Available online at www.sciencedirect.com
SCIENCE DIRECT
ACADEMIC PRESS
Journal of Sound and Vibration 261 (2003) 764–773
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

The regular and chaotic vibrations of an axially moving viscoelastic string based on fourth order Galerkin truncaton

Li-Qun Chen[a,b,*], Neng-Hui Zhang[a], Jean W. Zu[c]

[a] Department of Mechanics, Shanghai University, Shanghai 200436, China
[b] Shanghai Institute of Applied Mathematics & Mechanics, Shanghai University, Shanghai 200072, China
[c] Department of Mechanical and Industrial Engineering, University of Toronto, Ont., Canada M5S 3G8,

Received 18 August 2002; accepted 10 September 2002

1. Introduction

The transverse vibrations of axially moving strings are involved in many engineering devices, such as serpentine belts, fiber windings, magnetic tapes and thread lines. Much research has been done to understand the dynamical behaviors of such systems, which has been reviewed by Mote [1], Wickert and Mote [2], Abrate [3] and Chen and Zu [4]. One major problem is the occurrence of large transverse vibrations due to tension variation termed as parametric vibrations. However, the available studies [5–14] on transverse motions of parametrically excited moving strings are concentrated on equilibrium or periodic vibrations and their stability. The literature that is specially related to chaotic motion is very limited. To address the lack of research in this aspect, this paper studies chaotic behavior of a parametrically excited viscoelastic moving string with geometric non-linearity based on the Galerkin truncation of the equation of motion.

The Galerkin method has been applied to treat the transverse vibrations of parametrically excited axially moving strings [5–11]. However, only the low order Galerkin truncation is feasible if longtime non-linear dynamical behaviors (especially the chaotic behavior) of elastic or viscoelastic mechanisms and structures are concerned. So far there is no direct evidence to prove the plausibility of the low order Galerkin truncation, although it can be inferred from certain indirect evidence. The experimental work done by Moon and Holmes for a one-end clamped elastic beam has indicated that chaotic free-end motion obtained from an analog computer based on the first order Galerkin truncation is qualitatively consistent with the experimental measurement [15]. Numerical calculations done by Abhyankar et al. for a non-linear simply supported beam has shown that the numerical solutions of a partial differential equation and a differential equation deduced by the first order Galerkin truncation yield similar results [16]. In addition to the direct comparison with numerical and experimental results, the suitability of the

*Corresponding author. Department of Mathematics, Shanghai University, Hu Tai Road 555, Room 1101, Building 2, Shanghai 200070, China.

E-mail address: lqchen@online.sh.cn (L.-Q. Chen).

doi:10.1016/S0022-460X(02)01281-6

Galerkin truncation for research on regular or chaotic motions of non-linear mechanisms and structures can be approached by comparing the dynamical behaviors of different order truncation models simplified from the same mathematical model. In the case of viscoelastic columns [17] and beams [18], the numerical results indicate that the dynamical behaviors of first and second order Galerkin truncated models are qualitatively the same, but there are certain differences if the quantitative comparisons are concerned.

In this paper, the first, second, third and fourth order Galerkin truncated systems are, respectively, deduced from the partial-differential equation that governs the transverse vibrations of axially moving strings. The dynamics of those truncated systems is numerically compared. Regular and chaotic motions occur in these systems. The effects of the transport speed, the periodic perturbation amplitude, and the dynamic viscosity on the dynamical behaviors are numerically investigated.

2. Equation of motion and its simplifications

Consider that the viscoelastic string moving in the x direction is in a state of uniform initial stress, and only the transverse vibration in the y direction is taken into consideration. The material of the string obeys the linear viscoelastic differential constitutive relation. Then the equation of motion obtained by Newton's second law for the string is [12]

$$\rho\frac{\partial^2 V}{\partial t^2}+2\rho c\frac{\partial^2 V}{\partial t\partial x}+\left(\rho c^2-\frac{T}{A}\right)\frac{\partial^2 V}{\partial x^2}-\frac{\partial}{\partial x}\left(E^*\left(\frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2\right)\frac{\partial V}{\partial x}\right)=0, \tag{1}$$

where V is the displacement in the transverse direction, ρ is the mass per unit volume, c is the axially moving speed of the string, A is the area of cross-section of the string, T is the tension in the string, and E^* is the linear differential operator determined by the viscoelastic property of string materials. The string is subjected to the non-vibration boundary conditions

$$V(0,t)=0, \quad V(L,t)=0, \tag{2}$$

where L is the length of the string span. It is assumed that the tension T is characterized as a small periodic perturbation $T_1\cos\Omega t$ on the steady state tension T_0. The Kelvin viscoelastic model is chosen to describe the viscoelastic property of the string material. Thus,

$$E^*=E_0+\eta\frac{\partial}{\partial t}, \tag{3}$$

where E_0 is the stiffness constant of the string, and η is the dynamic viscosity of the dashpot.

Eqs. (1) and (2) can be transformed into the non-dimensional form

$$\begin{gathered}\frac{\partial^2 v}{\partial\tau^2}+2\gamma\frac{\partial v}{\partial\tau\partial\xi}+(\gamma^2-1-\alpha\cos\omega\tau)\frac{\partial^2 v}{\partial\xi^2}-\frac{3}{2}E_e\left(\frac{\partial v}{\partial\xi}\right)^2\frac{\partial^2 v}{\partial\xi^2},\\ -2E_v\frac{\partial v}{\partial\xi}\frac{\partial^2 v}{\partial\xi\partial\tau}\frac{\partial^2 v}{\partial\xi^2}-E_v\left(\frac{\partial v}{\partial\xi}\right)^2\frac{\partial^3 v}{\partial\xi^2\partial\tau}=0,\end{gathered} \tag{4}$$

$$v(0,t)=0, \quad v(1,t)=0 \tag{5}$$

where

$$v = \frac{V}{L}, \quad \xi = \frac{x}{L}, \quad \tau = \frac{t}{L}\sqrt{\frac{T_0}{\rho A}}, \quad \gamma = c\sqrt{\frac{\rho A}{T_0}}, \quad \alpha = \frac{T_1}{T_0},$$
$$\omega = \Omega L\sqrt{\frac{\rho A}{T_0}}, \quad E_e = \frac{E_0 A}{T_0}, \quad E_v = \frac{\eta}{L}\sqrt{\frac{T_0}{\rho A}}. \tag{6}$$

Eq. (4) is in the form of a continuous gyroscopic system with non-linear terms and a parameter excitation term.

The Galerkin method is employed to simplify Eq. (4). Under the boundary condition (5), the solution of Eq. (4) may be expanded into the following trial function:

$$v(\xi, \tau) = \sum_{n=1}^{m} q_n(\tau)\sin(n\pi\xi), \tag{7}$$

where the $q_n(\tau)$ are generalized displacements, and $\sin(n\pi\xi)$ is the nth eigenfunction of the simply supported stationary string. Taking the appropriate derivatives and substituting into Eq. (4), one obtains the residual

$$\begin{aligned} R(\xi, \tau) = &\sum_{n=1}^{m} \ddot{q}_n(\tau)\sin(n\pi\xi) + 2\gamma\sum_{n=1}^{m} n\dot{q}_n(\tau)\cos(n\pi\xi) \\ &- \pi^2(\gamma^2 - 1 - \alpha\cos\omega\tau)\sum_{n=1}^{m} n^2 q_n(\tau)\sin(n\pi\xi) \\ &+ \frac{3}{2}\pi^4 E_e\left(\sum_{n=1}^{m} n\dot{q}_n(\tau)\cos(n\pi\xi)\right)^2 \sum_{n=1}^{m} n^2 q_n(\tau)\sin(n\pi\xi) \\ &+ 2\pi^4 E_v\left(\sum_{n=1}^{m} nq_n(\tau)\cos(n\pi\xi)\right)\left(\sum_{n=1}^{m} n\dot{q}_n(\tau)\cos(n\pi\xi)\right)\sum_{n=1}^{m} n^2 q_n(\tau)\sin(n\pi\xi) \\ &+ \pi^4 E_v\left(\sum_{n=1}^{m} nq_n(\tau)\cos(n\pi\xi)\right)^2 \sum_{n=1}^{m} n^2 \dot{q}_n(\tau)\sin(n\pi\xi), \end{aligned} \tag{8}$$

where the derivative is with respect to the dimensionless time τ. Application of the Galerkin method requires that

$$\int_0^1 R(\xi, \tau)w_i(\xi)\,\mathrm{d}\xi = 0 \quad (i = 1, 2, \ldots, m), \tag{9}$$

where the weighting functions $w_i(\xi)$ are also chosen as the stationary string eigenfunctions

$$w_i(\xi) = \sin(i\pi\xi). \tag{10}$$

In the case that $m = 4$, inserting Eqs. (8) and (10) into Eq. (9) and integrating the resulting equation yield the simplified equations of motion based on the fourth order Galerkin truncation:

$$
\begin{aligned}
&\ddot{q}_1 - \pi^2(\gamma^2 - 1 - \alpha\cos\omega t)q_1 - \frac{16\gamma}{3}\dot{q}_2 - \frac{32\gamma}{15}\dot{q}_4 + \frac{3E_e\pi^4}{8}(q_1^3 + 8q_1q_2^2 + 3q_1^2q_3 \\
&\quad + 12q_2^2q_3 + 18q_1q_3^2 + 16q_1q_2q_4 + 48q_2q_3q_4 + 32q_1q_4^2) + \frac{E_v\pi^4}{4}[(3q_1^2 + 8q_2^2 \\
&\quad + 6q_1q_3 + 18q_3^2 + 16q_2q_4 + 32q_4^2)\dot{q}_1 + 8(2q_1q_2 + 3q_2q_3 + 2q_1q_4 + 6q_3q_4)\dot{q}_2 \\
&\quad + 3(q_1^2 + 4q_2^2 + 12q_1q_3 + 16q_2q_4)\dot{q}_3 + 16(q_1q_2 + 3q_2q_3 + 4q_1q_4)\dot{q}_4] = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\ddot{q}_2 - 4\pi^2(\gamma^2 - 1 - \alpha\cos\omega t)q_2 + \frac{16\gamma}{3}\dot{q}_1 - \frac{48\gamma}{5}\dot{q}_3 + 3E_e\pi^4(2q_2^3 + q_1^2q_2 \\
&\quad + 3q_1q_2q_3 + 9q_2q_3^2 + q_1^2q_4 + 6q_1q_3q_4 + 9q_3^2q_4 + 16q_2q_4^2) + 2E_v\pi^4[(2q_1q_2 \\
&\quad + 3q_2q_3 + 2q_1q_4 + 6q_3q_4)\dot{q}_1 + (q_1^2 + 6q_2^2 + 3q_1q_3 + 9q_3^2 + 16q_4^2)\dot{q}_2 + 3(q_1q_2 \\
&\quad + 6q_2q_3 + 2q_1q_4 + 6q_3q_4)\dot{q}_3 + (q_1^2 + 6q_1q_3 + 9q_3^2 + 32q_2q_4)\dot{q}_4] = 0, \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
&\ddot{q}_3 - 9\pi^2(\gamma^2 - 1 - \alpha\cos\omega t)q_3 + \frac{48\gamma}{5}\dot{q}_2 - \frac{96\gamma}{5}\dot{q}_4 + \frac{3E_e\pi^4}{8}(q_1^3 + 12q_1q_2^2 \\
&\quad + 18q_1^2q_3 + 72q_2^2q_3 + 81q_3^3 + 48q_1q_2q_4 + 144q_2q_3q_4 + 288q_3q_4^2) + \frac{3E_v\pi^4}{4}[(q_1^2 \\
&\quad + 4q_2^2 + 12q_1q_3 + 16q_2q_4)\dot{q}_1 + 8(q_1q_2 + 6q_2q_3 + 2q_1q_4 + 6q_3q_4)\dot{q}_2 + 3(2q_1^2 \\
&\quad + 8q_2^2 + 27q_3^2 + 16q_2q_4 + 32q_4^2)\dot{q}_3 + 16(q_1q_2 + 3q_2q_3 + 12q_3q_4)\dot{q}_4] = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\ddot{q}_4 - 16\pi^2(\gamma^2 - 1 - \alpha\cos\omega t)q_4 + \frac{32\gamma}{15}\dot{q}_1 + \frac{96\gamma}{7}\dot{q}_3 + 3E_e\pi^4(q_1^2q_2 + 6q_1q_2q_3 + 9q_2q_3^2 \\
&\quad + 4q_1^2q_4 + 36q_3^2q_4 + 16q_2^2q_4 + 32q_4^3) + 2E_v\pi^4[2(q_1q_2 + 3q_2q_3 + 4q_1q_4)x_1' + (q_1^2 + 6q_1q_3 \\
&\quad + 9q_3^2 + 32q_2q_4)\dot{q}_2 + 3(2q_1q_2 + 6q_2q_3 + 24q_3q_4)\dot{q}_3 + 4(q_1^2 + 4q_2^2 + 9q_3^2 + 24q_4^2)\dot{q}_4] = 0,
\end{aligned}
$$

which contains two gyroscopically coupled linear terms.

Setting $q_2 = q_3 = q_4 = 0$ in Eq. (11) leads to the simplification based on the first order Galerkin truncation. Setting $q_3 = q_4 = 0$ in Eq. (11) leads to the simplification based on the second order Galerkin truncation, which contains a gyroscopically coupled linear term. Setting $q_4 = 0$ in Eq. (11) leads to the simplification based on the third order Galerkin truncation.

3. Bifurcation diagrams of the Poincaré maps

The Poincaré map and the bifurcation diagram are the modern techniques used in the analysis of non-linear systems. The Poincaré map is a convenient tool to identify the dynamical behavior, especially chaos. The dynamics may be viewed globally over a range of parameter values, thereby allowing simultaneous comparison of regular and chaotic motions. The bifurcation diagram provides a summary of essential dynamics and is therefore a useful tool for acquiring this overview. In this research, we check the Poincaré maps of the non-dimensional displacement of the center of the moving string, respectively, determined by the first order Galerkin truncated system

$$v_1(nT, 0.5) = q_1(nT), \tag{12}$$

where q_1 is numerically integrated from the resulting equation by setting $q_2 = q_3 = q_4 = 0$ in Eq. (11), the second order Galerkin truncated system

$$v_2(nT, 0.5) = q_1(nT), \tag{13}$$

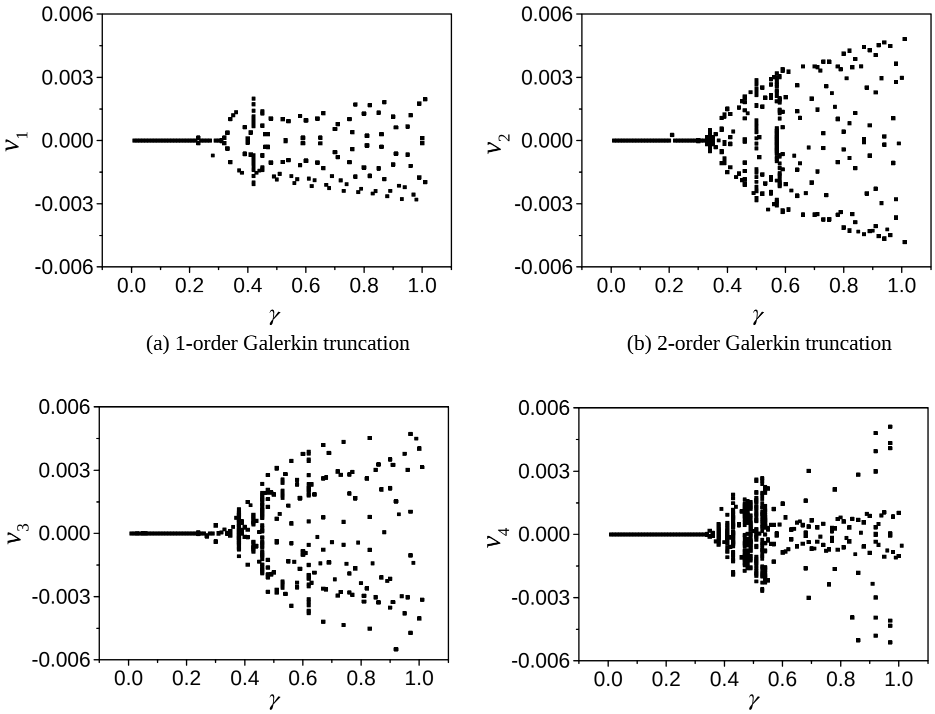
where q_1 is numerically integrated from the resulting equation by setting $q_3 = q_4 = 0$ in Eq. (11), the third order Galerkin truncated system

$$v_3(nT, 0.5) = q_1(nT) - q_3(nT), \tag{14}$$

where q_1 and q_3 are numerically integrated from the resulting equation by setting $q_4 = 0$ in Eq. (11), and the fourth order Galerkin truncated system

$$v_4(nT, 0.5) = q_1(nT) - q_3(nT), \tag{15}$$

where q_1 and q_3 are numerically integrated from Eq. (11). In Eqs. (12)–(15), $T = 2\pi/\omega$ and $n = 1, 2, 3, \ldots$. The fourth order Runge–Kutta routine is used for numerical integration. The



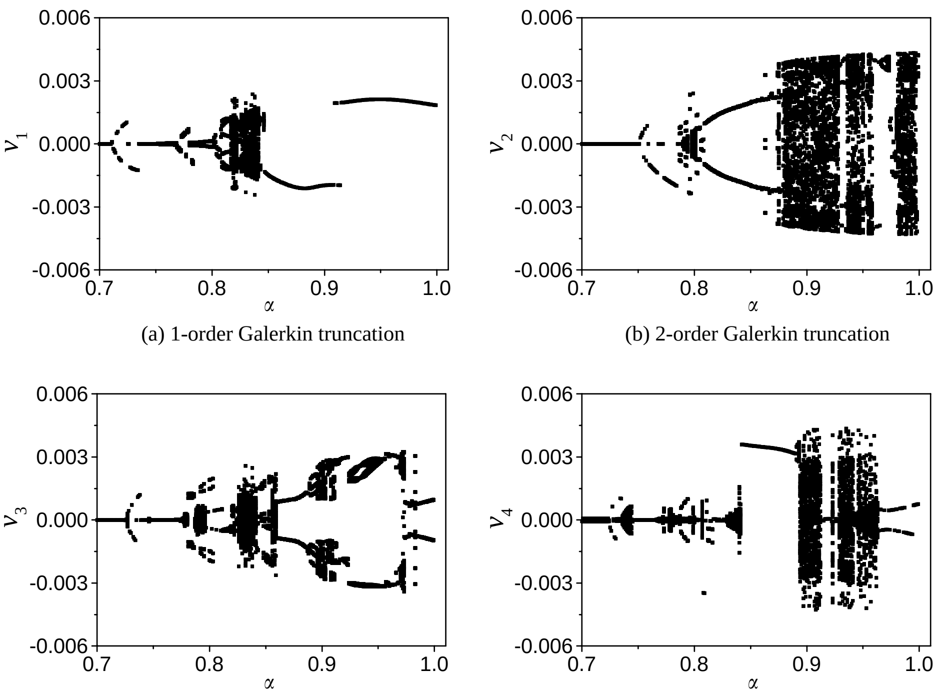
(a) 1-order Galerkin truncation
(b) 2-order Galerkin truncation
(c) 3-order Galerkin truncation
(d) 4-order Galerkin truncation

Fig. 1. Bifurcation versus the transport speed.

bifurcation diagrams are presented by varying, respectively, the non-dimensional transport speed γ, the non-dimensional amplitude of the periodic perturbation α, and the non-dimensional dynamic viscosity E_V while the non-dimensional frequency of the periodic perturbation ω and the non-dimensional stiffness of the string are, respectively, kept constant at $\omega = 0.125$ and $E_e = 400$. At each set of parameters, the first 2000 points of the Poincaré map are discarded in order to exclude the transient vibration, and the displacements for the next 50 points are, respectively, plotted on the bifurcation diagrams. Only the stable motions are considered.

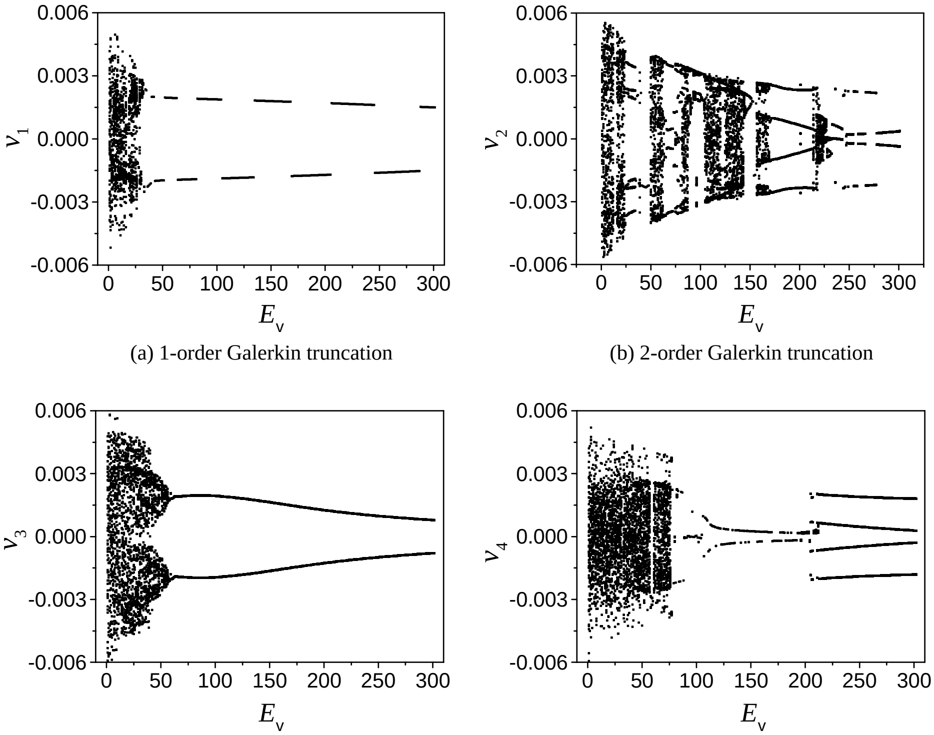
The bifurcation of the first, second, third, and fourth order truncated systems are detected by examining graphs of the non-dimensional center displacement v_i $(i = 1, 2, 3, 4)$ given by Eqs. (12)–(15) of the string versus γ for specific values of the parameters α, E_V, ω and E_e. The bifurcation diagrams for a specific value sets $\alpha = 0.9$ and $E_V = 120$ are presented in Fig. 1. For γ small enough, the system is asymptotically stable with its response tending to zero. With the increase of γ, the explosive bifurcation occurs. There is a discontinuous increase in the size and form of a strange attractor, the new enlarged attractor, after the bifurcation, which includes within itself the phase space regime of the old attractor. If γ is increased further, the chaotic motions and the regular motions alternately appear, and they are alternate more frequently in the second and fourth order Galerkin truncated systems than those in the first and third order ones.

The bifurcation diagrams in Fig. 2 shows the Poincaré maps of the non-dimensional center displacement v_i $(i = 1, 2, 3, 4)$ given by Eqs. (12)–(15) against the parameter α for fixed $\gamma = 0.5$ and $E_V = 50$. In this case, the system is asymptotically stable with its motion tending to zero for α



(a) 1-order Galerkin truncation
(b) 2-order Galerkin truncation
(c) 3-order Galerkin truncation
(d) 4-order Galerkin truncation

Fig. 2. Bifurcation versus the amplitude of the periodic perturbation.



(a) 1-order Galerkin truncation

(b) 2-order Galerkin truncation

(c) 3-order Galerkin truncation

(d) 4-order Galerkin truncation

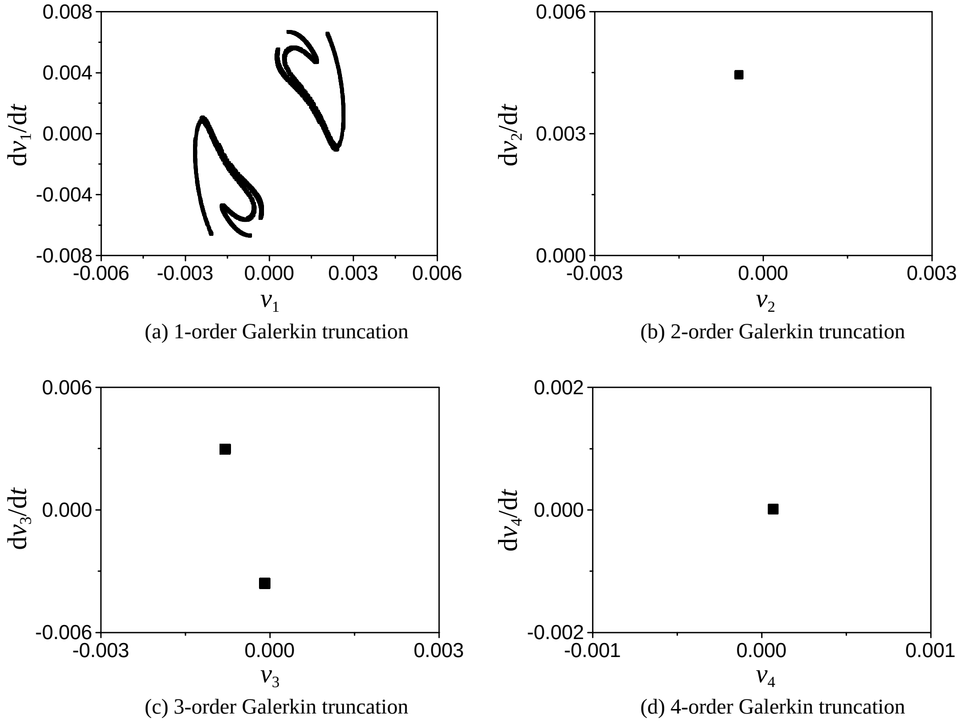
Fig. 3. Bifurcation versus the dynamic viscosity.

small enough, and the period-doubling bifurcation and the tangent bifurcation appear before the explosive bifurcation with the increase of α. However, there are significant differences among the bifurcation of the first, second, third and fourth order truncated systems, especially between the first and third order truncated systems and the second and fourth order counterparts.

The bifurcation of the non-dimensional center displacement v_i $(i = 1, 2, 3, 4)$, given by Eqs. (12)–(15), of the string against the non-dimensional dynamic viscosity E_V for fixed $\gamma = 0.5$ and $\alpha = 0.9$ is shown in Fig. 3. In this case, for $E_V < 25$ chaotic motion occurs. With the increase of E_V, the inverse explosive bifurcation occurs. The strange attractor suddenly disappears. For $E_V > 225$, the period-2 motion appears in the first and third order truncated systems, and the period-4 motion appears in the second and fourth order truncated systems. However, the bifurcation diagrams of first, second, third, and fourth order truncated systems are different for $25 < E_V < 225$. There is a fine structure of a cascade of bifurcation in Fig. 3(b), but the structure is not found in Fig. 3(a), (c) and (d) even if the local magnifications are made.

4. Periodic, quasi-periodic and chaotic vibrations

The bifurcation diagrams manifest that the transverse vibrations of axially moving strings may be regular or chaotic, depending on the transport speed, the amplitude of the periodic



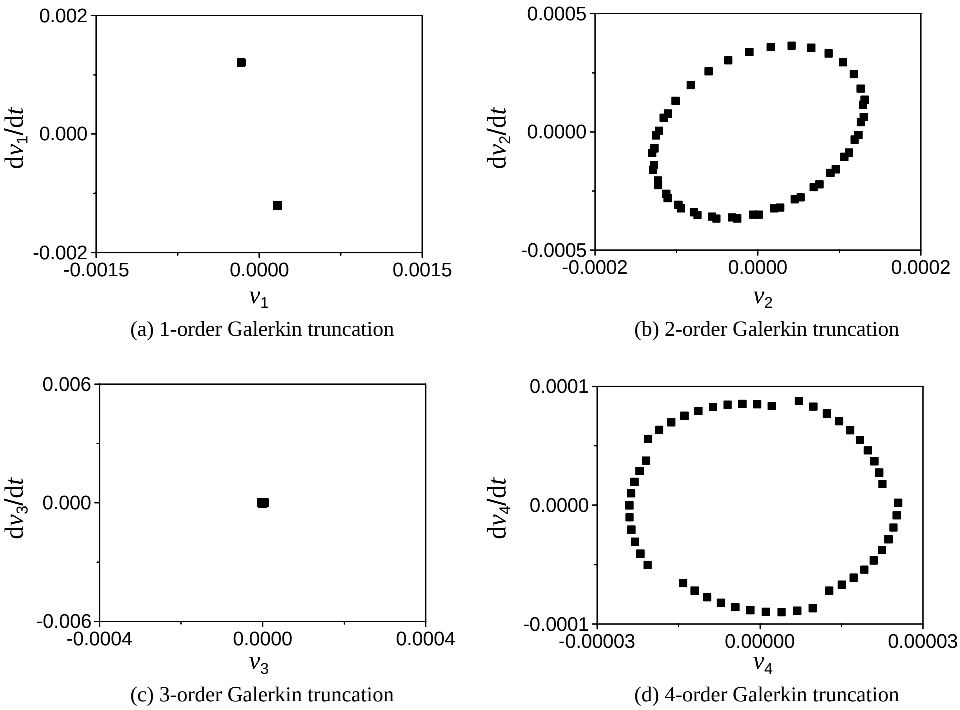
(a) 1-order Galerkin truncation

(b) 2-order Galerkin truncation

(c) 3-order Galerkin truncation

(d) 4-order Galerkin truncation

Fig. 4. Poincaré map of a periodic motion.



(a) 1-order Galerkin truncation

(b) 2-order Galerkin truncation

(c) 3-order Galerkin truncation

(d) 4-order Galerkin truncation

Fig. 5. Poincaré map of a quasi-periodic motion.

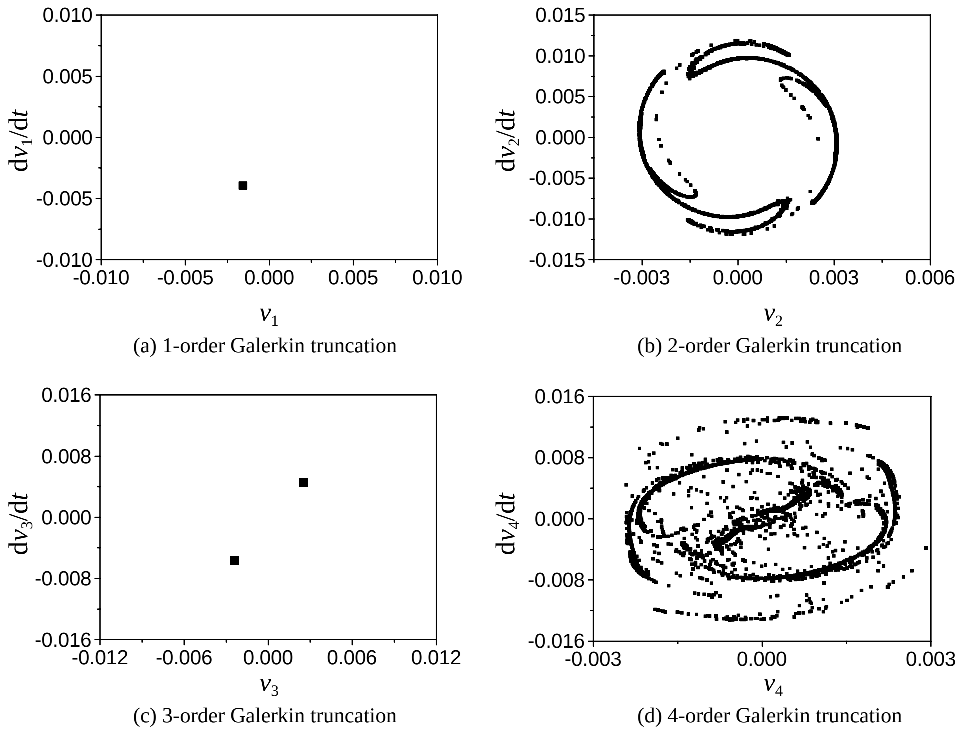


Fig. 6. Poincaré map of a chaotic motion.

perturbation or the dynamic viscosity. Particularly, there exist three types of motion, namely, periodic, quasi-periodic, and chaotic. Examples of those motions are presented as following, and the Poincaré maps are applied to identify the dynamical behaviors. The Poincaré map of a periodic motion is depicted in Fig. 4, in which $\gamma = 0.5$, $\alpha = 0.817$, $\omega = 0.125$, $E_e = 400$ and $E_V = 50$. The Poincaré map of a quasi-periodic motion is depicted in Fig. 5, in which $\gamma = 0.5$, $\alpha = 0.75$, $\omega = 0.125$, $E_e = 400$ and $E_V = 50$. The Poincaré map of a chaotic motion is depicted in Fig. 6, in which $\gamma = 0.54$, $\alpha = 0.9$, $\omega = 0.125$, $E_e = 400$ and $E_V = 120$.

It should be remarked that there is qualitative disagreement among the different order Galerkin truncations. The even order truncations are receivable because the gyroscopic coupling is taken into consideration.

5. Conclusions

Regular and chaotic vibrations of an axially moving viscoelastic string are investigated in this paper. The transverse motions of the string are governed by a non-linear partial-differential equation of motion. The Galerkin method is applied to simplify the equation into first, second, third, and fourth order truncated systems, respectively, defined by a set of ordinary differential equations. After the solutions of those differential equations are numerically calculated, the Poincaré maps are constructed to classify the vibrations. The bifurcation diagrams are obtained in the case that the transport speed, the amplitude of the periodic perturbation or the dynamic viscosity is, respectively, varied while other parameters are fixed. Examples of periodic, quasi-periodic, and chaotic vibrations are presented. Unlike the cases of non-linear viscoelastic structures such as columns [17] and beams [18] previously studied, the dynamical behaviors of the different order truncated systems may not be the same. To understand fully the non-linear dynamics of transverse vibrations of an axially moving viscoelastic string, the higher order truncation and the better trial function are needed, and confirmations through the direct numerical methods to solve the non-linear partial-differential equation had better been done.

Acknowledgements

The research is supported by the National Natural Science Foundation of China (Project No. 10172056).

References

[1] C.D. Mote Jr., Dynamic stability of axially moving materials, Shock and Vibration Digest 4 (1972) 2–11.
[2] J.A. Wickert, C.D. Mote Jr., Current research on the vibration and stability of axially moving materials, Shock and Vibration Digest 20 (1988) 3–13.
[3] A.S. Abrate, Vibration of belts and belt drives, Mechanism and Machine Theory 27 (1992) 645–659.
[4] L.-Q. Chen, J.W. Zu, Transverse vibration of axially moving strings and its control, Advance in Mechanics 31 (2001) 535–546.
[5] S. Mahalingam, Transverse vibrations of power transmission chains, British Journal of Applied Physics 8 (1957) 145–148.
[6] J.S. Huang, R.F. Fung, C.H. Lin, Dynamic stability of a moving string undergoing three-dimensional vibration, International Journal of Mechanical Science 37 (1995) 145–160.
[7] E.M. Mochensturm, N.C. Perkins, A.G. Ulsoy, Stability and limit cycles of parametrically excited, axially moving strings, Journal of Vibration and Acoustics 116 (1996) 346–351.
[8] M. Pakdemirli, H. Batan, Dynamic stability of a constantly accelerating string, Journal of Sound and Vibration 168 (1993) 371–378.
[9] M. Pakdemirli, A.G. Ulsoy, A. Ceranoglu, Transverse vibration of an axially accelerating string, Journal of Sound and Vibration 169 (1994) 179–196.
[10] M. Pakdemirli, A.G. Ulsoy, Stability analysis of an axially accelerating string, Journal of Sound and Vibration 203 (1997) 815–832.
[11] R.F. Fung, J.S. Huang, Y.C. Chen, The transient amplitude of the viscoelastic travelling string: an integral constitutive law, Journal of Sound and Vibration 201 (1997) 153–167.
[12] L. Zhang, J.W. Zu, Nonlinear vibration of parametrically excited moving belts, part 1: dynamic response, Journal of Applied Mechanics 66 (1999) 396–402.
[13] L. Zhang, J.W. Zu, Nonlinear vibration of parametrically excited moving belts, part 2: stability analysis, Journal of Applied Mechanics 66 (1999) 403–409.
[14] W.J. Zhao, L.Q. Chen, A numerical algorithm for non-linear parametric vibration analysis of a viscoelastic moving belt, International Journal of Nonlinear Science and Numerical Simulations 3 (2002) 129–134.
[15] F.C. Moon, P.J. Holmes, Amagnetoelastic strange attractor, Journal of Sound and Vibration 65 (1979) 285–296.
[16] N.S. Abhyankar, E.K. Hall, S.V. Hanagud, Chaotic vibrations of beams: numerical solution of partial differential equation, Journal of Applied Mechanics 60 (1993) 167–178.
[17] L.Q. Chen, C.J. Cheng, Dynamical behavior of nonlinear viscoelastic beams, Applied Mathematics and Mechanics 21 (2000) 995–1001.
[18] L.Q. Chen, C.J. Cheng, Dynamical behavior of nonlinear viscoelastic columns based on 2-order Galerkin truncation, Mechanics Research Communications 27 (2000) 413–419.