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Letter to the Editor

Vibration of orthotropic plates: discussion on the completeness of the solutions used in direct methods

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1. Introduction

The importance of the completeness of a set of solutions used in a direct variational method is herein discussed regarding a plate vibration application. The property is relevant to two issues: the claim of an "exact" solution and the avoidance of the loss of eigenfrequencies. A paper by Hurlebaus et al. [1] reports an "exact" series solution for calculating eigenfrequencies of orthotropic free plates. A trigonometric series is proposed and after many, not always simple, algebraic steps, a system of homogenous algebraic equations is derived. Although the approach and the algebra are formally correct, some of the steps require certain discussion. The authors of the present paper had previously, and in the same journal, published a study on free vibrations of free isotropic plates [2] in which this topic is discussed. A method named whole element method (WEM) is therein used. An extended trigonometric series for a two-dimensional domain is employed and justified, in particular, a complete series of sines. Its equivalent in a series of cosines is coincident with the solution reported in Ref. [1].

This issue is not tackled in Ref. [1] and neither is the question of uniform convergence of the solution and its essential functions (meaning functions involving derivatives up to order $k-1$, with $2k$ the largest order of derivation in the differential equation). Not complying with these two properties may lead to an approximate solution (Ritz method) or the loss of eigenvalues.

Hurlebaus and co-authors effectively employ a uniform convergence series for the deflection shape W that results from the linear combination of a complete set in the two-dimensional domain. However, in subsequent steps (Eqs. (32) and (33) of Ref. [1]) the uniform convergence of the slope is lost with the consequence of the probability that the eigenvalue converges to an approximate value and not to the exact one, or even more serious, as to jump to other eigenvalue. On the other hand, the reported numerical results for a square orthotropic plate are correct. We

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believe that some known functions that are replaced by infinite Fourier expansions (convergence in the mean) as in Eqs. (34) and (36) of Ref. [1] may fix the previous loss of uniform convergence. But these infinite expansions are not always possible, neither their effects predictable. Finite expressions are mixed with closed form functions (infinite expansions). Furthermore, the number of terms used in the numerical evaluations (size of the matrix) and whether the same number was used for all the modes, is not reported.

Additionally, we have adopted the concept of "arbitrary precision" numerical results since the eigenfrequencies are theoretically convergent to the exact values (i.e., knowing that the combined set is complete) but when found numerically, it is unavoidable to have a finite number of terms. Then knowing its convergence to the exact value, the technique consists in fixing a number of digits of desired accuracy and afterwards increasing the number of terms until these digits converge.

In short, in order to assert that the eigenvalues are theoretically exact (of arbitrary precision), it is necessary to ensure that the deflection shape and its first derivatives have uniform convergence to the classical solution (although unknown).

In what follows the effect of using an incomplete solution is shown and discussed and finally the problem of a free orthotropic plate is tackled with WEM and frequency values found with arbitrary precision are reported for various plate aspect ratios.

2. Illustration I: an incomplete solution

Let us show the use of a incomplete trigonometric series to find the first two symmetric–antisymmetric eigenfrequencies of a square (side a), isotropic free plate. The frequency parameter is $\Omega = \omega a^2\sqrt{\rho h/D}$ where the flexural rigidity is $D = Eh^3/12(1-\nu^2)$. The non-dimensional values are known to be $\Omega_1 = 34.82$ and $\Omega_2 = 61.13$ (found with WEM and 15 terms [2]). Leissa [3] reports $\Omega_1 = 35.02$ and $\Omega_2 = 61.53$, respectively, found with Ritz method using six beam functions.

As stated in Ref. [2], the following series constitutes a complete solution for a symmetric–antisymmetric mode:

$$w_{MN}(x,y) = \sum_{i=E}^{M}\sum_{j=O}^{N}\frac{A_{ij}s_i s_j}{\alpha_i\alpha_j} + \sum_{i=E}^{M}\frac{a_i s_i}{\alpha_i} + \left(x-\frac{1}{2}\right)\left[a_0 + \sum_{j=O}^{N}\frac{A_{0j}s_j}{\alpha_j}\right], \tag{1}$$

where $E = 2,4,6,\ldots,M$ (even indexes) and $O = 1,3,5,\ldots,N$ (odd indexes), $s_i \equiv \sin(\alpha_i x)$, $s_j \equiv \sin(\alpha_j y)$, $\alpha_m = m\pi$ and the non-dimensionalized domain [0, 1]. It may be shown that w_{MN} as well as its first derivatives are uniformly convergent. Moreover, since we are dealing with a free plate, it is necessary that the inertial equilibrium be satisfied. The three conditions (inertial force and two moments) in terms of the proposed solution are reduced to

$$\int\int w_{MN}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{2a}$$

$$\int\int w_{MN}(x,y)\;x\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{2b}$$

$$\int\int w_{MN}(x,y)\;y\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{2c}$$

Conditions (2a) and (2c) are satisfied identically by Eq. (1). On the other hand, condition (2b) forces the following relationship among the coefficients of the summations:

$$-2\sum_{i=E}^{M}\sum_{j=O}^{N}\frac{A_{ij}}{\alpha_i^2\alpha_j^2}-\sum_{i=E}^{M}\frac{a_i}{\alpha_i^2}+\frac{a_0}{12}+\frac{1}{6}\sum_{j=O}^{N}\frac{A_{0j}}{\alpha_j^2}=0. \tag{3}$$

It is possible to choose a subset that fulfills this condition. A possible one is selecting

$$A_{ij}=A_{0j}=0 \quad \Rightarrow \quad a_0=\sum_{i=E}^{M}\frac{a_i}{\alpha_i^2} \tag{4}$$

with which the proposed solution for symmetric–antisymmetric modes is

$$w_{MN}(x,y)=\sum_{i=E}^{M}\frac{a_i}{\alpha_i}\left[s_i+\frac{12(x-1/2)}{\alpha_i}\right]. \tag{5}$$

The satisfaction of statement (3) may be achieved by different sets of coefficients. However, an arbitrary selection, like (4) leads to the incomplete solution (5). It should be noted that the inertial equilibrium is not usually stated independently since it is included in the direct method. Here it was explicitly required so as to derive an incomplete solution. The application of WEM using this solution implies [2] the satisfaction of the following *pseudo-virtual work* statement:

$$\begin{aligned}&(w'',\delta w'')+\lambda^4(\bar{\bar{w}},\delta\bar{\bar{w}})-\lambda^2[(\bar{\bar{w}},\delta w'')+(w'',\delta\bar{\bar{w}})]\\&+\ 2\lambda^2(1-\nu)(\bar{w}',\delta\bar{w}')-\Omega^2(w,\delta w)|_{w_{MN}}=0,\end{aligned} \tag{6}$$

where the aspect ratio $\lambda=1$.

After the replacement of Eq. (5) in Eq. (6), the next trascendental equation results:

$$1+24\Omega^2\sum_{i=E}^{M}\frac{1}{\alpha_i^2(\alpha_i^4-\Omega^2)}=0. \tag{7}$$

The solution found with accuracy of 2 exact decimal digits yields $\Omega_1=61.67$ and $\Omega_2=199.86$ (in both cases found with $M=30$). Fig. 1 shows the resulting mode corresponding to $\Omega_1=61.67$. It is clear that the "true" first frequency was lost and that the second one (here the first) converges to an approximate value. At the point we stated Eq. (5) it seemed an appropriate choice. But the fact that we stated a function as the linear combination of an incomplete set leads to this incorrect answer. The correct mode shape obtained with the complete solution and with $\Omega_1=34.82$ is depicted in Fig. 2. Comparing both mode shapes, although they look quite similar, the slight difference led to the erroneous result.

In addition, another example of this situation is present in Gorman [4]. He employs the method of superposition using a linear combination of functions which are not a complete set in the

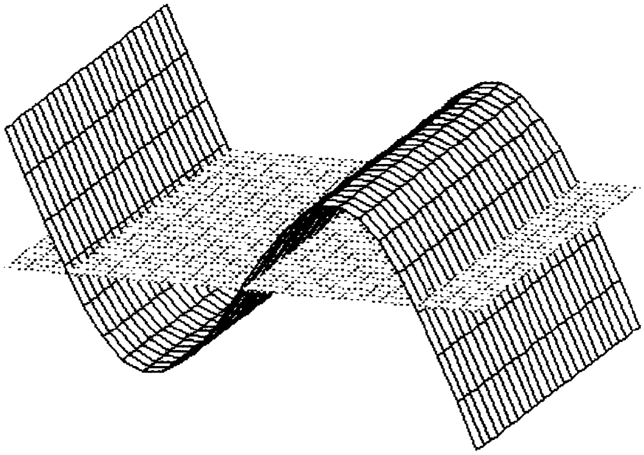

Fig. 1. First symmetric–antisymmetric mode found using an incomplete set.

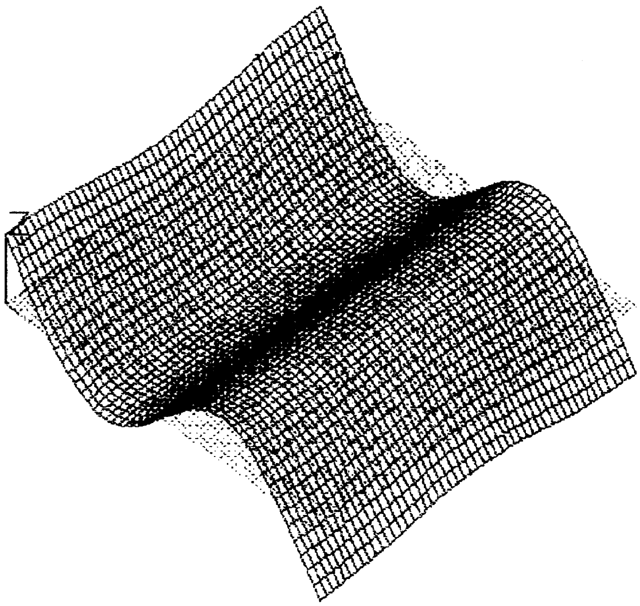

Fig. 2. First symmetric–antisymmetric mode found using a complete set [2].

domain. As a consequence, in the table of results reported in that reference, the second frequency of an isotropic square plate (doubly symmetric, $\Omega = 19.61$) is missing. A note of warning should be given in this regard. Furthermore, in a more recent paper [5] Gorman makes use of the superposition-Galerkin method. Again for the same reason, the first doubly symmetric mode is missing for the isotropic plate.

3. Illustration II: vibrations of an orthotropic plate using a complete solution

Let us introduce the following expansion—one among an infinite number—complete in a two-dimensional domain:

$$w_{MN}(x,y) = \sum_{i=1}^{M}\sum_{j=1}^{N} A_{ij}s_i s_j + x\left(a_0 + \sum_{j=1}^{N} A_{0j}s_j\right) + y\left(b_0 + \sum_{i=1}^{M} A_{i0}s_i\right) + A_{00}xy + \sum_{i=1}^{M} a_i s_i + \sum_{j=1}^{N} b_j s_j + k. \quad (8)$$

The application of WEM to the free vibration problem of an orthotropic plate requires that

$$(w'', \delta w'') + \beta\lambda^4(\bar{\bar{w}}, \delta\bar{\bar{w}}) + \lambda^2 v_y[(\bar{\bar{w}}, \delta w'') + (w'', \delta\bar{\bar{w}})] + 4\gamma\lambda^2(\bar{w}', \delta\bar{w}') - \Omega^2(w, \delta w)|_{w_{MN}} = 0, \quad (9)$$

where $\lambda = a/b$ is the plate aspect ratio, h is the thickness of the plate, $\beta = E_y/E_x$, E_x and E_y are the modulus of Young, v_x and v_y are the Poisson coefficients, $\gamma = D_{xy}/D_x$, $D_{xy} = G_{xy}h^3/12$, $D_x = E_x h^3/12(1 - v_x v_y)$, $D_y = E_y h^3/12(1 - v_x v_y)$.

When symmetric–symmetric modes are analyzed, both i and j are taken as odd indexes (0) and the WEM solution is as follows (also a complete subset):

$$w_{MN}(x,y) = \sum_{i=0}\sum_{j=O} A_{ij}s_i s_j + \sum_{i=0} a_i s_i + \sum_{j=O} b_j s_j + k. \quad (10)$$

It should be mentioned that this extended series is uniformly convergent, as well as its first derivatives. The application of the *pseudo-virtual work* (9) leads to the following set of equations:

$$a_i d_i^* + \sum_{q-0}^{M} b_q \gamma_{iq}^* = t_i^*, \quad (11)$$

$$b_j d_j^* + \sum_{p=0} a_p \mu_{pj}^* = u_j^*, \quad (12)$$

where

$$d_i^* = d_i(1 - 8SU_i) - 8\lambda^2 v_y \alpha_i^2 ZU_i, \quad d_i = \alpha_i^4 - \Omega_x^2, \quad (13)$$

$$d_j^* = d_j(1 - 8SV_j) - 8\lambda^2 v_y \alpha_j^2 ZV_j, \quad d_j = \alpha_i^4 \beta\lambda^4 - \Omega_x^2, \quad (14)$$

$$\gamma_{iq}^* = -8d_i m v_{iq} - 8\lambda^2 v_y \alpha_i^2 n v_{iq} + 8\lambda^2 v_y \alpha_i \alpha_q - 8\frac{\Omega_x^2}{\alpha_i \alpha_q}, \quad (15)$$

$$\mu_{iq}^* = -8d_j m u_{pj} - 8\lambda^2 v_y \alpha_j^2 n u_{pj} + 8\lambda^2 v_y \alpha_p \alpha_j - 8\frac{\Omega_x^2}{\alpha_p \alpha_j}, \quad (16)$$

$$t_i^* = k\left(32 d_i S1 W_i + 32\lambda^2 v_y \alpha_i^2 Z1 W_i + 4\frac{\Omega_x^2}{\alpha_i}\right), \quad (17)$$

$$u_j^* = k\left(32 d_j S2 W_j + 32\lambda^2 v_y \alpha_j^2 Z2 W_j + 4\frac{\Omega_x^2}{\alpha_j}\right). \quad (18)$$

The sums involved in the algebra are

$$SU_i = \sum_{q=0} \frac{U_{iq}}{D_{iq}\alpha_q}, \quad SV_j = \sum_{p=0} \frac{V_{pj}}{D_{pj}\alpha_p}, \tag{19}$$

$$ZU_i = \sum_{q=0} \frac{\alpha_q U_{iq}}{D_{iq}}, \quad ZV_j = \sum_{p=0} \frac{\alpha_p V_{pj}}{D_{pj}}, \tag{20}$$

$$S1W_i = \sum_{q=0} \frac{W_{iq}}{D_{iq}\alpha_q}, \quad S2W_j = \sum_{p=0} \frac{W_{pj}}{D_{pj}\alpha_p}, \tag{21}$$

$$Z1W_i = \sum_{q=0} \frac{\alpha_q W_{iq}}{D_{iq}}, \quad Z2W_j = \sum_{p=0} \frac{\alpha_p W_{pj}}{D_{pj}}, \tag{22}$$

$$D_{ij} = \alpha_i^4 + 2\lambda^2 v_y \alpha_i^2 \alpha_j^2 + \beta\lambda^4 \alpha_j^4 + 4\lambda^2 \gamma \alpha_i^2 \alpha_j^2 - \Omega_x^2, \quad W_{ij} = -\frac{\Omega_x^2}{\alpha_i \alpha_j}, \tag{23}$$

$$U_{ij} = \frac{\alpha_i^4}{\alpha_j} = \lambda^2 v_y \alpha_i^2 \alpha_j - \frac{\Omega_x^2}{\alpha_j}, \quad V_{ij} = \lambda^2 v_y \alpha_i \alpha_j^2 + \beta\lambda^4 \frac{\alpha_j^4}{\alpha_i} - \frac{\Omega_x^2}{\alpha_i}, \tag{24}$$

$$mu_{pj} = \frac{U_{pj}}{D_{pj}\alpha_p}, \quad mv_{iq} = \frac{V_{iq}}{D_{iq}\alpha_q}, \quad nu_{pj} = \frac{\alpha_p U_{pj}}{D_{pj}}, \quad nv_{iq} = \frac{\alpha_q V_{iq}}{D_{iq}}. \tag{25}$$

A numerical example was carried out and the arbitrary precision values of the frequency parameter $\Omega_x = \omega a^2 \sqrt{\rho h / D_x}$ are reported in Table 1. Also the frequency values in Hz found with the data reported in Ref. [1] and listed in Table 2, are depicted in Table 1 between parentheses.

Table 1
Natural frequency parameters Ω_x of an orthotropic plate for symmetric–symmetric modes[a]

Ω_x	$\lambda = a/b$		
	1	0.4	2.5
Ω_{x1}	6.3285	1.0124	22.3276
	(60.1737)[b]	(9.6262)	(212.2992)
Ω_{x2}	22.3601	3.8377	29.9740
	(212.6083)[b]	(36.4912)	(285.0041)
Ω_{x3}	32.0613	5.4729	39.6175
	(304.8509)	(52.0384)	(376.6981)

[a] Values in parentheses are frequencies in Hz found with the data of Table 2.
[b] Values of frequencies in Hz coincident with the ones reported in Ref. [1].

Table 2
Orthotropic plate data

a, b (m)	h (m)	ρ (kg/m^3)	E_x (GPa)	E_y (GPa)	G_{xy} (GPa)	v_x	v_y
0.254	0.001483	1584	127.9	10.27	7.312	0.22	0.0176654

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