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ACADEMIC PRESS
Journal of Sound and Vibration 261 (2003) 775–790
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

The sound power of an individual mode of a clamped–free annular plate

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Received 8 October 2001; accepted 8 May 2002

Abstract

The main aim of this study is an analysis of sound radiation of some clamped–free and free–clamped annular plates. The plates are embedded in some infinite perfectly rigid baffles and vibrate with the free-field condition satisfied. The Kirchhoff–Love linear theory of a perfectly elastic plate is used to solve the plates' equations of motion. The sound pressure at the plates' surfaces is expressed by its Hankel transform. The closed-path integral technique known from the literature and the stationary-phase method are used to find the standardized active and reactive sound power of an individual mode in the form of some high-frequency asymptotic formulae useful for some highly efficient engineering computations. Their non-oscillating and oscillating parts have been separated. It is assumed that the vibration and sound radiation are axisymmetric and time harmonic. Low fluid loading is also assumed. The complex sound power of an individual mode forms the basis for computing the total sound power of some excited and damped vibrations in the fluid. The total sound power is not discussed in this study.

1. Introduction

The problem of sound radiation of an acoustic system consisting of flat plates embedded in baffles is very important for several industrial applications, e.g., some reducing device systems in petroleum industry.

Exact expressions for the sound power were reported only for pistons of different shape embedded in baffles (cf., Refs. [1–4]). An exact analysis, in case of more complex sound sources such as plates, is very difficult or even impossible. Therefore, there are few studies dealing with approximate expressions for the sound power radiated by flat plates embedded in baffles. The

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PII: S0022-460X(02)00984-7

problem of sound radiation of a circular plate embedded in an infinite or finite baffle was solved by Ginsberg et al. [5]. A discussion on the vibrations of an elastically supported rectangular plate together with its directivity pattern were given by Lomas and Hayek [6] and the integral formulations for the radiation efficiency of the plate were presented by Berry et al. [7]. The equivalent area method was used to estimate the radiation efficiency of a clamped circular plate by Czarnecki et al. [8]. A comprehensive approach to find the radiation efficiency of a spinning annular disk was proposed by Lee and Singh [9]. A closed-path integral technique was used to derive the radiation efficiency of a clamped circular plate by Levine and Leppington [10]. The technique was applied by Rdzanek to provide some precise estimates of the sound power of an individual mode valid for the high frequencies [11]. The technique was further developed by Rdzanek et al. [12] to deal with the magnitude in the case of a simply supported circular plate and Rdzanek Jr. and Engel [13] for a clamped annular plate. The method of finding some of the high-frequency asymptotics for the sound power radiated by a clamped–free annular plate was signalled in Ref. [14]. The authors limited their considerations to the active sound power.

So far, to the best of the present author's knowledge, there have been no results reported on the high-frequency asymptotics of the active sound power of a free–clamped annular plate or on the reactive sound power in the case of clamped–free and free–clamped annular plates. Moreover, no comparative analysis of the sound power radiated by the clamped–free or free–clamped plates with that radiated by a clamped annular plate in the high frequency has been presented. Therefore, the main aim of this paper is to fill this literature gap by investigating the high-frequency asymptotics for the complex sound power of an individual in vacuo mode of the two different annular plates. One plate is clamped on the inside and completely free on the outside, and the other completely free on the inside and clamped on the outside. Further they will be referred to as the clamped–free plate and the free–clamped plate, respectively. Low fluid loading is assumed. A continuation of the closed-path integral technique is presented to deal with the sound radiation of both plates. First, the magnitudes are expressed using Hankel's transform to make possible their further integration. Based on the technique, asymptotic formulations for the active and reactive sound power radiated by the plates are derived whose non-oscillating and oscillating parts have been separated. The asymptotics are valid for the high frequencies, and make possible some fast engineering computations. Since the sound power of an individual mode makes the main contribution to the total sound power of the excited plate coupled with fluid, it can be used as a basis to derive the total sound power, not to be discussed herein. The total sound power of a clamped–clamped annular plate was given in Ref. [15].

2. Analysis assumptions

A thin annular plate is embedded in a perfectly rigid and infinite baffle. Two different configurations of the plate's boundaries are considered: (a) the internal edge is clamped and the external edge is free, (b) the opposite situation, further referred to as the clamped–free configuration and the free–clamped configuration, respectively. Based on the Kirchhoff–Love theory of a perfectly elastic plate, a modal analysis of its free vibration is carried out. It is assumed that all the configurations of the plate are axisymmetric, i.e., the plate's material is homogeneous, the plate's boundary conditions and the plate's geometric dimensions do not depend on the angle variable and the plate is excited by an axisymmetric force. Therefore, all the plate's magnitudes, as, e.g., the nth mode shape or a function characterizing the sound radiation associated with the nth mode, will be dependent on the radial variable only. The Kirchhoff–Love theory describes a linear model of the plate and therefore, in order to correctly express the plate's responses to some band-limited excitations, it will be enough to consider the mono-chromatic wave generated by the plate. Consequently, most of the magnitudes will be written in the amplitude form. Using the Kirchhoff–Love linear theory implies that the plate's thickness h and the amplitude of its transverse deflection $\eta(r)$ are small when compared with the remaining geometric dimensions of the plate such as the radii of its inner and outer edges, r_{in} and r_{out}, respectively. The nth mode shape can be written in the form of $v_n(r) = -\mathrm{i}\omega_n\eta_n(r)$ for some time-harmonic and axisymmetric processes, where $v_n(r,t) = v_n(r)\exp(-\mathrm{i}\omega t)$ and amplitude $\eta_n(r)$ is the solution of the homogeneous equation of the plate's motion written as

$$(k_n^{-4}\nabla_r^4 - 1)\eta_n(r) = 0. \tag{1}$$

The following denotations are used: $r \in [r_{in}, r_{out}]$ is the distance of the plate's point from its center, $n = 0, 1, 2, \ldots$ is the plate's individual mode number, $k_n^2 = \omega_n\sqrt{\varrho h/B}$ is the nth structural wavenumber raised to the second power, ω_n is the nth eigenfrequency, ϱ, E are the density and the Young's modulus of the plate, respectively, $B = Eh^3/[12(1-\nu^2)]$ is the plate's bending stiffness, ν is its Poisson's ratio, $\nabla_r^4 = (\partial^2/\partial r^2 + r^{-1}\partial/\partial r)^2$. The solution of Eq. (1) is predicted in the form (cf., Refs. [13,14,16–18])

$$\eta_n(r) = A_n[\mathrm{J}_0(k_n r) + B_n\mathrm{I}_0(k_n r) - C_n\mathrm{N}_0(k_n r) - D_n\mathrm{K}_0(k_n r)], \tag{2}$$

where A_n, B_n, C_n, D_n are the constants given in Ref. [19], $\mathrm{J}_0, \mathrm{I}_0, \mathrm{N}_0, \mathrm{K}_0$ are the Bessel's, modified Bessel's, Neumann's and McDonald's functions of zero order, respectively. The plate's boundary configurations imply that the value of solution $\eta_n(r)$ and its first order radial derivative are equal to zero for the clamped edge, thus

$$\eta_n(r_c) = 0, \quad \mathrm{d}\eta_n(r)/\mathrm{d}r|_{r=r_c}, \tag{3}$$

where $r_c = r_{in}$ or $r_c = r_{out}$ for the boundary configurations (a) and (b), respectively. The bending moment m_r and the transversely acting force q_r are equal to zero for the free edge

$$m_r(r_f) = -B\left[\frac{\mathrm{d}^2\eta_n(r)}{\mathrm{d}r^2} + (\nu/r)\frac{\mathrm{d}\eta_n(r)}{\mathrm{d}r}\right]_{r=r_f} = 0,$$
$$q_r(r_f) = -B\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}\eta_n(r)}{\mathrm{d}r}\right]\right)\right]_{r=r_f} = 0, \tag{4a, b}$$

where $r_f = r_{out}$ or $r_f = r_{in}$ for the boundary configurations (a) and (b), respectively (cf., Refs. [9,12–14,17,18]).

Inserting Eq. (2) into the boundary conditions (3) and (4) it can be deduced that for some clamped–free or free–clamped annular plates

$$C_n = \frac{r_f S(k_n r_f) + r_c S(k_n r_c) - 2\nu_n \mathrm{I}_1(k_n r_f)\mathrm{J}_1(k_n r_f)}{r_f T(k_n r_f) + r_c T(k_n r_c) - 2\nu_n \mathrm{I}_1(k_n r_f)\mathrm{N}_1(k_n r_f)}, \tag{5}$$

where $v_n = (1 - \nu)/k_n$, and

$$S(x) = \mathrm{J}_1(x)\mathrm{I}_0(x) + \mathrm{J}_0(x)\mathrm{I}_1(x), \quad T(x) = \mathrm{N}_1(x)\mathrm{I}_0(x) + \mathrm{N}_0(x)\mathrm{I}_1(x). \tag{6}$$

Deriving constant C_n together with the frequency equation of the plate and the corresponding eigenvalues were presented in detail in Ref. [19]. In this paper the nth eigenvalue is denoted by $k_n r_c$ for some clamped–free plates or by $k_n r_f$ for some free–clamped plates (cf., Refs. [9,13,17–19]). Constant A_n can be derived from the standardization condition (cf., Ref. [20])

$$A_n^{-2} = \frac{2}{r_f^2 - r_c^2}\left\{ r_f^2 G_0^2(k_n r_f) - r_c^2 G_0^2 - 2v_n G_1(k_n r_f)\left[r_f G_0(k_n r_f) + \frac{\nu}{k_n} G_1(k_n r_f)\right]\right\}, \tag{7}$$

where $G_0(u) = \mathrm{J}_0(u) - C_n \mathrm{N}_0(u)$ and $G_1(u) = \mathrm{J}_1(u) - C_n \mathrm{N}_1(u)$.

The values of the constants B_n and D_n are not necessary for the further analysis of the standardized sound power and they both have been presented in Refs. [14,19].

3. Integral formulae

The impedance approach to derive the plate's sound power requires that the nth mode shape $v_n(r)$ and the sound pressure $p_n(r)$ of the mode be known. The mode shape defined by $v_n(r) = -\mathrm{i}\omega_n \eta_n(r)$ and Eq. (2) is given in the previous section. The sound pressure $p_n(r)$ has to be derived to make possible the use of the definition for the sound power of the mode (cf., Refs. [21,22])

$$\Pi_n = \frac{1}{2}\int_S p_n v_n^* \,\mathrm{d}S, \tag{8}$$

where S is a surface that encloses the plate near its surface, $v_n^* = \mathrm{i}\omega_n \eta_n$ is the nth conjugate mode shape and $\mathrm{i} = \sqrt{-1}$.

3.1. Sound radiation

It is not possible to compute the sound power immediately from Eq. (8) using the impedance approach and therefore it is necessary to express the sound power in its Hankel representation (cf., Refs. [10,12,13]). For this purpose, the sound pressure at the plate's surface $p_n(r)$ associated with nth mode will first be expressed by the Hankel transform as

$$p_n(r) = k\varrho_0 c \int_0^{+\infty} W_n(\tau)\mathrm{J}_0(\tau r)\frac{\tau\,\mathrm{d}\tau}{\sqrt{k^2 - \tau^2}}, \tag{9}$$

where ϱ_0 is the density of the surrounding air column, c is the sound velocity in air, $k = 2\pi/\lambda$ is the acoustic wavenumber, λ is the radiated wavelength and $W_n(\tau)$ is a function characterizing sound radiation associated with the nth mode defined by (cf., Refs. [12,13])

$$W_n(\tau) = \int_{r_c}^{r_f} v_n(r)\mathrm{J}_0(\tau r) r\,\mathrm{d}r, \tag{10}$$

where τ is the complex wavenumber (cf., Refs. [23,24]). For some further analysis of the sound radiation using the impedance approach it is necessary to express integral (10) in its elementary form by substituting $k\sin\vartheta$ for $\boldsymbol{\tau}$ and by denoting $u = k\sin\vartheta$ to give

$$W_n(u) = \mathrm{i}\omega_n 2A_n w_n(u), \tag{11}$$

where

$$\begin{aligned} w_n(u) = & \frac{k_n^2}{k_n^4 - u^4}[k_n r_f G_1(k_n r_f)\mathrm{J}_0(r_f u) - r_f u G_0(k_n r_f)\mathrm{J}_1(r_f u) \\ & - k_n r_c G_1(k_n r_c)\mathrm{J}_0(r_c u) + r_c u G_0(k_n r_c)\mathrm{J}_1(r_c u)] \\ & - \frac{1}{k_n^2 + u^2}\{k_n r_f G_1(k_n r_f)\mathrm{J}_0(r_f u) - u[r_f G_0(k_n r_f) - v_n G_1(k_n r_f)]\mathrm{J}_1(r_f u)]\}, \end{aligned} \tag{12}$$

which is valid for both boundary configurations. Functionals $G_0(x) = \mathrm{J}_0(x) - C_n\mathrm{N}_0(x)$ and $G_1(x) = \mathrm{J}_1(x) - C_n\mathrm{N}_1(x)$, for any real $x \in \{k_n r_f, k_n r_c\}$, may be convenient for describing vibration and sound radiation of some planar annular plates (cf., Refs. [13,14,20]).

3.2. *The sound power*

The sound power of an individual in vacuo mode is analyzed below for some axisymmetric vibrations. The sound power is defined by (cf., Refs. [10–14])

$$\Pi_n = \pi\varrho_0 c k^2 \int_0^{\pi/2 - \mathrm{i}\infty} W_n(k\sin\vartheta) W_n^*(k\sin\vartheta)\sin\vartheta\,\mathrm{d}\vartheta, \tag{13}$$

where $\vartheta = \vartheta' + \mathrm{i}\vartheta'' \in \mathbb{C}$, ϑ', $\vartheta'' \in \mathbb{R}$. The integration path used in Eq. (13) was shown in Ref. [12]. The reference sound power of the nth mode, obtained for $k \to \infty$,

$$\Pi_n^{(\infty)} = (\pi/2)\varrho_0 c\omega_n (r_{out}^2 - r_{in}^2) \tag{14}$$

is used to standardize Eq. (13) as

$$\mathscr{P}_n = \mathscr{P}_{a,n} - \mathrm{i}\mathscr{P}_{r,n} = \frac{\Pi_n}{\Pi_n^{(\infty)}} = 8A_n^2 \frac{k^2 r_{in}^4}{r_{out}^2 - r_{in}^2} \int_0^{\pi/2 - \mathrm{i}\infty} w_n^2(k\sin\vartheta)\sin\vartheta\,\mathrm{d}\vartheta, \tag{15}$$

where the symbols $\mathscr{P}_{a,n}$ and $\mathscr{P}_{r,n}$ denote the standardized active and reactive sound power, respectively.

4. Asymptotics for the high frequencies

A number of acoustic systems are excited by frequencies much lower than their lowest eigenfrequencies. The values of eigenfrequencies of some planar plates are relatively low (cf., Ref. [18]) and the systems can also vibrate within their high frequency ranges (cf., Refs. [10,12–14]).

By substituting x for $\sin\vartheta'$ and 0 for ϑ'', Eq. (12) assumes the form of

$$w_n(kx) = \frac{\delta_n^2}{k r_{in}^2}\left[\frac{\xi_n(x)}{\delta_n^4 - x^4} - \frac{1}{\delta_n^2}\frac{\zeta_n(x)}{\delta_n^2 + x^2}\right], \tag{16}$$

where $\delta_n = k_n/k$ and the following denotations are used:

$$\xi_n(x) = \delta_n[r_f G_1(k_n r_f)\mathrm{J}_0(kr_f x) - r_c G_1(k_n r_c)\mathrm{J}_0(kr_c x)] - x[r_f G_0(k_n r_f)\mathrm{J}_1(kr_f x) - r_c G_0(k_n r_c)\mathrm{J}_1(kr_c x)], \tag{17a}$$

$$\zeta_n(x) = \delta_n r_f G_1(k_n r_f)\mathrm{J}_0(kr_f x) - x[r_f G_0(k_n r_f) - v_n G_1(k_n r_f)]\mathrm{J}_1(kr_f x). \tag{17b}$$

Integral (15) can be transformed to

$$\mathscr{P}_{a,n} = \frac{8A_n^2\delta_n^4}{r_f^2 - r_c^2}\int_0^1 \left[\frac{\xi_n(x)}{\delta_n^4 - x^4} - \frac{1}{\delta_n^2}\frac{\zeta_n(x)}{\delta_n^2 + x^2}\right]^2 \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} \tag{18}$$

for the active sound power, and

$$\mathscr{P}_{r,n} = \frac{8A_n^2\delta_n^4}{r_f^2 - r_c^2}\int_1^\infty \left[\frac{\xi_n(x)}{\delta_n^4 - x^4} - \frac{1}{\delta_n^2}\frac{\zeta_n(x)}{\delta_n^2 + x^2}\right]^2 \frac{x\,\mathrm{d}x}{\sqrt{x^2-1}} \tag{19}$$

for the reactive sound power.

4.1. The clamped–free boundary conditions

This subsection focuses on some results obtained from integrals (18) and (19) valid for an annular plate clamped on the inside and completely free on the outside. The method described in detail in Ref. [20] has been applied to this situation and after considerable algebraic manipulation it was found that, for the clamped–free boundary conditions, the non-oscillating active sound power for an individual *in vacuo* mode is

$$\bar{\mathscr{P}}_{a,n} = \frac{1}{\sqrt{1+\delta_n^2}} + \frac{q_n}{2}\left(\frac{1}{\sqrt{1-\delta_n^2}} - \frac{1}{\sqrt{1+\delta_n^2}}\right)\left\{1 + a_n^2(k_n r_f) - 2\frac{\varkappa_n^2(k_n r_f)}{1-v} - d_n^2[1 + a_n^2(k_n r_c)]\right\}, \tag{20}$$

where the following denotations and functionals, in terms of $u \in \mathbb{R}$, are introduced:

$$a_n(u) = G_1(u)/G_0(u), \quad d_n = r_c G_0(k_n r_c)/r_f G_0(k_n r_f), \quad \varkappa_n(u) = ((1-v)/u)a_n(u), \tag{21a}$$

$$q_n^{-1} = \left[\frac{2A_n^2 r_f^2}{r_f^2 - r_c^2} G_0^2(k_n r_f)\right]^{-1} = 1 - d_n^2 - 2\varkappa_n(k_n r_f)\left[1 + \frac{v}{k_n r_f} a_n(k_n r_f)\right] \tag{21b}$$

and the oscillating active sound power is

$$\tilde{\mathscr{P}}_{a,n} = \frac{2q_n}{k\sqrt{\pi k}} \frac{\delta_n^4}{(1+\delta_n^2)^2} \Bigg\{ (b_0^2 - b_1^2) \frac{\cos w_c}{r_c\sqrt{r_c}} + 2b_0 b_1 \frac{\sin w_c}{r_c\sqrt{r_c}} + (h_0^2 - h_1^2) \frac{\cos w_f}{r_f\sqrt{r_f}} + 2h_0 h_1 \frac{\sin w_f}{r_f\sqrt{r_f}} + \frac{2\sqrt{2}}{\sqrt{r_f r_c}} \Bigg[(h_1 b_0 - h_0 b_1) \frac{\cos w_{cf,-}}{\sqrt{r_f - r_c}} - (h_1 b_1 + h_0 b_0) \frac{\sin w_{cf,-}}{\sqrt{r_f - r_c}} + (h_1 b_1 - h_0 b_0) \frac{\cos w_{cf,+}}{\sqrt{r_f + r_c}} - (h_1 b_0 + h_0 b_1) \frac{\sin w_{cf,+}}{\sqrt{r_f + r_c}} \Bigg] \Bigg\}, \tag{22}$$

where

$$w_c = 2kr_c + \pi/4, \quad w_f = 2kr_f + \pi/4,$$
$$w_{cf,-} = (r_f - r_c)k + \pi/4, \quad w_{cf,+} = (r_f + r_c)k + \pi/4, \tag{23a}$$

$$b_0 = \frac{d_n}{(1-\delta_n^2)}, \; b_1 = \delta_n a_n(k_n r_c) b_0, \quad h_0 = \delta_n^2 \left[\frac{1}{(1-\delta_n^2)} - \varkappa_n(k_n r_f) \right],$$
$$h_1 = \frac{a_n(k_n r_f)}{\delta_n(1-\delta_n^2)}. \tag{23b}$$

The active sound power is the sum of the two parts together with the approximation error (cf., Ref. [12])

$$\mathscr{P}_{a,n} = \bar{\mathscr{P}}_{a,n} + \tilde{\mathscr{P}}_{a,n} + \mathscr{O}[\delta_n^4/(kr_{in})^{3/2}]. \tag{24}$$

The reactive sound power is computed differently, as described in detail in Ref. [20]. The method used results in the non-oscillating reactive sound power in the form of

$$\bar{\mathscr{P}}_{r,n} = \frac{q_n}{\pi k} \left[\frac{\alpha_{cf,1}}{(1-\delta_n^4)} + \frac{\alpha_{cf,2} \operatorname{arcsin} \delta_n}{2\delta_n(1-\delta_n^2)^{3/2}} + \frac{\alpha_{cf,3} \operatorname{arcsinh} \delta_n}{2\delta_n(1+\delta_n^2)^{3/2}} \right], \tag{25}$$

where

$$\alpha_{cf,1} = \delta_n^2 \left[\frac{a_n^2(k_n r_f)}{r_f} + d_n^2 \frac{a_n^2(k_n r_c)}{r_c} \right] + \left(\frac{1}{r_f} + \frac{d_n^2}{r_c} \right) - 2\frac{1-\delta_n^2}{r_f} \varkappa_n(k_n r_f)[1 - \varkappa_n(k_n r_f)], \tag{26a}$$

$$\alpha_{cf,2} = \frac{a_n^2(k_n r_f)}{r_f} - d_n^2 \frac{3-4\delta_n^2}{r_c} a_n^2(k_n r_c) + \frac{3-2\delta_n^2}{r_f} - d_n^2 \frac{1-2\delta_n^2}{r_c} - 4\frac{1-\delta_n^2}{r_f} \varkappa_n(k_n r_f), \tag{26b}$$

$$\alpha_{cf,3} = d_n^2 \frac{3+4\delta_n^2}{r_c} a_n^2(k_n r_c) - \frac{a_n^2(k_n r_f)}{r_f} + \frac{3+2\delta_n^2}{r_f} - d_n^2 \frac{1+2\delta_n^2}{r_c} - 4\frac{\varkappa_n(k_n r_f)}{r_f}[1 - \varkappa_n(k_n r_f) + 1 + \delta_n^2]. \tag{26c}$$

The oscillating reactive sound power has been computed in an analogous way as in the case of the oscillating part of the active sound power to give (cf., Ref. [20])

$$\tilde{\mathscr{P}}_{r,n} = -\frac{2q_n}{k\sqrt{\pi k}}\frac{\delta_n^4}{(1+\delta_n^2)^2}\left\{\left[(b_0^2-b_1^2)\frac{\sin w_c}{r_c\sqrt{r_c}} - 2b_0b_1\frac{\cos w_c}{r_c\sqrt{r_c}} + (h_0^2-h_1^2)\frac{\sin w_f}{r_f\sqrt{r_f}}\right.\right.$$
$$\left. -2h_0h_1\frac{\cos w_f}{r_f\sqrt{r_f}}\right] + \frac{2\sqrt{2}}{\sqrt{r_fr_c}}\left[(h_1b_0-h_0b_1)\frac{\sin w_{cf,-}}{\sqrt{r_f-r_c}} + (h_1b_1+h_0b_0)\frac{\cos w_{cf,-}}{\sqrt{r_f-r_c}}\right.$$
$$\left.\left. +(h_1b_1-h_0b_0)\frac{\sin w_{cf,+}}{\sqrt{r_f+r_c}} + (h_1b_0+h_0b_1)\frac{\cos w_{cf,+}}{\sqrt{r_f+r_c}}\right]\right\}. \tag{27}$$

The reactive sound power is the sum of the two parts together with the approximation error (cf., Eq. (24) valid for the active sound power or Ref. [12])

$$\mathscr{P}_{r,n} = \bar{\mathscr{P}}_{r,n} + \tilde{\mathscr{P}}_{r,n} + \mathscr{O}[\delta_n^4/(kr_{in})^{3/2}]. \tag{28}$$

4.2. *The free–clamped boundary conditions*

A procedure analogous to that used earlier leads to the following asymptotics for the non-oscillating active sound power for an individual *in vacuo* mode of an annular plate completely free on the inside and clamped on the outside

$$\bar{\mathscr{P}}_{a,n} = \frac{1}{\sqrt{1+\delta_n^2}}$$
$$+\frac{q_n}{2}\left(\frac{1}{\sqrt{1-\delta_n^2}} - \frac{1}{\sqrt{1+\delta_n^2}}\right)\left\{1 + a_n^2(k_nr_c) - d_n^2\left[1 + a_n^2(k_nr_f) - 2\frac{\varkappa_n^2(k_nr_f)}{(1-\nu)}\right]\right\} \tag{29}$$

and for the oscillating active sound power

$$\tilde{\mathscr{P}}_{a,n} = \frac{2q_n}{k\sqrt{\pi k}}\frac{\delta_n^4}{(1+\delta_n^2)^2}\left\{(b_0^2-b_1^2)\frac{\cos w_c}{r_c\sqrt{r_c}} + 2b_0b_1\frac{\sin w_c}{r_c\sqrt{r_c}} + (h_0^2-h_1^2)\frac{\cos w_f}{r_f\sqrt{r_f}}\right.$$
$$+ 2h_0h_1\frac{\sin w_f}{r_f\sqrt{r_f}} - \frac{2\sqrt{2}}{r_cr_f}\left[(h_1b_0-h_0b_1)\frac{\cos w_{fc,-}}{\sqrt{r_c-r_f}} + (h_1b_1+h_0b_0)\frac{\sin w_{fc,-}}{\sqrt{r_c-r_f}}\right.$$
$$\left.\left. - (h_1b_1-h_0b_0)\frac{\cos w_{fc,+}}{\sqrt{r_c+r_f}} + (h_1b_0+h_0b_1)\frac{\sin w_{fc,+}}{\sqrt{r_c+r_f}}\right]\right\}, \tag{30}$$

where

$$w_c = 2kr_c + \pi/4, \quad w_f = 2kr_f + \pi/4,$$
$$w_{cf,-} = (r_c - r_f)k + \pi/4, \quad w_{cf,+} = (r_c + r_f)k + \pi/4, \tag{31a}$$

$$d_n = \frac{r_f G_0(k_n r_f)}{r_c G_0(k_n r_c)}, \quad q_n^{-1} = 1 - d_n^2 \left\{ 1 - 2\varkappa_n(k_n r_f) \left[1 + \frac{v}{k_n r_f} a_n(k_n r_f) \right] \right\}, \tag{31b}$$

$$b_0 = \frac{1}{1-\delta_n^2}, \quad b_1 = \delta_n a_n(k_n r_c) b_0, \quad h_0 = \frac{d_n}{\delta_n^2} \left[\frac{1}{(1-\delta_n^2)} - \varkappa_n(k_n r_f) \right],$$
$$h_1 = \frac{d_n a_n(k_n r_f)}{\delta_n (1-\delta_n^2)}. \tag{31c}$$

Both parts, i.e., non-oscillating and oscillating, are summed using Eq. (24).

The non-oscillating reactive sound power assumes the form of

$$\bar{\mathscr{P}}_{r,n} = \frac{q_n}{\pi k} \left[\frac{\alpha_{fc,1}}{(1-\delta_n^4)} + \frac{\alpha_{fc,2} \arcsin \delta_n}{2\delta_n (1-\delta_n^2)^{3/2}} + \frac{\alpha_{fc,3} \operatorname{arcsinh} \delta_n}{2\delta_n (1+\delta_n^2)^{3/2}} \right] \tag{32}$$

with denotations

$$\alpha_{fc,1} = \delta_n^2 \left[\frac{a_n^2(k_n r_c)}{r_c} + d_n^2 \frac{a_n^2(k_n r_f)}{r_f} \right] + \frac{1}{r_c} + \frac{d_n^2}{r_f} - 2 d_n^2 \frac{1-\delta_n^2}{r_f} \varkappa_n(k_n r_f)[1 - \varkappa_n(k_n r_f)], \tag{33a}$$

$$\alpha_{fc,2} = \frac{d_n^2}{r_f} a_n^2(k_n r_f) - \frac{3 - 4\delta_n^2}{r_c} a_n^2(k_n r_c) + d_n^2 \frac{3 - 2\delta_n^2}{r_f} - \frac{1 - 2\delta_n^2}{r_c} - 4 d_n^2 \frac{1-\delta_n^2}{r_f} \varkappa_n(k_n r_f), \tag{33b}$$

$$\alpha_{fc,3} = \frac{3 + 4\delta_n^2}{r_c} a_n^2(k_n r_c) - \frac{d_n^2}{r_f} a_n^2(k_n r_f) + d_n^2 \frac{3 + 2\delta_n^2}{r_f} - \frac{1 + 2\delta_n^2}{r_c}$$
$$- 4 \frac{d_n^2}{r_f} \varkappa_n(k_n r_f)[2 + \delta_n^2 - \varkappa_n(k_n r_f)]. \tag{33c}$$

The oscillating reactive sound power assumes the form of

$$\begin{aligned} \tilde{\mathscr{P}}_{r,n} = & -\frac{2q_n}{k\sqrt{\pi k}} \frac{\delta_n^4}{(1+\delta_n^2)^2} \left\{ (b_0^2 - b_1^2) \frac{\sin w_c}{r_c \sqrt{r_c}} - 2 b_0 b_1 \frac{\cos w_c}{r_c \sqrt{r_c}} \right. \\ & + (h_0^2 - h_1^2) \frac{\sin w_f}{r_f \sqrt{r_f}} - 2 h_0 h_1 \frac{\cos w_f}{r_f \sqrt{r_f}} \\ & + \frac{2\sqrt{2}}{\sqrt{r_c r_f}} \left[-(h_1 b_0 - h_0 b_1) \frac{\sin w_{fc,-}}{\sqrt{r_c - r_f}} + (h_1 b_1 + h_0 b_0) \frac{\cos w_{fc,-}}{\sqrt{r_c - r_f}} \right. \\ & \left. \left. + (h_1 b_1 - h_0 b_0) \frac{\sin w_{fc,+}}{\sqrt{r_c + r_f}} + (h_1 b_0 + h_0 b_1) \frac{\cos w_{fc,+}}{\sqrt{r_c + r_f}} \right] \right\}. \end{aligned} \tag{34}$$

The non-oscillating and oscillating parts of the reactive sound power should be summed using Eq. (28).

4.3. The clamped–clamped boundaries

The asymptotics valid for the clamped–clamped boundaries are quoted below from Ref. [13] for comparing the three different boundary conditions, i.e., clamped–free, free–clamped and clamped–clamped. The non-oscillating and oscillating active sound power can be expressed by

$$\bar{\mathscr{P}}_{a,n} = \frac{1}{2}\left(\frac{1}{\sqrt{1-\delta_n^2}} + \frac{1}{\sqrt{1+\delta_n^2}}\right) + \frac{q_n}{2}\left(\frac{1}{\sqrt{1-\delta_n^2}} - \frac{1}{\sqrt{1+\delta_n^2}}\right)[a_n^2(k_n r_{out}) - d_n^2 a_n^2(k_n r_{in})] \quad (35)$$

and

$$\begin{aligned}\tilde{\mathscr{P}}_{a,n} = {} & \frac{2q_n}{k\sqrt{\pi k}}\frac{\delta_n^4}{(1+\delta_n^2)^2}\Bigg\{(b_0^2 - b_1^2)\frac{\cos w_{in}}{r_{in}\sqrt{r_{in}}} + 2b_0 b_1 \frac{\sin w_{in}}{r_{in}\sqrt{r_{in}}} \\ & + (h_0^2 - h_1^2)\frac{\cos w_{out}}{r_{out}\sqrt{r_{out}}} + 2h_0 h_1 \frac{\sin w_{out}}{r_{out}\sqrt{r_{out}}} \\ & - \frac{2\sqrt{2}}{\sqrt{r_{out} r_{in}}}\Bigg[(h_1 b_0 - h_0 b_1)\frac{\cos w_{io,-}}{\sqrt{r_{out} - r_{in}}} - (h_1 b_1 + h_0 b_0)\frac{\sin w_{io,-}}{\sqrt{r_{out} - r_{in}}} \\ & + (h_1 b_1 - h_0 b_0)\frac{\cos w_{io,+}}{\sqrt{r_{out} + r_{in}}} - (h_1 b_0 + h_0 b_1)\frac{\sin w_{io,+}}{\sqrt{r_{out} + r_{in}}}\Bigg]\Bigg\},\end{aligned} \quad (36)$$

respectively, where

$$w_{in} = 2kr_{in} + \pi/4, \quad w_{out} = 2kr_{out} + \pi/4, \quad w_{io,-} = (r_c - r_f)k + \pi/4, \quad (37a)$$

$$w_{io,+} = (r_c + r_f)k + \pi/4, \quad d_n = \frac{r_{in} G_0(k_n r_{in})}{r_{out} G_0(k_n r_{out})}, \quad q_n^{-1} = 1 - d_n^2, \quad (37b)$$

$$\begin{aligned} b_0 &= \frac{d_n}{(1-\delta_n^2)}, \quad b_1 = \delta_n a_n(k_n r_{in}) b_0, \\ h_0 &= \frac{1}{(1-\delta_n^2)}, \quad h_1 = \delta_n a_n(k_n r_{out}) h_0. \end{aligned} \quad (37c)$$

The non-oscillating and oscillating reactive sound power can be expressed by

$$\bar{\mathscr{P}}_{r,n} = \frac{q_n}{\pi k}\left[\frac{\alpha_{cc,1}}{(1-\delta_n^4)} - \frac{\alpha_{cc,2} \arcsin \delta_n}{2\delta_n(1-\delta_n^2)^{3/2}} + \frac{\alpha_{cc,3} \operatorname{arcsinh} \delta_n}{2\delta_n(1+\delta_n^2)^{3/2}}\right] \quad (38)$$

and

$$\tilde{\mathscr{P}}_{r,n} = -\frac{2q_n}{k\sqrt{\pi k}}\frac{\delta_n^4}{(1+\delta_n^2)^2}\Bigg\{(b_0^2-b_1^2)\frac{\sin w_{in}}{r_{in}\sqrt{r_{in}}} - 2b_0 b_1\frac{\cos w_{in}}{r_{in}\sqrt{r_{in}}}$$
$$+ (h_0^2-h_1^2)\frac{\sin w_{out}}{r_{out}\sqrt{r_{out}}} - 2h_0 h_1\frac{\cos w_{out}}{r_{out}\sqrt{r_{out}}}$$
$$+ \frac{2\sqrt{2}}{\sqrt{r_{out}r_{in}}}\Bigg[(h_1 b_0 - h_0 b_1)\frac{\sin w_{io,-}}{\sqrt{r_{out}-r_{in}}} + (h_1 b_1 + h_0 b_0)\frac{\cos w_{io,-}}{\sqrt{r_{out}-r_{in}}}$$
$$+ (h_1 b_1 - h_0 b_0)\frac{\sin w_{io,+}}{\sqrt{r_{out}+r_{in}}} + (h_1 b_0 + h_0 b_1)\frac{\cos w_{io,+}}{\sqrt{r_{out}+r_{in}}}\Bigg]\Bigg\}, \quad (39)$$

respectively, where

$$\alpha_{cc,2} = (3-4\delta_n^2)\hat{a}_1 + (1-2\delta_n^2)\hat{a}_0, \quad \alpha_{cc,3} = (3+4\delta_n^2)\hat{a}_1 - (1+2\delta_n^2)\hat{a}_0. \quad (40a)$$

$$\alpha_{cc,1} = \delta_n^2\hat{a}_1 + \hat{a}_0, \quad \hat{a}_1 = \frac{a_n^2(k_n r_{out})}{r_{out}} + d_n^2\frac{a_n^2(k_n r_{in})}{r_{in}}, \quad \hat{a}_0 = \frac{1}{r_{out}} + \frac{d_n^2}{r_{in}}. \quad (40b)$$

The integral formulae for the standardized sound power (18) and (19) are valid for the clamped–clamped boundaries if it is assumed that $\zeta_n(x) = 0$ instead of Eq. (17b).

4.4. Numerical analysis of the asymptotics

Several curves, illustrating the active and reactive sound power have been plotted to make it possible to compare the energetic behavior of an individual mode of an annular plate in the case of the three different boundary conditions, i.e., clamped–free, free–clamped and fully clamped (cf., Figs. 1–3). All the curves are plotted in terms of the acoustic wavenumber, k, related to the first structural wavenumber k_1, i.e., k/k_1, for three different values of the geometric parameter $r_{out}/r_{in} = 1.2, 2, 5$. The eigenvalue $k_1 r_{in}$ was chosen to make it possible to compare the curves valid for the clamped–free boundaries with those valid for the fully clamped ones, where the zero mode does not appear (cf., Ref. [13]).

Some strong mechanical interactions that occur between both edges of the plate can be observed especially for small values of r_{out}/r_{in}, when the edges are located very close to each other and there is a small difference in the values of terms depending on $(r_{out}/r_{in} - 1)\beta + \pi/4$ and $(r_{out}/r_{in} + 1)\beta + \pi/4$. In Figs. 1–3 the integral formulae are represented by the dotted lines and the asymptotic formulae are represented by the solid lines. The results obtained for the integrals and for the asymptotics show a good agreement for k/k_n being slightly greater than 1, i.e., the asymptotics are valid for the high frequencies. Consequently, the approximation error, represented by term $\mathcal{O}[\delta_n^4/(kr_{in})^{3/2}]$ in Eqs. (24) and (28), shows a considerable increase with decreasing k/k_n up to $\pm\infty$ for $k/k_n = 1$.

Several essential differences can be noted in the vibro-radiational behavior of some clamped–free, free–clamped and fully-clamped plates. Firstly, the reactive part of the standardized sound power tends to zero much more slowly with an increase in the value of k/k_n for all the

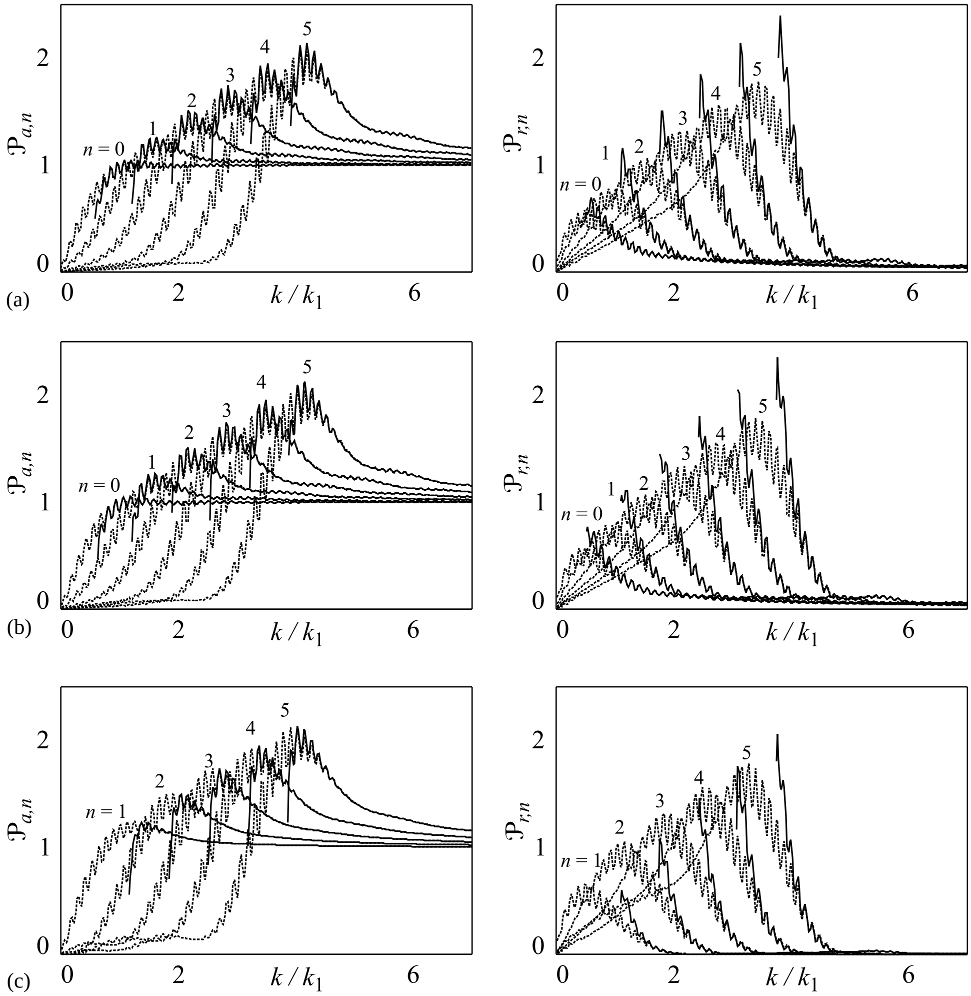

Fig. 1. The standardized active and reactive sound power of an individual *in vacuo* mode $\mathcal{P}_{a,n}$ and $\mathcal{P}_{r,n}$, respectively, for the geometric parameter $r_{out}/r_{in} = 1,2$ and for three different boundary conditions, namely: (a) clamped–free, (b) free–clamped, (c) clamped–clamped (cf., Ref. [13]). All the dotted curves in this figure are obtained from the integral formulae (18) and (19), and all the solid curves from the asymptotic formulae (24) and (25).

clamped–free and free–clamped plates than for the fully-clamped plates, for all the mode numbers and values of the geometric parameter r_{out}/r_{in} taken into account. Secondly, the fully-clamped plates do not have the zero mode. There is no main maximum in the curve representing the active sound power of the zero mode of the plates with one edge free. The zero mode is particularly important for the radiational behavior of the clamped–free plate, because in this case the bending moment is rather weak in the whole plate, especially in its region located close enough to its free edge, as compared with the fully-clamped plate.

The curve shapes representing the standardized sound power of the nth mode are similar to one another for the higher modes for all the three boundary configurations under consideration

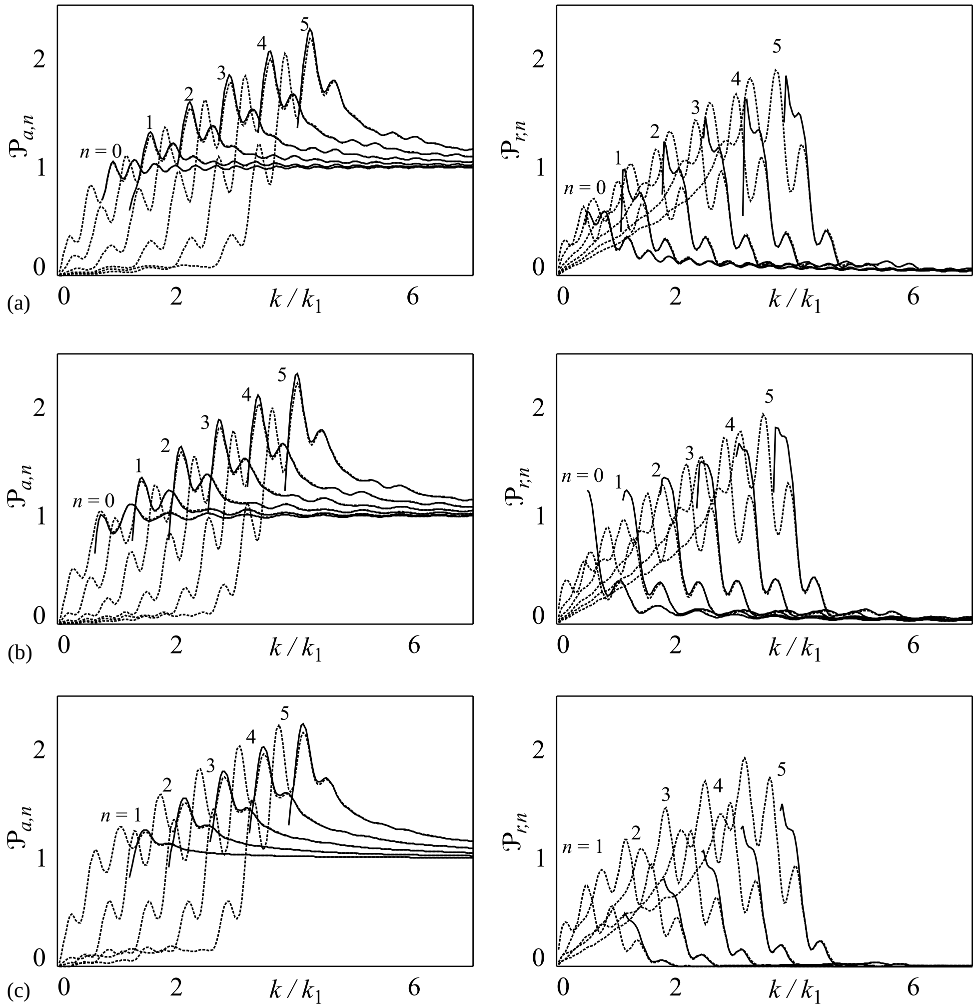

Fig. 2. The standardized active and reactive sound power of an individual *in vacuo* mode $\mathcal{P}_{a,n}$ and $\mathcal{P}_{r,n}$, respectively, for the geometric parameter $r_{out}/r_{in} = 2.0$ and for three different boundary conditions, namely: (a) clamped–free, (b) free–clamped, (c) clamped–clamped (cf., Ref. [13]). All the dotted curves in this figure are obtained from the integral formulae (18) and (19), and all the solid curves from the asymptotic formulae (24) and (25).

(cf., Fig. 1) when r_{out}/r_{in} is considerably small. The number of oscillations observed in the curves decreases for the increasing value of r_{out}/r_{in}, especially for the free–clamped and fully clamped plates. An increase in the value of r_{out}/r_{in} results in a significant difference in the energetic behavior of the three plates (cf., Fig. 3). However, with an increase in the value of r_{out}/r_{in} of an annular plate its geometric shape, mode shapes and energetic behavior become more similar for both annular and circular plates, except for the zero mode which does not exist for a clamped circular plate and the corresponding standardized sound power. Thus, the energetic magnitudes valid for a clamped circular plate (cf., Refs. [8,10–13]) can be used as a rough approximation for the corresponding magnitudes valid for a free–clamped annular plate only, described by a considerably large value of r_{out}/r_{in}.

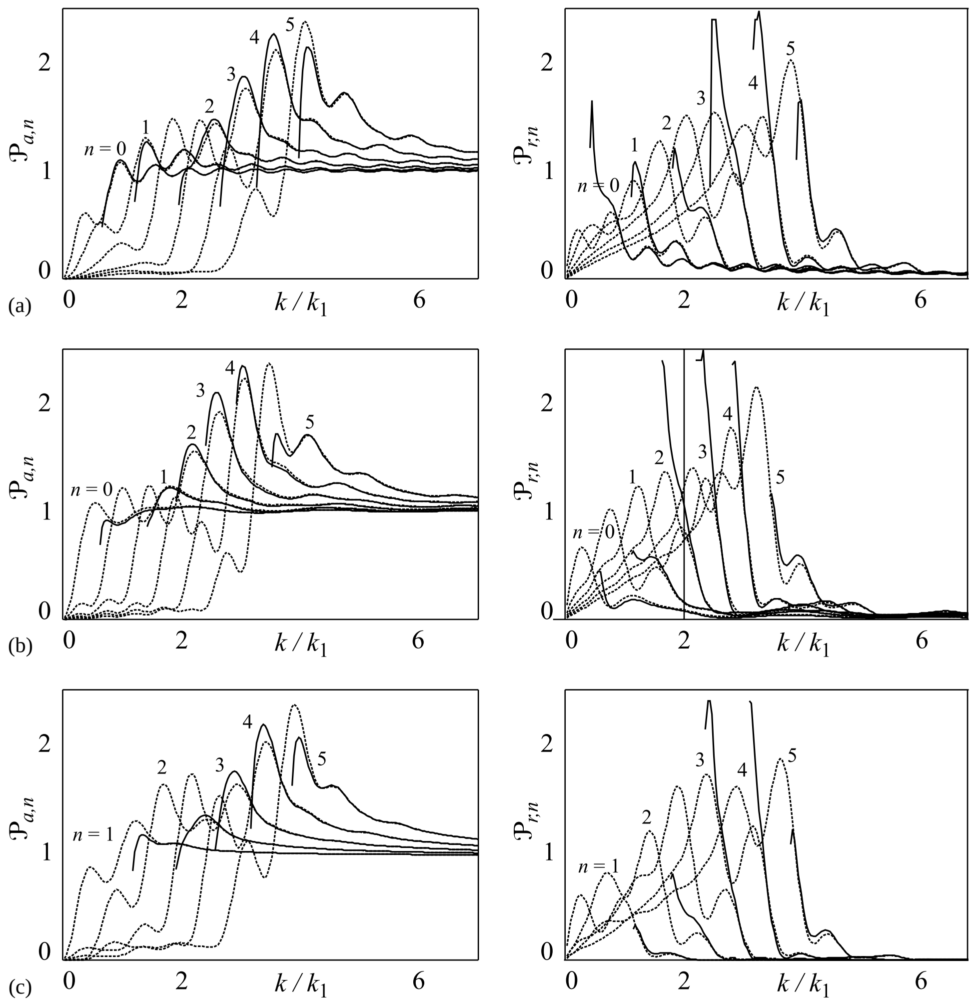

Fig. 3. The standardized active and reactive sound power of an individual *in vacuo* mode $\mathscr{P}_{a,n}$ and $\mathscr{P}_{r,n}$, respectively, for the geometric parameter $r_{out}/r_{in} = 5.0$ and for three different boundary conditions, namely: (a) clamped–free, (b) free–clamped, (c) clamped–clamped (cf., Ref. [13]). All the dotted curves in this figure are obtained from the integral formulae (18) and (19), and all the solid curves from the asymptotic formulae (24) and (25).

5. Conclusions

The results obtained in this study illustrate the vibro-radiational behavior of the plates investigated. Formulations for the standardized active and reactive sound power of an individual in vacuo mode of an annular plate have been derived. Two different boundary configurations have been considered, i.e., clamped–free and free–clamped. First, the sound power has been expressed in the form of the integral formulae, i.e., in its Hankel representation (cf., Eqs. (15), (18) and (19), and then in the form of the asymptotic formulae of a very small error in the high frequencies (cf., Eqs. (24) and (28)). However, the integral formulae that are valid in the full frequency range are considerably time consuming. Therefore, they have only been used to test the accuracy of the asymptotic formulae. The results obtained from the asymptotic formulae show a good agreement with those obtained from the integral formulae for frequencies higher than the successive eigenfrequencies (cf., Figs. 1–3). The asymptotic formulae are easy to express in terms of some computer code and do not require much processor capacity.

The influence of some three different boundary configurations of the plate on the sound power radiated has been analysed. Some significant differences in the radiational behavior of clamped–free, free–clamped and clamped–clamped plates have been found. The results presented in this study can be used for further computation of the total sound power radiated by plates excited in an acoustic fluid.

Acknowledgements

The author would like to acknowledge his gratitude to the Polish State Committee for Scientific Research[1], who have partially supported the investigations presented herein by Grant No. 7T07B-051-18.

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