac{-G - \sqrt{G^2 + 4H^3}}{2},$$

satisfying

$$rs = -H.$$

Hence, roots of Eq. (19) are given by

$$m_n^2 = z_n - \frac{T_1}{3T_0}, \quad n = 1, 2, 3. \tag{24}$$

The eigenvectors $X(q,p)$ corresponding to different eigenvalues $\pm m_1$, $\pm m_2$ and $\pm m_3$ are obtained as

$$X_i^{\mathrm{T}} = [-k_i', l_i', -m_i', n_i'], \quad \text{for } m = m_i,$$
$$X_{i+3}^{\mathrm{T}} = [-k_i', l_i', -m_i', n_i'], \quad \text{for } m = -m_i, \ \ i = 1, 2, 3, \tag{25}$$

where

$$n_i' = n_{i1}n_{i2} - n_{i3}n_{i4}, \quad m_i' = n_{i1}m_{i1} - n_{i3}m_{i2},$$
$$l_i' = \frac{1}{n_{i1}}[n_{i4}m_i' - m_{i2}n_i'], \quad k_i' = \frac{1}{k_{i4}}[k_{i1}l_i' - k_{i2}m_i' + k_{i3}n_i'],$$
$$k_{i1} = a^2m_i^2 - H_{12}, \quad k_{i2} = qm_ib^2,$$
$$k_{i3} = b^2m_i^2 - H_{22}, \quad k_{i4} = qm_if^2,$$
$$m_{i1} = m_i\{f^2q(f^2q^2 + H_{12}) + b^2q(d^2m_i^2 - c^2q^2 - H_{11})\},$$
$$m_{i2} = f^2(a^2m_i^2 - H_{12}) - e^2(b^2m_i^2 - H_{22}),$$
$$n_{i1} = f^2(m_i^2 - d^2q^2 - H_{11}) - e^2(a^2m_i^2 - H_{12}),$$
$$n_{i2} = (f^2q^2 + H_{12})^2 + (b^2q^2 + H_{22})(d^2m_i^2 - c^2q^2 - H_{11}),$$
$$n_{i3} = m_i\{e^2q(f^2q^2 + H_{12}) + a^2q(d^2m_i^2 - c^2q^2 - H_{11})\},$$
$$n_{i4} = a^2f^2qm_i - b^2e^2qm_i, \quad i = 1, 2, 3. \tag{26}$$

Thus, the general solution of the plain axisymmetric problem in transversely isotropic liquid-saturated porous medium, in the transformed form, can be written as

$$V(q,z,p) = \sum_{i=1}^{3}\{B_i X_i(q,p)\mathrm{e}^{m_i z} + B_{i+3}X_{i+3}(q,p)\mathrm{e}^{-m_i z}\}, \tag{27}$$

where B_i $(i = 1,2,3,4,5,6)$ are arbitrary constants to be determined from the boundary conditions.

Substituting Eq. (7) into Eq. (18), we get the characteristic equation of the corresponding problem in isotropic liquid-saturated porous medium. Hence, use of expressions (7) in other relevant expressions provide us the general solution of the plain axisymmetric problem of isotropic liquid-saturated porous medium.

5. Application

Let us consider an instantaneous force of magnitude $F = -F_0\delta(r)\delta(t)/2\pi r$ is acting at the origin of the cylindrical polar co-ordinate system along z-axis in an infinite transversely isotropic liquid-saturated porous medium. The appropriate boundary conditions in the present case at $z = 0$ are given as

$$\begin{aligned}
&u(r,0^+,t) - u(r,0^-,t) = 0, \quad w(r,0^+,t) - w(r,0^-,t) = 0,\\
&W(r,0^+,t) - W(r,0^-,t) = 0, \quad \sigma_{zz}(r,0^+,t) - \sigma_{zz}(r,0^-,t) = -F_0\frac{\delta(r)\delta(t)}{2\pi r},\\
&\sigma_{rz}(r,0^+,t) - \sigma_{rz}(r,0^-,t) = 0, \quad \sigma(r,0^+,t) - \sigma(r,0^-,t) = F_0\frac{\delta(r)\delta(t)}{2\pi r}.
\end{aligned} \tag{28}$$

Using Eq. (9) and applying the Laplace and then the Hankel transformation defined by Eqs. (12) and (13), respectively, on these boundary conditions (28). We obtain the non-dimensional transformed form of boundary conditions (28), after suppressing the dashes as

$$\begin{aligned}
&\hat{u}(q,0^+,p) - \hat{u}(q,0^-,p) = 0, \quad \hat{w}(q,0^+,p) - \hat{w}(q,0^-,p) = 0,\\
&\hat{W}(q,0^+,p) - \hat{W}(q,0^-,p) = 0, \quad \hat{\sigma}_{zz}(q,0^+,p) - \hat{\sigma}_{zz}(q,0^-,p) = -\frac{F_0}{2\pi},\\
&\hat{\sigma}_{rz}(q,0^+,p) - \hat{\sigma}_{rz}(q,0^-,p) = 0, \quad \hat{\sigma}(q,0^+,p) - \hat{\sigma}(q,0^-,p) = \frac{F_0}{2\pi},
\end{aligned} \tag{29}$$

Now, from Eqs. (5) and (6), after reducing them to non-dimensional transformed form by using Eqs. (9), (10), (12) and (13), along with Eqs. (27), the displacement and stress components in the transformed form (after suppressing the dashes) can be written as for $z \geqslant 0$

$$\begin{aligned}
\hat{u} &= -(B_4 k_1' e^{-m_1 z} + B_5 k_2' e^{-m_2 z} + B_6 k_3' e^{-m_3 z}),\\
\hat{w} &= (B_4 l_1' e^{-m_1 z} + B_5 l_2' e^{-m_2 z} + B_6 l_3' e^{-m_3 z}),\\
\hat{U} &= -(B_4 m_1' e^{-m_1 z} + B_5 m_2' e^{-m_2 z} + B_6 m_3' e^{-m_3 z}),\\
\hat{W} &= (B_4 n_1' e^{-m_1 z} + B_5 n_2' e^{-m_2 z} + B_6 n_3' e^{-m_3 z}),\\
\hat{\sigma}_{zz} &= B_4 Q_1 e^{-m_1 z} + B_5 Q_2 e^{-m_2 z} + B_6 Q_3 e^{-m_3 z},\\
\hat{\sigma}_{xz} &= B_4 G_1 e^{-m_1 z} + B_5 G_2 e^{-m_2 z} + B_6 G_3 e^{-m_3 z},\\
\hat{\sigma} &= B_4 J_1 e^{-m_1 z} + B_5 J_2 e^{-m_2 z} + B_6 J_3 e^{-m_3 z},
\end{aligned} \tag{30}$$

for $z \leqslant 0$

$$\begin{aligned}
\hat{u} &= -(B_1 k_1' e^{m_1 z} + B_2 k_2' e^{m_2 z} + B_3 k_3' e^{m_3 z}),\\
\hat{w} &= (B_1 l_1' e^{m_1 z} + B_2 l_2' e^{m_2 z} + B_3 l_3' e^{m_3 z}),\\
\hat{U} &= -(B_1 m_1' e^{m_1 z} + B_2 m_2' e^{m_2 z} + B_3 m_3' e^{m_3 z}),\\
\hat{W} &= (B_1 n_1' e^{m_1 z} + B_2 n_2' e^{m_2 z} + B_3 n_3' e^{m_3 z}),\\
\hat{\sigma}_{zz} &= B_1 P_1 e^{m_1 z} + B_2 P_2 e^{m_2 z} + B_3 P_3 e^{m_3 z},\\
\hat{\sigma}_{xz} &= B_1 R_1 e^{m_1 z} + B_2 R_2 e^{m_2 z} + B_3 R_3 e^{m_3 z},\\
\hat{\sigma} &= B_1 H_1 e^{m_1 z} + B_2 H_2 e^{m_2 z} + B_3 H_3 e^{m_3 z},
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
P_i &= l_i' m_i - q(e^2 - d^2) k_i' + a^2 n_i' m_i - q a^2 m_i',\\
Q_i &= -l_i' m_i - q(e^2 - d^2) k_i' - a^2 n_i' m_i - q a^2 m_i',\\
R_i &= -(k_i' m_i + q l_i') d^2, \quad G_i = (k_i' m_i - q l_i') d^2,\\
H_i &= a^2 l_i' m_i - q f^2 k_i' + b^2 n_i' m_i - q b^2 m_i',\\
J_i &= -a^2 l_i' m_i - q f^2 k_i' - b^2 n_i' m_i - q b^2 m_i'.
\end{aligned} \tag{32}$$

Making use of the transformed displacements and stresses given by Eqs. (30) and (31) in the transformed boundary conditions (29), we obtain a system of six equations in six unknowns B_1, B_2, B_3, B_4, B_5 and B_6, which on solving gives

$$\begin{aligned}
B_4 = B_1 &= -s_1 \frac{(t_3 + r_3)s_2 - (t_2 + r_2)s_3}{\Delta} \frac{F_0}{2\pi},\\
B_5 = B_2 &= s_1 \frac{(t_3 + r_3)s_1 - (t_1 + r_1)s_3}{\Delta} \frac{F_0}{2\pi},\\
B_6 = B_3 &= -s_1 \frac{(t_2 + r_2)s_1 - (t_1 + r_1)s_2}{\Delta} \frac{F_0}{2\pi},
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
\Delta &= (t_2 s_1 - t_1 s_2)(r_3 s_1 - r_1 s_3) - (r_2 s_1 - r_1 s_2)(t_3 s_1 - t_1 s_3),\\
r_i &= Q_i - P_i, \quad s_i = G_i - R_i, \quad t_i = J_i - H_i.
\end{aligned} \tag{34}$$

Thus, expressions (30) and (31) with the help of Eqs. (33)–(34) represent the displacement and stress components in the transformed form for an infinite transversely isotropic fluid-saturated porous medium due to an instantaneous force acting at the origin along z-axis. These displacement and stress components in the transformed form on inversion enable us to give the displacement and stress components in the physical form. To invert the Laplace and Hankel transforms in the transformed domain expressions, we make use of a numerical inversion technique to get the results in the physical form, numerically.

6. Inversion of the transforms

The transformed solutions are functions of depth variable 'z', the parameters of Laplace and Hankel transforms p and q, respectively, and hence are of the form $\hat{f}(q,z,p)$. To get the function $f(r,z,t)$ in the physical domain, first we invert the Hankel transform by using

$$\bar{f}(r,z,p) = \int_0^\infty q\hat{f}(q,z,p) J_n(qr)\,\mathrm{d}q. \tag{35}$$

Thus, expression (35) gives us the Laplace transform $\bar{f}(r,z,p)$ of the function $f(r,z,t)$. Now, the function $\bar{f}(r,z,p)$ in expression (35) can be considered as the Laplace transform $\bar{g}(p)$ of some function $g(t)$. Following Honig and Hirdes [36], the Laplace transformed function $\bar{g}(p)$ can be inverted as given below.

The function $g(t)$ can be obtained by using

$$g(t) = \frac{1}{2\pi i}\int_{C-\mathrm{i}\infty}^{C+\mathrm{i}\infty} \mathrm{e}^{pt}\bar{g}(p)\,\mathrm{d}p, \tag{36}$$

where C is an arbitrary real number greater than all the real parts of the singularities of $\bar{g}(p)$. Taking $p = C + \mathrm{i}y$, we get

$$g(t) = \frac{e^{Ct}}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ty}\bar{g}(C+\mathrm{i}y)\,\mathrm{d}y, \tag{37}$$

Now, taking $\mathrm{e}^{-Ct}g(t)$ as $h(t)$ and expanding it as Fourier series in $[0, 2L]$, we obtain approximately the formula

$$g(t) = g_\infty(t) + E_D, \tag{38}$$

where

$$g_\infty(t) = \frac{C_0}{2} + \sum_{k=1}^{\infty} C_k, \quad 0 \leqslant t \leqslant 2L, \tag{39}$$

$$C_k = \frac{\mathrm{e}^{Ct}}{L}\mathrm{Re}\left[\mathrm{e}^{\mathrm{i}k\pi t/L}\bar{g}\left(C + \frac{\mathrm{i}k\pi}{L}\right)\right],$$

E_D is the discretization error and can be made arbitrarily small by choosing C large enough.

Since the infinite series in Eq. (39) can be summed up only to a finite number of N terms, so the approximate value of $g(t)$ becomes

$$g_N(t) = \frac{C_0}{2} + \sum_{k=1}^{N} C_k, \quad 0 \leqslant t \leqslant 2L. \tag{40}$$

Now we introduce a truncation error E_T as

$$E_T = \sum_{k=N+1}^{\infty} C_k,$$

that must be added to the discretization error to produce the total approximate error in evaluating $g(t)$ using the above formula. The discretization error is reduced by using the 'Korrektur method' and then the 'ε-algorithm' is used to reduce the truncation error and hence to accelerate the convergence.

The Korrektur method formula, to evaluate the function $g(t)$ is

$$g(t) = g_\infty(t) - \mathrm{e}^{-2CL} g_\infty(2L + t) + E'_D,$$

where

$$|E'_D| \ll |E_D|.$$

Thus, the approximate value of $g(t)$ becomes

$$g_{N_k}(t) = g_N(t) - \mathrm{e}^{-2CL} g_{N'}(2L + t), \tag{41}$$

where N' is an integer such that $N' < N$.

We shall now describe the ε-algorithm which is used to accelerate the convergence of series (40). Let N be a natural number and $s_m = \sum_{k=1}^{m} C_k$ be the sequence of partial sums of Eq. (40). we define the ε-sequence by

$$\varepsilon_{0,m} = 0, \quad \varepsilon_{1,m} = s_m,$$

$$\varepsilon_{n+1,m} = \varepsilon_{n-1,m+1} + \frac{1}{\varepsilon_{n,m+1} - \varepsilon_{n,m}}, \quad n, m = 1, 2, 3, \ldots .$$

The sequence $\varepsilon_{1,1}, \varepsilon_{3,1}, \ldots, \varepsilon_{N,1}$ converges to $g(t) + E_D - C_0/2$ faster than the sequence of partial sums s_m, $m = 1, 2, 3, \ldots$. The actual procedure to invert the Laplace transform consists of Eq. (41) together with the ε-algorithm. The values of C and L are chosen according to the criteria outlined by Honig and Hirdes [36].

The last step is to calculate the integral in Eq. (35). The method for evaluating the integral is described by Press et al. [37], which involves the use of Romberg's integration with adaptive step size. This also uses the results from successive refinements of the extended trapezoidal rule followed by extrapolation of the results to the limit when the step size tends to zero.

7. Numerical results and discussion

Using the numerical inversion technique described above, the displacement and stress components in the solid and liquid parts of the porous aggregate have been computed separately, for a particular model. In the model considered, we choose the following values of elastic constants:

$$A = 4.43 \times 10^{10}\,\text{dyn/cm}^2, \quad F = fA,$$
$$Q = 0.743 \times 10^{10}\,\text{dyn/cm}^2, \quad M = mQ,$$
$$N = 2.765 \times 10^{10}\,\text{dyn/cm}^2, \quad L = lN,$$
$$R = 0.326 \times 10^{10}\,\text{dyn/cm}^2, \quad C = F + 2L.$$

So that for $f = m = l = 1.0$, these constants become the elastic constants for isotropic kerosene saturated sandstone, Fatt [38].

The dynamical coefficients are taken as

$$\rho_{11} = 1.926\,\text{gm/cm}^3, \quad \rho_{12} = -0.00214\,\text{gm/cm}^3,$$
$$\rho_{22} = 0.21534\,\text{gm/cm}^3,$$

and porosity

$$\beta = 0.26.$$

Also, we have considered

$$\frac{F_0}{C} = 1.0.$$

Here, for numerical calculations, the anisotropy is considered due to

$$f = 1.5, \quad m = 1.2, \quad l = 2.0,$$

but we can take some other set of values, also.

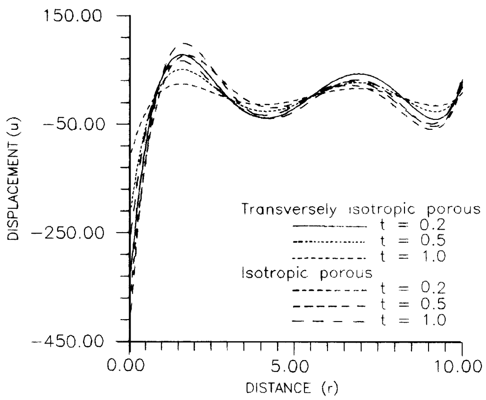

Fig. 1. Displacement distribution due to impulsive source.

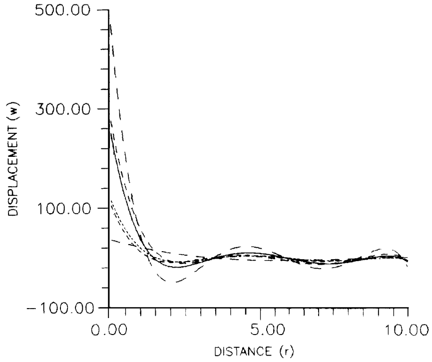

Fig. 2. Displacement distribution due to impulsive source. Key as for Fig. 1.

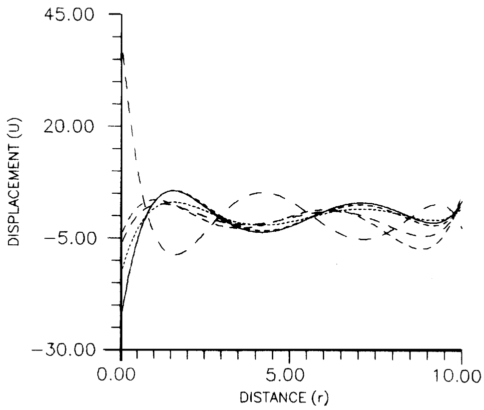

Fig. 3. Displacement distribution due to impulsive source. Key as for Fig. 1.

The elastodynamic responses represented by displacement and stress components are calculated on the plane $z = 1$, against distance 'r' for the following values of time $t = 0.2$, 0.5 and 1.0, using a computer programme in FORTRAN-IV on a PC. The medium considered is transversely isotropic liquid-saturated porous solid, as well as isotropic liquid-saturated porous solid. The distribution curves are shown in Figs. 1–7. Figs. 1–4 represent the distribution of displacement components due to impulsive source and Figs. 5–7 represent the distribution of stress components due to impulsive source.

It is observed that the maximum (absolute) displacements are occurring corresponding to the minimum distance 'r' and minimum time 't', i.e., the impact of an impulsive force is maximum as soon as it applies, and at the point of observation very near to the point of application of the force. From Figs. 1–7, it is clear that if we fix the point of observation, i.e., the value of distance 'r', the displacement and stress components increase or decrease with the passage of time. With further increase in time, it reveals that the displacements and stresses, following an oscillatory pattern about zero value, become zero, ultimately. But, this oscillatory pattern is not uniform for all the distances 'r' and for all the displacement and stress components. This is due to the fact that the medium considered is liquid-saturated porous, which is a two-phase medium involving an elastic solid matrix with pores saturated with fluid. The disturbances travelling through these different

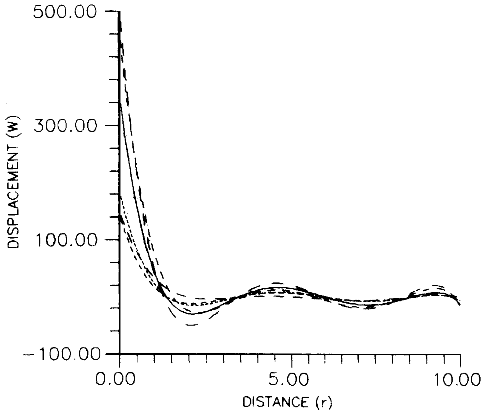

Fig. 4. Displacement distribution due to impulsive source. Key as for Fig. 1.

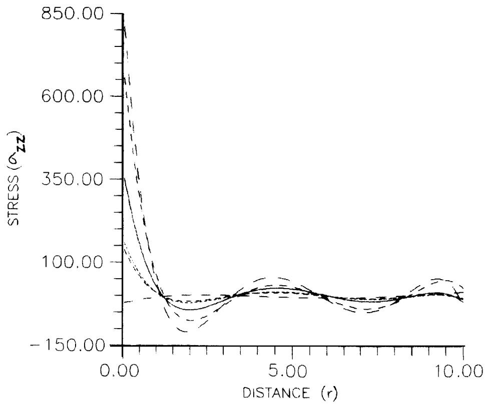

Fig. 5. Stress distribution due to impulsive source. Key as for Fig. 1.

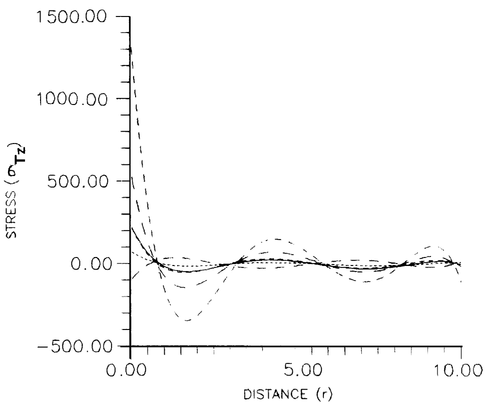

Fig. 6. Stress distribution due to impulsive source. Key as for Fig. 1.

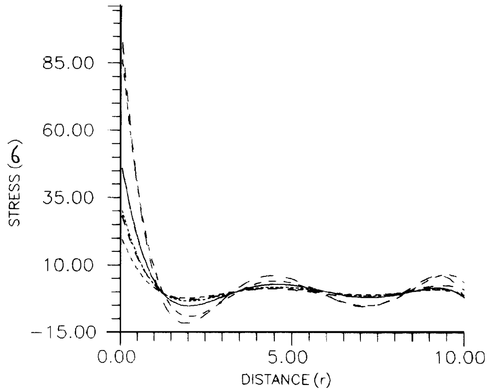

Fig. 7. Stress distribution due to impulsive source. Key as for Fig. 1.

constituents of the medium suffer sudden changes, resulting in an inconsistent/non-uniform pattern. Otherwise, when it passes through either solid or liquid, it shows a uniform pattern.

It is observed that the magnitude of displacement and stress components decrease along an oscillatory path with the increase in distance 'r'. With further increase in distance, it reveals that all the displacements become zero, ultimately. This means that all the displacements vanish as the point of observation is far away from the point of application of the force, which justifies the radiation conditions.

It is seen that the trend of curves in the two constituents of the liquid-saturated porous medium, solid and liquid, for the corresponding components are almost same with difference in the magnitudes of their values. This means that the disturbances in the two constituents are in the same phase.

Figs. 1–7 also show the effect of anisotropy on the disturbances produced due to impulsive force. It is observed that anisotropy affects mainly the magnitudes of displacements and stresses, and not the trend of curves. Thus, it reveals that the anisotropic effect is mainly quantitative in nature.

It is concluded that the trend of curves exhibits the properties of liquid-saturated porous medium and satisfy the requisite conditions of the problem. The disturbances produced in the liquid-saturated porous medium are affected by anisotropy of the medium, which in turn will effect the various phenomena, e.g., wave propagation. This study will provide the base for tackling similar types of problems in relation to other materials, for instance, it can be extended to partially saturated porous media.

Acknowledgements

One of the authors (N.R. Garg) is thankful to U.G.C. for supporting a minor research project.

Appendix A. Nomenclature

A, N, F, M, C, Q, L, R	elastic constants for transversely isotropic liquid-saturated porous medium
$\rho_{11}, \rho_{12}, \rho_{22}$	dynamical coefficients

ρ_s, ρ_f, ρ	mass densities of the solid, fluid, bulk material
β	porosity
ω	angular frequency
t	time variable
x, y, z	Cartesian co-ordinates
r, ϑ, z	cylindrical co-ordinates
u_i, U_i	components of displacements in the solid and liquid parts of the porous aggregate along x_i direction
$\vec{u}, \vec{U}$	displacement vectors in the solid and liquid parts of the porous aggregate
u, w	radial and normal components of the displacement in the solid part
U, W	radial and normal components of the displacement in the liquid part
f_p	pressure in the fluid
σ_{ij}	components of stress in the solid part
σ	stress in the fluid part
e_{ij}	components of strain in the solid part
e	dilatation in the solid part
ε	dilatation in the liquid part
h	a quantity having the dimension of length

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