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Stochastic dynamics of parametrically excited two d.o.f. systems with symmetry

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Abstract

In this paper we investigate the non-linear dynamics of a two-degree-of-freedom system with symmetries subject to random parametric excitation. The study of this non-linear near-Hamiltonian system is simplified by using the symmetry and separation of scales present in the problem. To this end, we study the equations as a random perturbation of a four-dimensional weakly dissipative Hamiltonian system. We achieve the model-reduction through *stochastic averaging* and the reduced process is simply a Markov process on a line. Examination of the reduced Markov process on the line yields many important results, namely, probability density functions, and stochastic bifurcations. The steady state dynamics is computed explicitly. Phenomenological and dynamical bifurcations are investigated. The approach adopted in this paper can in principle, be applied to any four-dimensional integrable system.

1. Introduction

In many engineering systems, the physical problems have an underlying symmetry. The presence of certain types of symmetry in the governing equations of these systems can greatly simplify their analysis. In this paper we explore the role of symmetry in the dimensional reduction of random dynamical systems and hence in the study of stochastic bifurcation problem. In order to show such symmetries exist, we need to demonstrate that the governing equations are unaffected by a particular group of symmetry operations. Mechanical problems such as vibrating strings, square plates, spinning discs, and spherical pendulum have reflection and rotational symmetries and it is such symmetries that we shall consider in this paper. As a concrete example,

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we consider the non-linear response of a taut string to an excitation parallel to its axis, neglecting its longitudinal inertia as given in Appendix A. Non-linear dynamics of a circular spinning disc subject to random parametric excitation is another concrete example.

To this end we consider two degree-of-freedom (d.o.f.) non-gyroscopic problems encountered in mechanical systems. The discretized equations of motion in the neighborhood of an equilibrium configuration can be written in the form

$$\ddot{q} + \nabla U(q, \xi(t)) + \zeta \Lambda \dot{q} = 0, \tag{1}$$

where $q \in \mathbb{R}^2$ is a generalized co-ordinate, Λ, is a 2×2 matrix, U is a scalar potential function and $\xi(t)$ represents a random parametric perturbation. This study examines systems that are rotationally symmetric.

In order to discuss symmetry in a coherent manner, let us define the following:

Let $\mathbb{T}_\gamma$ be the matrix representation of the element γ of a group Γ. If a potential $U : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ can be written as

$$U(\mathbb{T}_\gamma q, \lambda) = U(q, \lambda), \quad \forall_\gamma \in \Gamma,$$

then the potential U is invariant under the symmetry group Γ.

If a vector field $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is derived from a potential $U : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ that is *invariant* under the symmetry group Γ then it can be shown [1] that the vector field is *equivariant* under the symmetry operations, i.e.,

$$\mathbb{T}_\gamma f(q, \lambda) = f(\mathbb{T}_\gamma q, \lambda), \quad \forall_\gamma \in \Gamma.$$

Mathematically, one would say that the corresponding equations of motion have a *group structure*. Eq. (1) is assumed to be invariant under the action of the group $\mathbf{R}_\gamma$,

$$\mathbf{R}_\gamma.(q_1, q_2) \to (q_1', q_2'),$$

where (q_1', q_2') are co-ordinates in a reference frame which is obtained by rotating the (q_1, q_2) reference frame by an angle γ, and $\mathbf{R}_\gamma$ is given by

$$\mathbf{R}_\gamma = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix}, \quad \forall \gamma \in [0, 2\pi].$$

This property is denoted by $\mathbf{S}^1$ symmetry, and implies that the system dynamics remain unchanged in the rotated co-ordinate system (q_1', q_2'). We observe that this symmetry implies that the matrix Λ must be diagonal and of the form λI, and also places restrictions on the non-linear potential $U(x)$ in Eq. (1). Due to $\mathbf{S}^1$ symmetry, Eq. (1) is also invariant under the following transformations:

$$(q_1, q_2) \to (-q_1, -q_2).$$

This property is referred to as *Reflection Symmetry* and is denoted as $\mathbf{Z}_2$ symmetry, and this symmetry implies that $U(q)$ is an even function of q. The general form of the non-linear potential, $U(q)$ satisfying S^1 (and $\mathbf{Z}_2$) invariance can be obtained in terms of *two* real constants ω, a and is given as

$$U(q, 0) = \frac{\omega^2}{2}(q_1^2 + q_2^2) + \frac{a}{4}(q_1^2, q_2^2)^2 + \mathbf{O}(q^6). \tag{2}$$

The linear equations possess a rotational symmetry, provided the natural frequencies are the same in both degrees of freedom. This has various physical implications depending on the problem under consideration. In addition, the equations of motion also remains invariant under the *diagonal reflection or flip symmetry*,

$$(q_1, q_2) \to (q_2, q_1).$$

The random parametric perturbations are assumed to be of $\mathbf{O}(\epsilon)$ and the damping is assumed to be of $\mathbf{O}(\epsilon^2)$. This scaling is motivated by the perturbation technique that we use in obtaining various results. The ordering described above makes the effects of the random perturbations and dissipation behave in such a way that the leading order diffusion part balances the leading order drift term. Hence Eq. (1) reads

$$\begin{aligned} &\ddot{q}_1 + \omega^2 q_1 + a q_1(q_1^2 + q_2^2) + \varepsilon k \xi(t) q_1 + \varepsilon^2 \zeta \dot{q}_1 = 0, \\ &\ddot{q}_2 + \omega^2 q_2 + a q_2(q_1^2 + q_2^2) + \varepsilon k \xi(t) q_2 + \varepsilon^2 \zeta \dot{q}_2 = 0, \end{aligned} \tag{3}$$

where $k = \gamma\sigma_1$ and σ_1 is the magnitude of the random perturbation, $\sigma_1\xi(t)$, and $\xi(t)$ represents the noise. We assume a Gaussian white noise model for the random parametric perturbation. Eq. (3) represent, for example, two-mode approximation of the non-linear response of a taut string to an excitation parallel to its axis or a circular thin spinning disc subject to random parametric excitation [2].

In the absence of noise and dissipation the dynamics of Eqs. (3) are conservative and have a Hamiltonian. The rotational symmetry yields an integral of motion. This results in a reduction of the dimension of the phase space on which the dynamics occurs. A suitable canonical transformation of variables is made resulting in a reduced set of equations governing the dynamics of the unperturbed system. As usual, the first order form of Eqs. (3) will be interpreted as a set of Stratonovich stochastic differential equations (SDE). Owing to our choice of scaling of the noise and dissipative terms in the equations of motion, the dynamics can be viewed as consisting of a fast motion along the trajectories of the unperturbed system and a slow diffusion across the trajectories. The existence of a distinct fast and slow motion makes the set of equations amenable to averaging thus affording further reduction of the dynamics. The study of the dynamics of the stochastically perturbed coupled oscillators (3) is thus reduced to the examination of the associated one-dimensional diffusion problem.

In Section 2, we discuss the stochastic stability of the 2-d.o.f. system. We briefly review here, the concepts of almost-sure and moment stability and their relation to the top Lyapunov exponent and the moment Lyapunov exponent, respectively. We then compute the top Lyapunov exponent and moment Lyapunov exponents explicitly by reducing the linearized system to phase-amplitude form and applying stochastic averaging. In Section 3 which constitutes the main part of this paper, we discuss first the dynamics of the unperturbed system. This includes determination of the integrals of motion, and reduction of the dynamics of the system through a canonical change of variables to simplify the dynamics. The resulting system is then further reduced using stochastic averaging to yield a one-dimensional Markov diffusive process. In Section 4, we verify the existence of a stationary probability measure and the stationary probability density corresponding to the one-dimensional diffusion is computed. We present the results and discussions of the analysis in Section 5. The phenomenological and dynamical types of stochastic bifurcations are described. Finally, in Section 6 we present the conclusions of this work with some recommendation for possible future work.

2. Stochastic stability

Stochastic stability can be qualified at least in three different ways. We can think of stability of random dynamical systems in the sense of convergence in probability, convergence in the mean, and almost-sure convergence. In addition, moment-stability is another important way of describing the stability of stochastic systems. Sample or almost-sure stability of a stationary solution of a random dynamical system implies that all samples except for a set of measure zero tend to the stationary solution as time goes to infinity. Almost-sure stability depends upon the sign of the maximal Lyapunov exponent which is an exponential growth rate of the solution of the randomly perturbed dynamical system. A negative sign of the maximal Lyapunov exponent implies almost-sure stability whereas a non-negative value indicates instability.

2.1. Lyapunov exponents

In analyzing the stability of solutions of random dynamical systems both, almost-sure stability and moment stability are widely used. Almost-sure stability is described by the maximal Lyapunov exponent which is defined as

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \|x(t;x_0)\|,$$

where $x(t;x_0)$ is the solution process of a linear dynamical system. It gives the exponential growth rate of the solution. If $\lambda<0$, then, by definition, $\|x(t;x_0)\|\to 0$ as $t\to\infty$ almost-surely, and $\lambda>0$ implies instability of the solution in the almost-sure sense. The exponential growth rate $\mathbb{E}\|x(t;x_0)\|^p$ is provided by the moment Lyapunov exponent defined as

$$g(p;x_0) = \lim_{t\to\infty} \frac{1}{t} \ln \mathbb{E}\|x(t;x_0)\|^p, \tag{4}$$

where $\mathbb{E}$ denotes the expectation. If $g(p;x_0)<0$, then, by definition, $\mathbb{E}\|x(t;x_0)\|^p\to 0$ as $t\to\infty$ and this condition is referred to as pth moment stability. The moment Lyapunov exponent provides us with finer stability properties of the random dynamical system [3–6]. The connection between the moment Lyapunov exponent and the associated large-deviation problem for linear systems perturbed by white noise was first established by Stroock [7]. An alternative approach to the study of moment Lyapunov exponents and large deviations was presented by Baxendale [6].

The Lyapunov exponent can be viewed as a stochastic analogue of the real part of the eigenvalues of a linear deterministic system. A corresponding linear space analogous to the eigenspace of a matrix also exists. The proof of existence of the Lyapunov exponents and the splitting of the phase space into a direct sum of measurable subspaces (referred to as Oseledec spaces) is given by the multiplicative ergodic theory [8].

2.2. Linearized equations of motion

Linearizing equation (3) about the trivial solution, we obtain

$$\ddot{q}_i + \omega^2 q_i + \epsilon^2 \zeta \dot{q}_i + \epsilon k q_i \xi_t = 0, \quad i = 1, 2. \tag{5}$$

Since the linearized equations (5) are identical and decoupled, it is sufficient to consider only one of the 2 d.o.f. to study the behavior of the linear system. Therefore, we compute the maximal and moment Lyapunov exponents for the system

$$\ddot{q} + \omega^2 q + \epsilon^2 \zeta \dot{q} + \epsilon k q \xi_t = 0. \tag{6}$$

In this paper, we calculate the maximal and moment Lyapunov exponents by transforming the linear system to amplitude-phase form, and then using stochastic averaging to obtain an averaged amplitude equation which by itself represents a diffusive Markov process. The maximal and moment Lyapunov exponents are calculated both, for the case of white noise excitation and for the case when the random perturbation is due to real noise where the random process ξ_t is assumed to satisfy certain strong mixing conditions.

In the case of ergodic but non-white noise excitation, the results are due to [9–11]. Hence, for the case of *real noise excitation* with certain strong mixing condition the maximal Lyapunov exponent is obtained by the Furstenberg–Khas'minskii Formula as in Ref. [10]

$$\lambda = -\varepsilon^2 \zeta \left[-\frac{1}{2} + \frac{\alpha}{8} S(2\omega) \right], \quad \alpha = \frac{k^2}{\zeta \omega^2}, \tag{7}$$

where the cosine spectrum is defined as

$$S(2\omega) = 2 \int_0^\infty R_{\xi\xi}(\tau) \cos(2\omega\tau)\, \mathrm{d}\tau \quad \text{and} \quad R_{\xi\xi}(\tau) = \mathbb{E}[\xi_t \xi_{t+\tau}].$$

For the case of *white noise excitation* with unit intensity, Eq. (7) reduces to

$$\lambda = -\varepsilon^2 \zeta \left[-\frac{1}{2} + \frac{\alpha}{8} \right]. \tag{8}$$

In order to determine the pth moment stability we solve for the pth moment of the amplitude response and compute the moment Lyapunov exponent defined by (4). For the linearized system subject to *real noise excitation* we obtain the pth-moment Lyapunov exponent as in [12]

$$g(p) = \varepsilon^2 p \zeta \left(-\frac{1}{2} + \frac{(p+2)}{16} \alpha S(2\omega) \right), \tag{9}$$

while for the case of *white noise excitation* with unit intensity,

$$g(p) = \varepsilon^2 p \zeta \left[-\frac{1}{2} + \frac{(p+2)}{16} S_0 \alpha \right]. \tag{10}$$

A negative value of the Lyapunov exponent implies almost-sure or sample-stability of the trivial solution. Hence for the white noise-driven system, almost-sure stability is given by the condition $\alpha < 4$, where α in physical terms, represents the ratio of the intensity of noise to the damping. Similarly, a negative value for the pth moment Lyapunov exponent implies stability in the pth norm.

3. Reduction of noisy non-linear systems

In this section, firstly, we shall discuss the dynamics of the unperturbed system. The rotational $\mathbf{S}^1$ symmetry of the system results in an additional integral of motion. We shall make use of the symmetry to reduce the dynamics of the system. A suitable canonical transformation of the variables is made making use of the $\mathbf{S}^1$ invariance of the unperturbed system. The perturbed system is described in terms of these canonical variables as a set of Ito stochastic differential equations. Further, we shall make use of the separation of scales in the system to obtain reduction through the method of stochastic averaging. The dynamics of the original system is thus reduced to a one-dimensional Markov diffusive process.

3.1. Unperturbed system ($\epsilon = 0$)

The system represented by Eq. (3) is conservative in the absence of noise and damping. The equations of motion of the unperturbed system are

$$\ddot{q}_1 + \omega^2 q_1 + a q_1(q_1^2 + q_2^2) = 0, \quad \ddot{q}_2 + \omega^2 q_2 + a q_2(q_1^2 + q_2^2) = 0. \tag{11}$$

It may be noted that the 2 d.o.f. are linearly decoupled and are coupled only through the non-linear stiffness term. On transforming (q_1, q_2) to polar co-ordinates as $(q_1, q_2) \mapsto (r\cos\theta, r\sin\theta)$, and obtaining the Lagrange's equations of motion, we find θ to be an ignorable or cyclic co-ordinate. The corresponding conjugate momentum $I_\theta = r^2\dot{\theta}$ would then be an integral of motion. Letting $\dot{q}_i = p_i,\ i = 1, 2$, we obtain $I_\theta = p_2 q_1 - p_1 q_2$.

The Hamiltonian for this system denoted $H(p, q)$ would be

$$H(p,q) \coloneqq \tfrac{1}{2}(p_1^2 + p_2^2) + \frac{\omega^2}{2}(q_1^2 + q_2^2) + \frac{a}{4}(q_1^2 + q_2^2)^2. \tag{12}$$

In terms of (p, q), Eq. (11) can be represented by the following set of first order differential equations:

$$\begin{aligned} \dot{q}_1 &= p_1, \quad \dot{p}_1 = -\omega^2 q_1 - a q_1(q_1^2 + q_2^2), \\ \dot{q}_2 &= p_2, \quad \dot{p}_2 = -\omega^2 q_2 - a q_2(q_1^2 + q_2^2). \end{aligned} \tag{13}$$

Owing to the symmetry present in the system we can simplify the dynamics using an appropriate set of canonical transformations. The following canonical transformation of coordinates (see [13])

$$\begin{aligned} q_1 &= r\cos\theta, \quad p_1 = P\cos\theta - \frac{I}{r}\sin\theta, \\ q_2 &= r\sin\theta, \quad p_2 = P\sin\theta + \frac{I}{r}\cos\theta, \end{aligned} \tag{14}$$

can easily be obtained by solving the Hamilton–Jacobi equation. In terms of the new set of canonical variables (14), the Hamiltonian for the unperturbed system takes the form

$$H(P, I, r, \theta) = \tfrac{1}{2}\left(P^2 + \frac{I^2}{r^2}\right) + \tfrac{1}{2}\omega^2 r^2 + \frac{a}{4}r^4 \tag{15}$$

and the Hamiltonian system of equations in terms of the new set of variables can be obtained as

$$\dot{r} = P, \quad \dot{P} = -\omega^2 r - ar^3 + \frac{I^2}{r^3},$$
$$\dot{\theta} = \frac{I}{r^2}, \quad \dot{I} = 0. \tag{16}$$

The Hamiltonian is independent of θ, and therefore, the variable θ in Eq. (16) does not play any role in describing the dynamics of the system. The reduced system can thus be described in terms of the variables r, P, I. Also, from the third of equations (16), I is a constant that depends on the initial value, and therefore, the dynamics of the unperturbed system is reduced to a phase space represented by (r, P) with I as a parameter. For most structural materials the non-linear stiffness coefficient a is positive. In this paper we consider only the case $a > 0$. For $a > 0$ and $I = 0$, the only fixed point of the system is $(r, P) = (0, 0)$. A linear stability analysis of the system reveals the fixed point to be a center. For $I \neq 0$ and $a > 0$, a fixed point exists at $(r_0, 0)$ where, r_0 is the real positive solution of the equation $ar^6 + \omega^2 r^4 = I^2$. The trajectories for the unperturbed system are periodic. The phase portrait for the case when $I = 0$ is shown in the Fig. 1. We remark here that for $I = 0$, the initial velocities and amplitudes for the two waves traveling in opposite directions have the same ratio. $I := (p_2 q_1 - p_1 q_2) = 0$ implies $q_2 = kq_1$, where k is a constant depending on the initial conditions. Thus in this case the two equations (11) are identical up to a scaling. The unperturbed dynamics can thus be reduced to motion on a two-dimensional manifold. We now consider the system perturbed by the addition of small damping and small multiplicative white noise.

3.2. Addition of small noise and damping

The system perturbed by addition of small noise $\mathbf{O}(\varepsilon)$ and small damping $\mathbf{O}(\epsilon^2)$ is described in Eq. (3). The physical reason behind these perturbations has been described earlier in Section 2. In

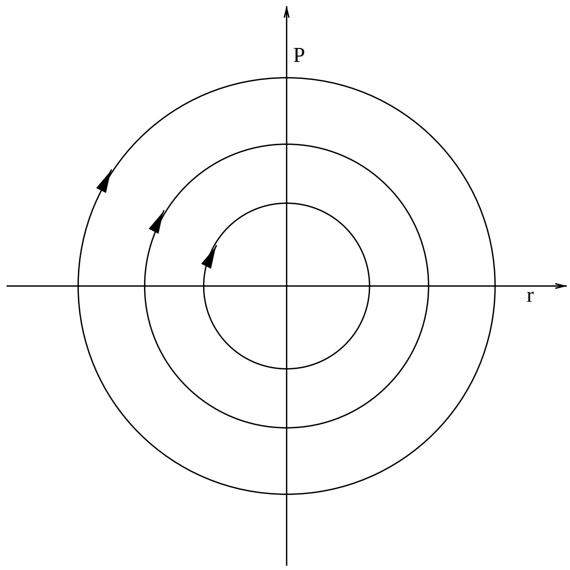

Fig. 1. Phase-portrait for different energy levels H of the unperturbed system for $I = 0$.

terms of the canonical variables Eqs. (14) and (3) can be written as a set of stochastic differential equations in the Stratonovich form as

$$\begin{aligned} \mathrm{d}r &= P\,\mathrm{d}t, \\ \mathrm{d}P &= \left(-\omega^2 r - ar^3 + \frac{I^2}{r^3} - \varepsilon^2\zeta P\right)\mathrm{d}t - \varepsilon kr \circ \mathrm{d}W(t), \\ \mathrm{d}\theta &= \frac{I}{r^2}, \quad \mathrm{d}I = -\epsilon^2\zeta I\,\mathrm{d}t. \end{aligned} \tag{17}$$

Here, the equations in r, P, and I are independent of the variable θ, and therefore, these equations by themselves completely represent the dynamics of the perturbed system. Thus excluding the equation in θ the perturbed dynamics (17) can be described by the following set of Ito stochastic differential equations:

$$\begin{aligned} \mathrm{d}r &= P\,\mathrm{d}t, \\ \mathrm{d}P &= \left(-\omega^2 r - ar^3 + \frac{I^2}{r^3} - \varepsilon^2\zeta P\right)\mathrm{d}t - \varepsilon kr\,\mathrm{d}W(t), \\ \mathrm{d}I &= -\epsilon^2\zeta I\,\mathrm{d}t. \end{aligned} \tag{18}$$

It is important to note that the correction term is identically zero in the Ito SDEs denoted by Eq. (18). From Eq. (15), we have

$$P = \sqrt{\rho(r,H) - \frac{I^2}{r^2}}, \quad \text{where } \rho(r,H) = 2H - \omega^2 r^2 - \frac{a}{2}r^4. \tag{19}$$

Our primary goal is to obtain the steady state dynamics of the perturbed system. We observe that the conjugate momentum I decays exponentially as given by Eq. (18). Therefore, we shall study the dynamics of the perturbed system (18), with $I = 0$. It is worth noting that in the presence of additive noise the evolution of I would also be governed by a stochastic differential equation.

Further, by transforming the variables (r, P) into (r, H) using Eq. (19), we can represent the dynamics of the perturbed system given by Eq. (18) in terms of the variables (r, H). By doing so, we note that the dynamics can now be described as consisting of a fast motion in terms of the variable r and a slow motion in terms of the Hamiltonian H. This makes it possible to use the results of stochastic averaging to reduce the perturbed system by averaging the motion over the fast variable. Therefore, the dynamics of the perturbed system can finally be reduced to a one-dimensional diffusive Markov process in the Hamiltonian (slow variable). Applying Ito's lemma to the Hamiltonian we obtain

$$\mathrm{d}H(t) = \varepsilon^2\left(-\zeta P^2 + \tfrac{1}{2}k^2 r^2\right)\mathrm{d}t - \varepsilon krP\,\mathrm{d}W(t). \tag{20}$$

The equations in (r, H) are therefore, given by

$$\begin{aligned} \mathrm{d}r &= \sqrt{\rho(r,H)}\,\mathrm{d}t, \\ \mathrm{d}H &= \varepsilon^2\left(-\zeta\rho(r,H) + \tfrac{1}{2}k^2 r^2\right)\mathrm{d}t - \varepsilon kr\sqrt{\rho(r,H)}\,\mathrm{d}W(t). \end{aligned} \tag{21}$$

Eq. (21) is in the standard form for averaging. We can therefore apply stochastic averaging and reduce Eq. (21) to a one-dimensional diffusion.

3.3. Stochastic averaging

The motion of the perturbed Hamiltonian system (21) has two components, a fast motion along the periodic trajectories of the unperturbed dynamical system in the (r, P)-plane, and a slow displacement in the direction transverse to the periodic trajectories. The distribution of the fast component would be close to the occupational measure on the closed trajectories of the unperturbed system. By averaging over the fast motion with respect to the invariant distribution concentrated on the closed trajectories, the slow motion can be approximated by the trajectory of the averaged system which would be represented by an Ito equation in the slow variables. Thus, averaging would result in a reduction of the dynamics. For our problem, we obtain a Ito stochastic differential equation in terms of the Hamiltonian. This can be shown to represent a Markov diffusive process, using the results of Khas'minskii [14]. Rigorous results at an elliptic fixed point are also given by Namachchivaya and Sowers [15]. The averaging procedure used here is similar to the one presented in Liang and Sri Namachchivaya [16].

Carrying out the averaging as described above for Eq. (21) with $I = 0$ we obtain an Ito stochastic differential equation in the slow variable H, of the form

$$\mathrm{d}H = \mu_H\,\mathrm{d}t + \sigma_H\,\mathrm{d}W(t). \tag{22}$$

The drift and diffusion terms in Eq. (22) are obtained as

$$\mu_H(H(m)) = \zeta\frac{\omega^4}{a}\mu_1(m),\quad \sigma_H^2(H(m)) = k^2\frac{\omega^6}{a^2}\sigma_1^2(m), \tag{23, 24}$$

where

$$\begin{aligned}\mu_1(m) = &\frac{1}{(1-2m^2)K(m)}\Big\{\alpha[E(m)-(1-m^2)K(m)]\\ &-\frac{1}{(1-2m^2)}\big[\tfrac{1}{2}(1-(1-2m^2)^2)K(m) - 2(1-2m^2)[E(m)-(1-m^2)K(m)]\\ &-\tfrac{2}{3}[(2-3m^2)(1-m^2)K(m) - 2(1-2m^2)E(m)]\big]\Big\},\end{aligned}$$

$$\begin{aligned}\sigma_1^2(m) = &\frac{1}{(1-2m^2)^3K(m)}\Big\{(1-m^2)\big[-1+(1-2m^2)\big(\tfrac{7}{15}-\tfrac{6}{5}m^2\big)\\ &+\tfrac{12}{5}m^2(1-m^2)\big]K(m) + \big[1-\tfrac{7}{15}(1-2m^2)^2 - \tfrac{12}{5}(1-m^2)m^2\big]E(m)\Big\}\end{aligned}$$

and α is as defined earlier in Eq. (7); $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively, and m is the modulus of the elliptic integrals given by

$$m^2 = \frac{1}{2}\left[1 - \frac{\frac{\omega^2}{a}}{\sqrt{4\frac{H}{a}+\frac{\omega^4}{a^2}}}\right]. \tag{25}$$

Since Eqs. (23) and (24) are in terms of m it would be much easier to consider the diffusion in m. Therefore, subsequently we shall be considering the diffusion equation in terms of m. Applying Ito's rule to $m(H)$ given in Eq. (25), we obtain the one-dimensional diffusion problem in terms of

m as

$$\mathrm{d}m = \mu(m)\,\mathrm{d}t + \sigma(m)\,\mathrm{d}W(t), \tag{26}$$

where the drift and diffusion terms $\mu(m)$ and $\sigma(m)$, respectively, are given by

$$\mu(m) = \frac{1}{2}\zeta\left\{\frac{(1-2m^2)^3}{m}\mu_1(m) - \frac{1}{4}\alpha\frac{(1-2m^2)^5}{m^3}(1+10m^2)\sigma_1^2(m)\right\},$$
$$\sigma^2(m) = \frac{1}{4}\zeta\alpha\frac{(1-2m^2)^6}{m^2}\sigma_1^2(m). \tag{27}$$

The domain $\mathscr{D}$ of the one-dimensional diffusion is the open interval $\mathscr{D} = \{m \in (0, 1/\sqrt{2})\}$. The stationary probability density function for the diffusive process (26) is then given by the solution of the Fokker–Planck equation

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\,m^2}[\sigma^2(m)p_\alpha] - \frac{\mathrm{d}}{\mathrm{d}m}[\mu(m)p_\alpha] = 0, \tag{28}$$

along with appropriate boundary conditions.

4. Analysis of the reduced system

We had seen from the stability analysis in Section 2 that the invariant measure associated with the trivial fixed point loses stability at $\alpha = 4$ while for $\alpha < 4$ a δ-measure exists at the trivial fixed point. This suggests that a stationary probability density exists for $\alpha > 4$. In this section we shall verify using Khas'minskii's results that a stationary probability measure indeed exists for $\alpha > 4$. In order to determine the boundary conditions for the diffusion process we shall carry out an asymptotic analysis at the boundaries, $m = 0$ and $1/\sqrt{2}$. We then compute the stationary probability density corresponding to the one-dimensional diffusive process in m.

4.1. Boundary behavior

Since $m = 0$ is not a physical boundary, the boundary conditions that need to be imposed are not obvious. In order to define this we shall examine the boundary behavior. The first passage probabilities help in classifying the boundaries as accessible and inaccessible. We refer the reader to Refs. [18,19] for an exhaustive treatment of one-dimensional Markov diffusive processes. A point m in the state space of the diffusion is said to be *singular* if $\sigma(m) = 0$. If in addition, $\mu(x) = 0$ then the point m is said to be a *trap*. For the diffusion problem under consideration $\sigma(0) = \mu(0) = 0$. *Thus, the boundary point $m = 0$ is both, a singular point and a trap.*

In order to describe the boundary behavior of the one-dimensional diffusion process (26), we define the *scale function* [18,19]

$$\mathscr{S}[a,b] := \int_a^b s(\eta)\;\mathrm{d}\eta \quad \text{where } s(m) = \exp\left[-\int^m \frac{2\mu(\eta)}{\sigma^2(\eta)}\mathrm{d}\eta\right] \tag{29}$$

and *speed measure*

$$M[a,b] := \int_a^b \mathscr{M}(\eta)\,\mathrm{d}\eta, \quad \text{where } \mathscr{M}(m) := \frac{1}{\sigma^2(m)s(m)}, \tag{30}$$

and where $\mathscr{M}(m)$ is called the *speed density*. We define $\Sigma(l)$ as

$$\Sigma(l) = \int_l^x \left\{ \int_\eta^x \mathscr{M}(\zeta)\,\mathrm{d}\zeta \right\} s(\eta)\,\mathrm{d}\eta. \tag{31}$$

$\Sigma(l)$ in rough terms represents the time it takes to reach the boundary l starting from an interior point $x<b$. A boundary for which $\Sigma(l) = \infty$ is said to be *unattainable*. We also define another quantity at the boundary l, namely $N(l)$ as

$$N(l) = \int_l^x \mathscr{S}[\eta, x]\,\mathrm{d}M(\eta). \tag{32}$$

$M(l,x]$ measures the speed of the process near the boundary l and $N(l)$ roughly measures the time to reach an interior point x in (l,r) starting at the boundary l, where r represents the right boundary. It can also be easily shown that any interior point of (l,r) can be reached starting from the boundary l if and only if $N(l)<\infty$. Analogous definitions can be made for the right boundary. With $\mathscr{S}(l,x]$, $\Sigma(l)$, $N(l)$, and $M(l,x]$ as defined above we have the following result: $\mathscr{S}(l,x] = \infty$ implies $\Sigma(l) = \infty$, $\Sigma(l)<\infty$ implies $\mathscr{S}(l,x]<\infty$, $M(l,x] = \infty$ implies $N(l) = \infty$, $N(l)<\infty$ implies $M(l,x]<\infty$.

In the following we shall compute these quantities and evaluate them at the two boundaries. We shall thus describe the behavior of the one-dimensional diffusion process completely in terms of the quantities $\mathscr{S}, M, \Sigma$, and N defined above. However, the calculation of these quantities would involve the drift and diffusion terms in the averaged diffusion equation (22) which are complex functions. This makes it difficult to evaluate these integrals in closed form. A diffusion process is said to be *regular* if starting from any point in the interior of the domain $\mathscr{D}$ any other point in the interior of $\mathscr{D}$ may be reached with positive probability. It can be easily shown that the diffusion process governed by Eq. (22) is regular. This makes it sufficient to evaluate these integrals in a close neighborhood of the singular boundaries. We thus investigate the boundaries at $m = 0$ and $1/\sqrt{2}$ using an asymptotic approach.

4.1.1. Asymptotic analysis near $m = 0$

For m close to zero, the asymptotic expressions for the complete elliptic integrals $K(m)$ and $E(m)$ are given by

$$E(m) = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 m^2 - \left(\frac{1}{2}\cdot\frac{3}{4}\right)^2 \frac{m^4}{3} - \cdots\right],$$

$$K(m) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 m^2 + \left(\frac{1}{2}\cdot\frac{3}{4}\right)^2 m^4 + \cdots\right].$$

The asymptotic approximations for the drift and diffusion terms given by Eq. (27) as $m \to 0$ can then be obtained as

$$\mu(m) \approx \frac{\zeta}{2}\left(\frac{3}{8}\alpha - 1\right)m, \quad \sigma^2(m) \approx \frac{1}{8}\zeta\alpha m^2. \tag{33}$$

The infinitesimal generator for the diffusion process close to $m = 0$ is then given by

$$L = \frac{\zeta}{2}\left(\frac{3}{8}\alpha - 1\right)m\frac{\partial}{\partial m} + \frac{1}{2}\left(\frac{1}{8}\zeta\alpha m^2\right)\frac{\partial^2}{\partial m^2}. \tag{34}$$

The various quantities $\mathscr{S}(0, x]$, $M(0, x]$, $\Sigma(0)$, and $N(0)$ can then easily be computed. The results are summarized in Table 1.

Thus we conclude that for $\alpha \geqslant 4$ the boundary $m = 0$ is not attainable in finite time starting from any point in the interior of $(0, 1/\sqrt{2})$, and also starting from $m = 0$ no point in the interior of $(0, 1/\sqrt{2})$ can be accessed within a finite duration of time. According to the Feller classification, such a boundary is termed as a *natural boundary*. Hence, $m = 0$ *is a natural boundary*.

4.1.2. Asymptotic analysis near $m = 1/\sqrt{2}$

Carrying out the asymptotics close to $m = 1/\sqrt{2}$ we obtain the infinitesimal generator for the diffusion process as

$$L = -0.66\zeta\left(\frac{1}{\sqrt{2}} - m\right)\frac{\partial}{\partial m} + 0.501\zeta\alpha\left(\frac{1}{\sqrt{2}} - m\right)^3\frac{\partial^2}{\partial m^2}. \tag{35}$$

An asymptotic approximation as in the previous subsection can be carried out close to the boundary at $m = 1/\sqrt{2}$. The results obtained for this case are tabulated in Table 2. Since

Table 1
Summary of scale and speed measures at $m = 0$

	$\mathscr{S}(0, x]$	$\Sigma(0)$	$M(0, x]$	$N(0)$
$\alpha \geqslant 4$ $\lambda^{\epsilon} \geqslant 0$	∞ (not attracting)	∞ (not attainable)	$< \infty$	∞
$\alpha < 4$ $\lambda^{\epsilon} < 0$	$< \infty$ (attracting)	∞ (not attainable)	∞	∞

Table 2
Summary of scale and speed measures at $m = 1/\sqrt{2}$

$\mathscr{S}(0, x]$	$\Sigma(0)$	$M(0, x]$	$N(0)$
∞ (not attracting)	∞ (not attainable)	$< \infty$	∞

$\sum(1/\sqrt{2}) = \infty$, the boundary at $m = 1/\sqrt{2}$ cannot be reached in finite time starting from a point in the interior of the domain. Further, $N(1/\sqrt{2}) = \infty$ which implies that starting from the boundary at $m = 1/\sqrt{2}$ a point in the interior of the domain cannot be attained. By the Feller classification the boundary at $m = 1/\sqrt{2}$ is a *natural boundary*.

4.2. Stationary probability density

We shall use the asymptotic expressions obtained to establish the recurrence and non-explosion of process (26) defined on the state space $\mathscr{E} =^{\text{def}} (0, 1\sqrt{2})$. Then according to Khas'minskii's results [17, Chapter 3], for recurrence and non-explosion, and finiteness of the mean recurrence time, it suffices to show that there exists a non-negative function $V(m)$ such that $LV(m) \leqslant -1$ outside a fixed compact subset of the state space $\mathscr{E}$, where L is the generator of process (26).

To this end, consider a domain $\mathscr{D} =^{\text{def}} (\delta_1, 1/\sqrt{2} - \delta_2) \subset (0, 1/\sqrt{2})$. Khas'minskii's results imply the recurrence of process (26) relative to the domain $\mathscr{D}$, i.e., conditions under which the sample paths starting from any point $m \in \mathscr{E} \backslash \mathscr{D}$ almost surely reach the set $\mathscr{D}$, if there exists a non-negative function $V(m)$ such that

$$LV(m) \leqslant -1, \qquad m \in (0, \delta_1), \quad m \in \left(1/\sqrt{2} - \delta_2, 1/\sqrt{2}\right) \tag{36}$$

and such that $V(m) \to \infty$ as $m \to 0$ and for $m \to \frac{1}{\sqrt{2}}$. Let

$$V(m) = -\ln m + \frac{1}{(1/\sqrt{2} - m)}. \tag{37}$$

Then, using Eqs. (34) and (35) for the generator of the process $m(t)$ at the boundaries $m = 0$ and $m = 1/\sqrt{2}$, respectively, one can easily verify that condition (36) holds for $\alpha > 4$ for $V(m)$ given by Eq. (37). Under the additional condition that the diffusion term is non-singular in some domain $\mathscr{D} \subset (0, 1/\sqrt{2})$ and in some neighborhood $\mathscr{U}$ of $\mathscr{D}$, existence of a stationary probability density can be established [17, Chapter 4]. We have thus verified the existence of a stationary probability density for the diffusion process $m(t)$ given by Eq. (26) for $\alpha > 4$. It is important to note that the condition $\alpha > 4$ is indeed the condition that we obtain from the stability analysis in Section 2 for the existence of a stationary probability density. Thus Khas'minskii's results provides a verification of the fact that a stationary probability density exists only for $\alpha > 4$.

The stationary probability density for a one-dimensional diffusion is given by [18,19]

$$p(m) = \mathscr{M}(m)[C_1 \mathscr{S}(0, m] + C_2], \tag{38}$$

where $\mathscr{M}(m)$ is the speed measure and $\mathscr{S}(0, m]$ is the scale function defined in Eqs. (29) and (30), respectively. Using the asymptotic expressions for the drift and diffusion terms from Eq. (33) it can easily be verified that for m small, $\mathscr{S}(0, m] = \infty$ for $\alpha > 4$. However, since $\mathscr{S}(0, m]$ is a monotonically increasing function it follows that

$$\mathscr{S}(0, m] = \infty, \quad m \in (0, 1/\sqrt{2}). \tag{39}$$

Therefore, we require that $C_1 = 0$ in a Eq. (38). The stationary probability density for $\alpha > 4$ is then given by

$$p(m) = C_2 \mathscr{M}(m), \quad m \in \left(0, 1/\sqrt{2}\right), \tag{40}$$

where C_2 is the normalization factor given by

$$C_2 = \frac{1}{\int_0^{1/\sqrt{2}} \mathscr{M}(\eta)\, d\eta}. \tag{41}$$

The stationary probability density at small values of $m(m \to 0)$ can be computed using the asymptotic expressions given by Eq. (33) in Eqs. (29), (30) and (40). We obtain the asymptotic expression for the stationary probability density as

$$p(m) \approx k_1 m^{1-8/\alpha}, \quad m \to 0. \tag{42}$$

Therefore, as $m \to 0$ we have

$$p(m) \to \begin{cases} \infty, & 4 < \alpha < 8, \\ k_1, & \alpha = 8, \\ 0, & \alpha > 8. \end{cases} \tag{43}$$

For $\alpha \leqslant 4$, $M(m) = \infty$ for all $m \in (0, 1/\sqrt{2})$. Therefore, clearly from Eq. (40), $p(m)$ is not normalizable for $\alpha < 4$. However, as seen from Eq. (8), the invariant measure at the trivial fixed point is *stable* for $\alpha < 4$. Hence, we expect a *δ-invariant measure* at $m = 0$ for $\alpha < 4$. We note that a phenomenological change in the stationary density occurs at $\alpha = 8$, which will be discussed in detail in the following section.

5. Results and discussion

We present here the results for three cases, namely $\alpha = 5$, 8, and 10. These results are obtained by numerically computing the stationary probability density (40) using Eqs. (27), (29) and (30). The stationary probability density for $\alpha = 5$ and 8 is shown in Figs. 2 and 3 respectively, and that for $\alpha = 10$ is shown in Fig. 4.

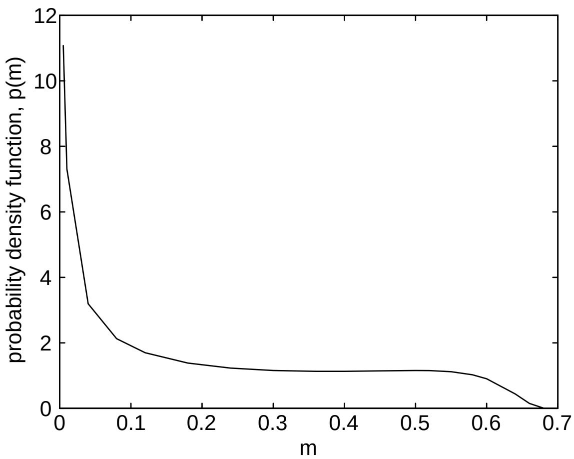

Fig. 2. Stationary probability density for $\alpha = 5$.

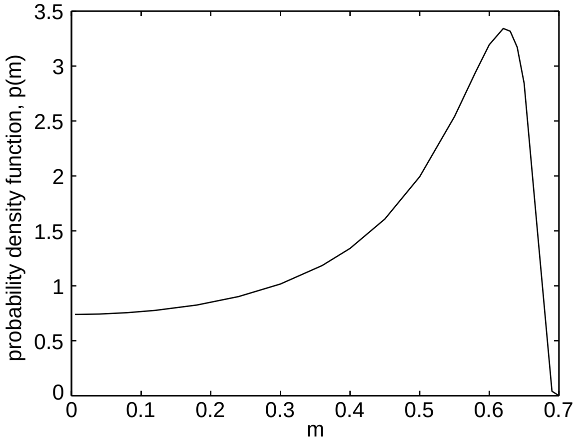

Fig. 3. Stationary probability density for $\alpha = 8$.

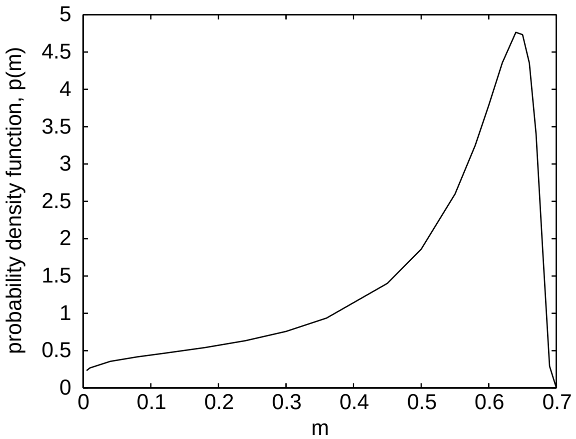

Fig. 4. Stationary probability density for $\alpha = 10$.

The concept of P-bifurcation (*phenomenological*) is associated with qualitative changes of the probability density p_α. A point $\alpha = \alpha_0$ in each neighborhood of which there exists *non-equivalent densities* in the sense of Zeeman [20,21], is called a "P-bifurcation point" [22,23]. The reader is referred to [22–24] for details. Baxendale [25] proved that P-bifurcation occurs at a parameter value α for which the non-trivial zero of the moment Lyapunov exponent, $g(p, \alpha)$, is at $-d$ where, d is the dimension of the state space. In other words the value of the bifurcation parameter at which the P-bifurcation can occur is given by the solution of $g(-d, \alpha) = 0$.

Referring to the stationary probability densities for the various values of α presented in Figs. 2–4 we observe that the probability density changes from one without a peak for $4 < \alpha < 8$ to a smooth function with a single peak for $\alpha > 8$. This is also evident from the asymptotic behavior of the stationary probability density for various values of the parameter α as $m \to 0$, as shown in

Eq. (43). This suggests that a phenomenological change or a P-bifurcation takes place at $\alpha = 8$. This result can also be obtained from Baxendale's conjecture [25] based on moment Lyapunov exponent as shown below.

Using Eq. (33) we obtain the linear variational equation about $m = 0$ as

$$\mathrm{d}m = \frac{\zeta}{2}\left(\frac{3}{8}\alpha - 1\right)m\,\mathrm{d}t + \sqrt{\frac{\zeta\alpha}{8}}\,m\,\mathrm{d}W(t). \tag{44}$$

The expression for the moment Lyapunov exponent can then easily be obtained as

$$g(P) = \frac{1}{16}\alpha\zeta p^2 + \frac{\zeta}{2}\left(\frac{\alpha}{4} - 1\right)p. \tag{45}$$

Clearly, $g(-1) = 0$ for $\alpha = 8$. Therefore, as observed earlier, a P-bifurcation occurs at $\alpha = 8$. The P-bifurcation described above however, is a *static* concept and cannot be related to the stability of the invariant measure.

In the rest of this section we shall discuss the stability of the invariant measure associated with the process $m(t)$ based on Lyapunov exponent. For a family $(\Phi_\alpha)_{\alpha\in\mathscr{R}}$ of random dynamical systems with invariant measure μ_α, the point $(\alpha_0, \mu_{\alpha 0})$ is called a *dynamical* or *D-bifurcation* point if for each α in a neighborhood of α_0, there is a Φ_α-invariant measure $v_\alpha \neq \mu_\alpha$ for which $v_\alpha \to \mu_{\alpha_0}$ weakly as $\alpha \to \alpha_0$ [5,22].

For our problem it was not possible to calculate the flow explicitly. However, we note that for $\alpha < 4$, there exists a δ-measure at zero which is stable ($\lambda < 0$). At the parameter value $\alpha = 4$, the fixed point at zero loses stability, and for $\alpha > 4$, we obtain a stationary distribution of the probability in the domain $m \in (0, 1/\sqrt{2})$. We therefore, conjecture that a *dynamical* or *D-bifurcation* occurs at $\alpha = 4$. The stationary probability density represents the mean proportion of time spent at any state $m \in (0, 1/\sqrt{2})$ (or, $H \in (0, \infty)$). Thus, the state corresponding to the peak of the density function represents the most probable state. With regard to the transverse motion of the taut string, this represents the amplitude response exhibited by the randomly perturbed taut string for most part of a given duration of time. In the original phase space (q_1, q_2, p_1, p_2), (q_1, q_2) represents the amplitudes whereas (p_1, p_2) denotes the amplitude velocities of the two modes. The trivial solution of the unperturbed equations of motion represents a fixed point in the (q_1, q_2, p_1, p_2) phase space while a nonzero value of the Hamiltonian (H) represents a periodic trajectory of the unperturbed motion.

Notice that, for $\alpha < 4$, a δ-invariant measure exists at $H = 0$, i.e, the trivial fixed point is stable. For the linearized system, as soon as $\alpha > 4$ ($\lambda > 0$) all trajectories go to infinity exponentially fast (almost-surely) so there is no invariant probability for the *linearized system* on $\mathbb{R}^2 - \{0, 0\}$. For $\alpha > 4$, the non-linear system is bounded to build up a stationary measure on $\mathbb{R}^2 - \{0, 0\}$ leading to bifurcation of a new invariant measure. However, the trajectories of the noisy non-linear system still spends a large amount of time in the small punctured neighborhoods of 0. For $4 < \alpha < 8$, the mass of the occupation measure is large around 0 compared to the mass away from zero due to the fact that the non-linear effects are not sufficiently strong enough to keep the trajectories close to the new nontrivial bifurcating measure. Hence, it is worth noting that for $4 < \alpha < 8$, $m = 0$ is the *most probable* state of the non-linear system although $m = 0$ is *almost-surely unstable* for the linearized system. This is quite unlike the deterministic case where the trajectories tend to move away from an unstable equilibrium.

6. Conclusion

We have examined in this paper the non-linear behavior of a two-degree-of-freedom system subject to random parametric excitation. The $\mathbf{S}^1$-symmetry present in the physical system was exploited in reducing the dimension of the phase space on which the dynamics occurs. Further, by averaging the system over the motion of the fast variables, the perturbed dynamics was reduced to a one-dimensional Markov diffusive process. The procedure adopted here can in principle, be applied to any general near-integrable system.

Analysis of the linearized system of equations showed that the trivial solution became unstable for values of the bifurcation parameter $\alpha > 4$. For $\alpha < 4$, the equilibrium at the origin is asymptotically stable and a δ-invariant measure exists at this fixed point. For values of $\alpha > 4$, the trivial solution loses stability and results in a new invariant measure. We conjecture that a D-bifurcation occurs at $\alpha = 4$.

The existence of a stationary probability density for the process $m(t)$ was shown for values of $\alpha > 4$. The stationary probability density was computed numerically for some chosen values of the bifurcation parameter. A phenomenological or P-bifurcation was shown to occur at $\alpha = 8$.

Our work can further be extended to include random external excitation. This can be reduced to the problem of a two-dimensional Markov diffusive process in the two conserved quantities namely, the Hamiltonian and the conserved momentum corresponding to the S^1 symmetry of the system. We shall consider the case of noisy external excitations separately. Real-noise models for the random perturbations can also be considered.

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Appendix A. Equations of motion for a taut string

As a concrete example, we consider the non-linear response of a taut string of length l (neglecting its longitudinal inertia) to an excitation parallel to its axis. Let the string be subjected to a constant tension N_0 and free to oscillate in the transverse directions. Let c_1 and c_2 denote the speeds of the longitudinal and transverse linear waves, respectively. These speeds are related to constant tension N_0 and the string properties, such as Young's modulus E, strain e_0, density ρ and cross-sectional area of the string and are given by

$$c_1^2 = \frac{E(1+e_0)}{\rho} \quad \text{and} \quad c_2^2 = \frac{N_0}{\rho A}. \tag{A.1}$$

When the speed c_1 is much larger than the speed c_2, the dynamics of the string is governed by [26]

$$v_{tt}(x,t) + 2\bar{\mu} v_t(x,t) - (c_2^2 + h(t)) v_{xx}(x,t)$$

$$= \frac{c_1^2}{2l} v_{xx}(x,t) \int_0^t (v_x^2(x,t) + w_x^2(x,t))\, \mathrm{d}x + f_v(x,t),$$

$$w_{tt}(x,t) + 2\bar{\mu} w_t(x,t) - (c_2^2 + h(t)) w_{xx}(x,t)$$

$$= \frac{c_1^2}{2l} w_{xx}(x,t) \int_0^t (v_x^2(x,t) + w_x^2(x,t))\, \mathrm{d}x + f_w(x,t), \tag{A.2}$$

where $v(x,t)$ and $w(x,t)$ are the displacements in the transverse directions, $\bar{\mu}$ is the damping coefficient, $h(t)$ is the applied longitudinal force, and $f_v(x,t)$ and $f_w(x,t)$ are applied transverse forces. For a string with fixed ends, the boundary conditions provide linear undamped mode shapes which can be used in the Galerkin procedure to yield

$$\ddot{\eta}_n(t) + n^2\eta_n(t) + n^2\xi(t)\eta_n(t) + 2\mu\eta_n(t) + \frac{n^2}{4}\pi^4\eta_n(t)\sum_{m=1}^{\infty} m^2(\eta_m^2(t) + \zeta_m^2(t)) = g_{v_n}(t),$$

$$\ddot{\zeta}_n(t) + n^2\zeta_n(t) + n^2\xi(t)\zeta_n(t) + 2\mu\zeta_n(t) + \frac{n^2}{4}\pi^4\zeta_n(t)\sum_{m=1}^{\infty} m^2(\eta_m^2(t) + \zeta_m^2(t)) = g_{w_n}(t),$$

where

$$\xi(t) = \frac{h(t)}{c_2^2}, \quad g_{v_n,w_n}(t) = \frac{2}{\pi^2 c_2^2}\int_0^l f_{v,w}(x,t)\sin\frac{n\pi x}{l}\,\mathrm{d}x,$$

$$\mu = \frac{2}{\pi c_2}\int_0^l \bar{\mu}\sin\frac{n\pi x}{l}\,\mathrm{d}x.$$

We are mainly interested in the parametrically excited system with $g_{v_n,w_n}(t) = 0$.

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