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Letter to the Editor

About the stability of non-conservative undamped systems

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1. Introduction

This paper deals with the stability of non-conservative undamped linear systems of the form $M\ddot{x} + Kx = 0$, where **M** and **K** are arbitrary square matrices and the damping matrix is absent. Sometimes, these systems of ordinary differential equations with asymmetric stiffness matrices are called non-self-adjoint boundary problems [1].

Theoretical interest in such systems has been stimulated by a number of flutter-related instability phenomena: instability of tubes conveying the flow of gas or liquid is a classical example [2]. In aeronautics, an important non-conservative problem can be found in the instability of wings in air flow. Namely, it consists of the bending-torsion mode of dynamic instability of aircraft wings in the air flow [3]. Another example of a non-conservative problem is given by cantilever beams subjected to a follower load. A large amount of work has been done to study the transition between stability and instability. Zigler [4], among others, produced stability diagrams in terms of the load versus non-conservative loading parameters for a 2-degree-of-freedom (d.o.f.) model (inverted double pendulum). His results have been resumed and extended by many other researchers [5].

The stability analysis of most of the aforesaid non-conservative problems, has been carried out by eigenvalue calculations. However, when the number of d.o.f. of the system is high (three or more), it is often impossible to obtain a closed-form solution for the eigenvalue calculation. In the following sections, a powerful tool for carrying out stability conditions expressed by means of a set of non-linear inequalities, no matter how high the number of d.o.f. of the system is, will be shown. Thus, numerical calculation of eigenvalues is avoided.

It is well known that, when inertial and stiffness matrices of an undamped conservative system are symmetric positive definite, the system is stable in a BIBO sense (also called weakly stable) [6]. In such a case, the system simply vibrates, that is, performs harmonic oscillations about the equilibrium point.

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On the other hand, in the case of an undamped non-conservative system, instability could occur. In fact, the system could be either weakly stable, or unstable by flutter, or unstable by divergence. As it has been previously said, a possible method for studying the stability of these systems consists in numerically calculating the eigenvalues. Although this method can be easily employed when the elements of the matrices are numerical values, it is less useful when the elements of the matrices are expressed by means of functions of some physical parameters. The main reason is that, in this case, one would desire to express the eigenvalues directly in terms of the physical parameters, in order to see the effect of the parameters on the stability. Therefore, closed-form solution of the characteristic equation is required. The solution cannot always be given in closed form, especially when the degree of the polynomial characteristic equation is high.

Another tool for investigating the stability is the Routh–Hurvitz criterion. Nowadays, this criterion finds a lot of applications in system theory because it predicts stability without calculating the eigenvalues. Unfortunately, it cannot be employed to predict the weak stability of undamped systems as it has been highlighted by Afolabi [7]. The main reason is that the Routh–Hurwitz criterion gives stability conditions in an asymptotical sense, but it does not predict the weak stability. To overcome the Routh–Hurwitz criterion's limits, Afolabi proposed an alternative criterion to predict the stability of non-conservative linear undamped systems. This criterion gives necessary but not sufficient conditions for an undamped non-conservative system to be weakly stable.

In this paper, necessary and sufficient conditions will be given for a linear non-conservative undamped system to be stable, in a BIBO sense (weak stability). An original theorem, which extends the results obtained by Afolabi, will be introduced. The theorem is a complete tool for stability analysis. It has three main features: it is complete, in the sense that it gives both necessary and sufficient conditions as far as weak stability is concerned; it does not require eigenvector calculation; stability conditions, involving characteristic polynomial coefficients, are given by a set of non-linear inequalities.

The effectiveness of the method will be shown by means of three illustrative numerical examples. Such examples aim to give a clear explanation of how to use the theorem in a practical scenario. They also stress the difference between flutter and divergence instability. Eventually, the theorem will be employed to study the stability of two linear non-conservative undamped systems that represent the dynamic behavior of two real mechanical phenomena: the mode-coupling chatter in machining, and the instability of a set of three cantilever beams subjected to a follower load.

2. Theory

Non-conservative undamped linear systems are mostly expressed in the form

$$M\ddot{x} + Kx = 0, \tag{1}$$

where **M** and **K** are arbitrary real square matrices of order n and x is the state vector. Such a system is called undamped since the damping matrix is absent.

The characteristic polynomial associated with system (1) has the structure

$$\left|M\lambda^2 + K\right| = a_0\lambda^{2n} + a_1\lambda^{2(n-1)} + \cdots + a_{n-1}\lambda^2 + a_n, \tag{2}$$

where $a_0, a_1, \ldots, a_n$ are real coefficients.

If system (1) was conservative, **K** and **M** would be symmetric and positive definite. If so, all polynomial roots λ_i^2 $(i = 1, \dots, n)$ would be negative real and their square roots $\lambda_i = \pm \mathrm{j}\omega_i$, would be purely imaginary numbers. ω_i $(i = 1, \dots, n)$ are the natural frequencies of the free vibrations and $\mathrm{j} = \sqrt{-1}$. As a consequence, the system would be always *weakly stable*. In such a case, the system will perform harmonic oscillations about the equilibrium point.

On the other hand, as it has been explained in the introduction, there exist a number of dynamic systems where **K** and/or **M** are not symmetric. The dynamic behavior of such a system can be classified into three categories [6]:

(a) *The system is weakly stable*: This case occurs when all roots λ_i^2 are real negative. The system is dynamically stable in the sense that the motion is harmonic and bounded (BIBO stability).
(b) *The system loses stability via divergence*: This case occurs when all the λ_i^2 are real numbers and at least one $\lambda_k^2 > 0$. As a consequence, the real positive eigenvalue $\lambda_k > 0$ will give rise to an aperiodic, exponentially growing motion.
(c) *The system loses stability via flutter*: This case occurs when at least one of the roots λ_i^2 is complex. If λ_i^2 is complex, solutions of the eigenproblem occur in two eigenvalues with positive real part. In fact, if $\lambda_i^2 = (\alpha_i^2 - \omega_i^2) + \mathrm{j}(2\alpha_i\omega_i)$ is the complex root, where both α_i and ω_i are real positive, and its conjugate is $\overline{\lambda_i^2} = (\alpha_i^2 - \omega_i^2) - \mathrm{j}(2\alpha_i\omega_i)$, the four associated eigenvalues are $\pm\alpha_i \pm \mathrm{j}\omega_i$. Two of these eigenvalues have positive real parts α_i, yielding exponentially growing oscillations. This is called periodic exponential instability or flutter instability.

It is possible to rewrite polynomial (2) by replacing $\mu = \lambda^2$. The characteristic equation becomes

$$f(\mu) = a_0\mu^n + a_1\mu^{n-1} + \cdots + a_{n-1}\mu + a_n = 0. \tag{3}$$

Polynomial (3) is referred to as a *reduced polynomial* in the variable μ.

Essentially, the whole theory is based on the consideration that systems (1) is weakly stable if and only if all the roots of the reduced polynomial (3) are real and non-positive.

There are different methods to check whether all the roots of a polynomial are real. In this paper, the method proposed by Yang [8] will be employed and briefly reviewed as follows.

Some useful definitions are given, which differ slightly from Yang's ones.

Definition 1. Given a polynomial $f(\mu) = a_0\mu^n + a_1\mu^{n-1} + \cdots + a_{n-1}\mu + a_n$, the following $(2n \times 2n)$ matrix is called *discriminant matrix*:

$$\Delta(f) = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & & \\ 0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} & 0 & & \\ & a_0 & a_1 & \cdots & a_{n-1} & a_n & 0 & \\ & & na_0 & \cdots & 2a_{n-1} & a_{n-1} & & \\ & & & \cdots & \cdots & & & \\ & & & \cdots & \cdots & & & \\ & & & 0 & a_0 & a_1 & \cdots & a_n \\ & & & & 0 & na_0 & \cdots & a_{n-1} \end{bmatrix}. \tag{4}$$

The discriminant matrix in turn can be thought of as associated with the undamped system or with the characteristic polynomial of the undamped system.

Definition 2. The sequence

$$\{D_1, \dots, D_n\}, \tag{5}$$

where D_i is the determinant of the sub-matrix of the *discriminant matrix* formed by the first $2i$ rows and $2i$ columns, is called a *discriminant sequence* of the polynomial $f(\mu) = a_0\mu^n + a_1\mu^{n-1} + \cdots + a_{n-1}\mu + a_n$. Sometimes, the D_i's are called *sub-discriminants* or *principal sub-resuetants*.

Theorem 1. *Given a polynomial $f(\mu) = a_0\mu^n + a_1\mu^{n-1} + \cdots + a_{n-1}\mu + a_n$, with real coefficients, a necessary and sufficient condition for the polynomial to have only real roots, is that the elements of the discriminant sequence are all non-negative*

$$D_i \geqslant 0, \quad i = 1, \dots, n. \tag{6}$$

The proof can be found in Yang [8] or Grantmacher [9]. Note that if just one sub-discriminant D_i was negative, at least one root of the reduced polynomial associated to the system would be complex. As a consequence, the system would have an eigenvalue λ_i with positive real part and non-null complex part. Therefore, the system would lose stability via flutter.

On the contrary, if all $D_i \geqslant 0$, in order to complete the stability analysis, it is necessary to check for the negativeness of all the roots of the reduced polynomial.

This test can be carried out by means of Theorem 2. In conclusion, Theorem 1 gives sufficient conditions for an undamped system to be weakly stable and necessary and sufficient conditions for an undamped system to loose stability via flutter.

Theorem 2. *A reduced polynomial $f(\mu) = a_0\mu^n + a_1\mu^{n-1} + \ldots + a_{n-1}\mu + a_n$ with real coefficient is given. Suppose that all the roots of the reduced polynomial are real* (*roots' realness can be verified by means of Theorem* 1). *A necessary and sufficient condition for all the roots of the reduced polynomial to be negative is that polynomial coefficients are either all non-positive or all non-negative.*

Proof of necessity. The polynomial can be factorized as follows:

$$a_0\mu^n + a_1\mu^{n-1} + \cdots + a_{n-1}\mu + a_n = a_0\prod_{i=1}^{n}(\mu - \mu_i) = 0, \tag{7}$$

where μ_i are the roots of the reduced polynomial. Each coefficient of the polynomial can be expressed by sums and products of positive terms $-\mu_i \geqslant 0$ multiplied by the coefficient a_0. Therefore, the coefficients are all non-positive or all non-negative. The sign of the coefficients depends on the sign of a_0.

Proof of sufficiency. If all the coefficients of the polynomial are non-negative (with at least one coefficient positive), it yields

$$f(\mu) > 0, \quad \forall \mu > 0.$$

Table 1
Stability analysis of an undamped non-conservative system by means of sub-discriminants D_i and polynomial coefficients a_i

Theorem 1	Theorem 2	
	$a_i \leqslant 0,\ i = 1, \dots, n$ or $a_i \geqslant 0,\ i = 1, \dots, n$	$\exists a_i > 0,\ a_j < 0,\ i \neq j$
$D_i \geqslant 0,\ i = 1, \dots, n$	Weak stability μ_i real, $\mu_i \leqslant 0$ λ_i pure complex	Divergence μ_i real, $\exists \mu_i > 0$ $\exists \lambda_i$ with positive real part and null complex part
At least one $D_i < 0$	Flutter $\exists \mu_i$ complex $\exists \lambda_i$ with positive real part	Flutter μ_i real, $\mu_i \leqslant 0$ λ_i pure complex

On the other hand, if all the coefficients of the polynomial are non-positive (with at least one coefficient negative), it yields

$$f(\mu) > 0, \quad \forall \mu > 0.$$

In either cases, $f(\mu) \neq 0,\ \ \forall \mu > 0$. Therefore, the polynomial does not have positive real roots.

Note that the hypothesis of Theorem 2 (all real roots) requires that $D_i \geqslant 0$. Combining Theorems 1 and 2, the following final theorem on the weak stability can be introduced. The proof is a direct consequence of the first two theorems:

Theorem. *Consider a linear undamped system. Let its polynomial characteristic be* $a_0\lambda^{2n} + a_1\lambda^{2n-1} + \cdots + a_{n-1}\lambda^2 + a_n$ *and its reduced polynomial be* $f(\mu) = a_0\mu^n + a_1\mu^{n-1} + \cdots + a_{n-1}\mu + a_n$. *A necessary and sufficient condition for the system to be weakly stable is that all the elements of the discriminant sequence of the reduced polynomial are non-negative and that all the coefficients of the polynomial are all non-positive or all non-negative. The results are summarized by means of Table 1.*

3. Examples

The proposed method will be illustrated by means of some numerical examples. The three examples show how the stability of undamped systems can be studied without calculating the roots of the characteristic polynomial. Moreover, Examples 2 and 3 provide evidence that the method can be employed to determine the kind of instability: *divergence* and *flutter*.

Example 1. Suppose that the characteristic polynomial of an undamped system is

$$f(\lambda) = \lambda^6 + 6\lambda^4 + 11\lambda^2 + 6 = 0.$$

Theorems 1 and 2 can be applied to the following reduced polynomial:

$$g(\mu) = f(\sqrt{\mu}) = \mu^3 + 6\mu^2 + 11\mu + 6 = 0.$$

Its discriminant matrix is

$$\Delta(g) = \begin{bmatrix} 1 & 6 & 11 & 6 & 0 & 0 \\ 0 & 3 & 12 & 11 & 0 & 0 \\ 0 & 1 & 6 & 11 & 6 & 0 \\ 0 & 0 & 3 & 12 & 11 & 0 \\ 0 & 0 & 1 & 6 & 11 & 6 \\ 0 & 0 & 0 & 3 & 12 & 11 \end{bmatrix}.$$

Computing the discriminant sequence, it yields

$$\left\{ D_1 = \det\begin{bmatrix} 1 & 6 \\ 0 & 3 \end{bmatrix} = 3, D_2 = \det\begin{bmatrix} 1 & 6 & 11 & 6 \\ 0 & 3 & 12 & 11 \\ 0 & 1 & 6 & 11 \\ 0 & 0 & 3 & 12 \end{bmatrix} = 6, D_3 = \det[\Delta(g)] = 4 \right\}.$$

Since all the elements of the discriminant sequence are positive, the roots of the polynomial are real according to Theorem 1. Moreover, since all the coefficients of the polynomial are positive, all the roots of the reduced polynomial are also negative, according to Theorem 2. In conclusion, the undamped system is weakly stable. This conclusion is in accordance with the roots of the reduced polynomial

$$\mu_1 = -3, \mu_2 = -2, \mu_3 = -1$$

that are all negative.

Example 2. Suppose that the characteristic polynomial of an undamped system is

$$f(\lambda) = \lambda^6 + 6\lambda^4 + 11\lambda^2 + 7 = 0 \quad \Rightarrow \quad g(\mu) = f(\sqrt{\mu}) = \mu^3 + 6\mu^2 + 11\mu + 7 = 0.$$

The discrimination matrix is

$$\Delta(g) = \begin{bmatrix} 1 & 6 & 11 & 7 & 0 & 0 \\ 0 & 3 & 12 & 11 & 0 & 0 \\ 0 & 1 & 6 & 11 & 7 & 0 \\ 0 & 0 & 3 & 12 & 11 & 0 \\ 0 & 0 & 1 & 6 & 11 & 7 \\ 0 & 0 & 0 & 3 & 12 & 11 \end{bmatrix},$$

while the discriminant sequence is

$$\{D_1 = 3, D_2 = 6, D_3 = -23\}.$$

Since an element of the discriminant sequence is negative, it can be immediately inferred that the system loses stability via flutter (see Theorem 1). This conclusion is in accordance with the roots of the reduced polynomial, since two of them are complex:

$$\mu_1 = -3.32472, \quad \mu_2 = -1.33764 - \mathrm{i}0.56228, \quad \mu_3 = -1.33764 + \mathrm{i}0.56228.$$

Example 3. Suppose that the characteristic polynomial of an undamped system is

$$f(\lambda) = \lambda^6 + 1.5\lambda^4 - 0.25\lambda^2 - 0.375 = 0 \;\Rightarrow\; g(\mu) = f(\sqrt{\mu}) = \mu^3 + 1.5\mu^2 - 0.25\mu - 0.375 = 0.$$

The discriminant matrix is

$$\Delta(g) = \begin{bmatrix} 1 & 1.5 & -0.25 & -0.375 & 0 & 0 \\ 0 & 3 & 3 & -0.25 & 0 & 0 \\ 0 & 1 & 1.5 & -0.25 & -0.375 & 0 \\ 0 & 0 & 3 & 3 & -0.25 & 0 \\ 0 & 0 & 1 & 1.5 & -0.25 & -0.375 \\ 0 & 0 & 0 & 3 & 3 & -0.25 \end{bmatrix},$$

while the discriminant sequence

$$\{D_1 = 3, D_2 = 6, D_3 = 4\}.$$

As in Example 1, since all the elements of the discriminant sequence are positive, the roots of the reduced polynomial are real (see Theorem 1). For Theorem 2, since the two coefficients of the polynomial are negative, the reduced polynomial has a positive real root. It can be inferred that the undamped system associated with the reduced polynomial loses stability via divergence. This conclusion is in accordance with the roots of the reduced polynomial that are all real but one:

$$\mu_1 = -1.5, \quad \mu_2 = -0.5, \quad \mu_3 = 0.5.$$

Example 4. The following example comes from the problem of self-excited vibrations that occur during many machining operations [10]. Gasparetto studied the cutting process by means of a linear 2-d.o.f. dynamic model, namely one of the form $M\ddot{x} + Kx = 0$, where

$$M = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, \quad K = \begin{bmatrix} k_x + k\sin(\gamma)\cos(\gamma) & -k\cos^2(\gamma) \\ k\sin^2(\gamma) & k_y - k\sin(\gamma)\cos(\gamma) \end{bmatrix},$$

where M is the cutting tool mass, K_x, K_y are stiffness coefficients associated with the machine structure, and k, γ are parameters associated with the cutting process.

Note that the stiffness matrix is asymmetric. Gasparetto carried out the stability conditions by evaluating the eigenvalues and eigenvectors of the system. It will be shown how to lead to the same stability conditions, while avoiding eigenvalue calculation.

The characteristic polynomial of the undamped system and the reduced polynomial are, respectively,

$$\begin{aligned} f(\lambda) &= M^2\lambda^4 + M(k_x+k_y)\lambda^2 + k_xk_y - k(k_x-k_y)\sin(\gamma)\cos(\gamma) = 0, \\ g(\mu) &= f(\sqrt{\mu}) = a_0\mu^2 + a_1\mu + a_2 \\ &= M^2\mu^2 + M(k_x+k_y)\mu + k_xk_y - k(k_x-k_y)\sin(\gamma)\cos(\gamma) = 0. \end{aligned}$$

The discriminant matrix is

$$\Delta(g) = \begin{bmatrix} M^2 & M(k_x+k_y) & k_xk_y - k(k_x-k_y)\sin(\gamma)\cos(\gamma) & 0 \\ 0 & 2M^2 & M(k_x+k_y) & 0 \\ 0 & M^2 & M(k_x+k_y) & k_xk_y - k(k_x-k_y)\sin(\gamma)\cos(\gamma) \\ 0 & 0 & 2M^2 & M(k_x+k_y) \end{bmatrix}.$$

Computing the discriminant sequence, it yields

$$D_1 = 2M^4, \quad D_2 = (k_x-k_y)(k_x-k_y+2k\sin(2\gamma)).$$

All the elements of the discriminant sequence as well as the reduced polynomial coefficients have to be positive, in order for the system to be weakly stable. Note that $a_0 > 0,\ a_1 > 0,\ D_1 > 0$ Therefore, the only two necessary and sufficient conditions for stability are

$$a_0 \geqslant 0 \;\Leftrightarrow\; k_xk_y - k(k_x-k_y)\sin(\gamma)\cos(\gamma) \geqslant 0,$$

$$D_2 \geqslant 0 \;\Leftrightarrow\; (k_x-k_y)(k_x-k_y+2k\sin(2\gamma)) \geqslant 0.$$

These conditions agree with the conditions carried out by Gasparetto. It is reminded that Gasparetto carried out the conditions by eigenvalue calculations.

Example 5. Consider the system of three beams of Fig. 1. Three beams are connected to each other by means of friction-free revolute joints and torsional springs whose stiffness constant is k. Beams are assumed to be light and rigid. Each beam is l long. In addition, small angular displacements are assumed. As a consequence, the approximations $\sin(\vartheta_1) \cong \vartheta_1$, $\cos(\vartheta_1) \cong 1$ are adopted. Only three masses are taken into account, namely $2m$, $2m$ and m. They are located at each revolute joint. The last beam is subjected to a tangential follower load P_F and to a constant direction load P_C at its free end. For the sake of simplicity, the parameters $P = P_C + P_F$ and $\alpha = P_F/P$ are introduced. α is referred to as the *non-conservative loading parameter*. As α varies, divergence instability or flutter instability can occur.

The dynamic equations are derived as follows. All the forces acting on the system are schematized in Fig. 1, inertial forces included. Momentum balances with respect to points A (considering the external forces acting on the three beams), B (considering the external forces acting only on the last two beams) and C (considering the external forces acting only on the

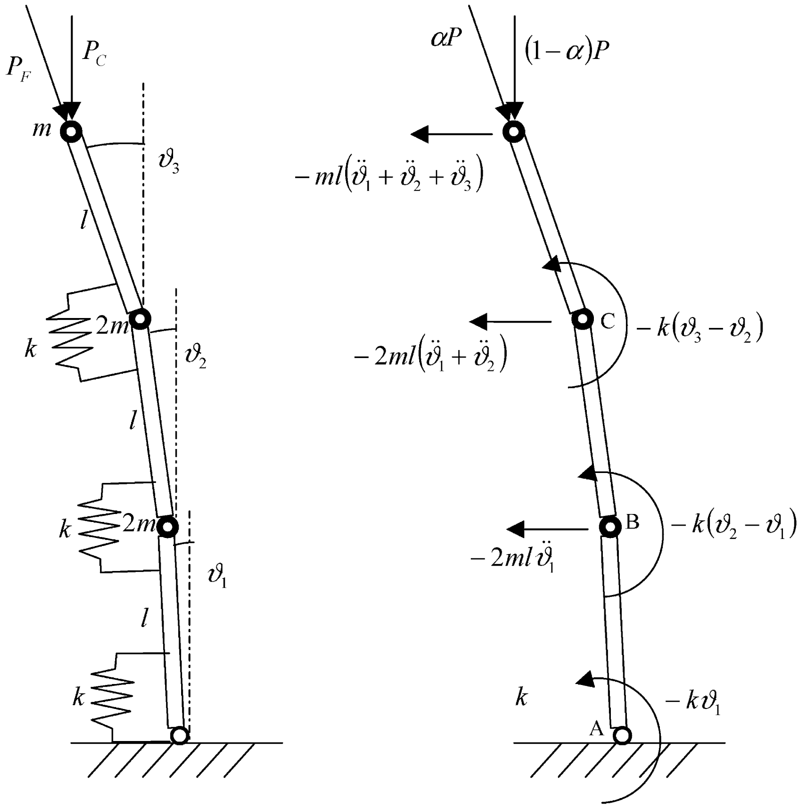

Fig. 1. Dynamic model of the system subjected to a follower load and a constant direction load.

last beam) are (see Fig. 1)

$$
\begin{aligned}
&-ml(\ddot{\vartheta}_1+\ddot{\vartheta}_2+\ddot{\vartheta}_3)3l-2ml(\ddot{\vartheta}_1+\ddot{\vartheta}_2)2l-2ml\ddot{\vartheta}_1 l-k\vartheta_1\\
&\quad+(1-\alpha)Pl(\vartheta_1+\vartheta_2+\vartheta_3)-\alpha Pl(-\vartheta_1-\vartheta_2+2\vartheta_3)=0,\\
&-ml(\ddot{\vartheta}_1+\ddot{\vartheta}_2+\ddot{\vartheta}_3)2l-2ml(\ddot{\vartheta}_1+\ddot{\vartheta}_2)l-k(\vartheta_2-\vartheta_1)\\
&\quad+(1-\alpha)Pl(\vartheta_2+\vartheta_3)-\alpha Pl(\vartheta_3-\vartheta_2)=0,\\
&-ml(\ddot{\vartheta}_1+\ddot{\vartheta}_2+\ddot{\vartheta}_3)l-k(\vartheta_3-\vartheta_2)+(1-\alpha)Pl\vartheta_3=0.
\end{aligned}
$$

Using a matrix notation

$$
M\begin{Bmatrix}\ddot{\vartheta}_1\\ \ddot{\vartheta}_2\\ \ddot{\vartheta}_3\end{Bmatrix}+K\begin{Bmatrix}\vartheta_1\\ \vartheta_2\\ \vartheta_3\end{Bmatrix}=0,
$$

where

$$
M=ml^2\begin{bmatrix}9&7&3\\4&4&2\\1&1&1\end{bmatrix},\quad K=\begin{bmatrix}(k-Pl)&-Pl&Pl(3\alpha-1)\\-k&(k-Pl)&Pl(2\alpha-1)\\0&-k&Pl(\alpha-1)+k\end{bmatrix}.
$$

The reduced polynomial of the system is

$$g(\mu) = f(\sqrt{\mu}) = t_0\mu^3 + t_1\mu^2 + t_2\mu + t_3 = 0,$$

where the parameters

$$\begin{aligned} t_0 &= (4l^6m^3)/k^3, \\ t_1 &= (2l^4m^2(13 + 2(\alpha - 3)\beta))/k^2, \\ t_2 &= (l^2m(26 + 18(\alpha - 2)\beta + (9 - 6\alpha)\beta^2))/k, \\ t_3 &= 1 + (\alpha - 1)(6\beta - 5\beta^2 + \beta^3) \end{aligned}$$

are the polynomial coefficients and $\beta = Pl/k$ is termed the *critical load*.

Using the new notation, the discriminant matrix becomes

$$\Delta(g) = \begin{bmatrix} t_0 & t_1 & t_2 & t_3 & 0 & 0 \\ 0 & 3t_0^2 & 2t_1 & t_2 & 0 & 0 \\ 0 & t_0 & t_1 & t_2 & t_3 & 0 \\ 0 & 0 & 3t_0^2 & 2t_1 & t_2 & 0 \\ 0 & 0 & t_0 & t_1 & t_2 & t_3 \\ 0 & 0 & 0 & 3t_0^2 & 2t_1 & t_2 \end{bmatrix},$$

while the discriminant sequence is

$$D_1 = 2t_1^2, \quad D_2 = t_0^2[(6t_0 - 4)t_1^2 + 3(1 - 3t_0)t_0^2t_2],$$
$$D_3 = -t_0^2[(2 - 3t_0)\,t_1^2t_2^2 + 4(3t_0 - 2)\,t_1^3t_3 + 18(1 - 2t_0)t_0^2\,t_1t_2t_3 + t_0((1 - 3t_0^2)t_2^2 + 27t_0^4t_3^2)].$$

Now, the stability can be analyzed with respect to the parameters α and β in the range $(-1, 5)$ of the parameter α and $(-4, 6)$ of the parameter β. The results are represented by plots in Fig. 2. The first two plots of Fig. 2 show the regions where the sub-discriminants D_2 and D_3 are positive. Note that D_1 is always positive for each value of α and β. Therefore, it is not necessary to represent the diagram of the region where D_1 is positive. Also, the first plot shows that D_2 is positive for $\alpha \in [-1, 5]$ and $\beta \in [-4, 6]$. The second plot shows that there exists a region where D_3 is negative. According to Theorem 1, when α and β belong to that region, the reduced polynomial has at least a complex root. Therefore, the system has at least an eigenvalue with positive real part: the system will be unstable and the system will lose stability via flutter. Outside that region, all the roots of the reduced polynomial are real.

Stability analysis will be completed by applying Theorem 2. The system will be weakly stable if and only if all the reduced polynomial coefficients t_i are non-negative. Note that t_0 is always positive. Therefore it is necessary to study only the sign of t_1, t_2 and t_3. Plots on the second row of Fig. 2 represent the regions where t_1, t_2 and t_3 are positive. In conclusion, according to Table 1, the only region of weak stability is given by the intersection of the regions where t_1, t_2, t_3, D_2 and D_3 are positive. Such a region is represented by the bottom plot of Fig. 2. The diagram perfectly agrees with the one obtained by Gasparini et al. [5]. Remember that diagrams obtained in this paper have been carried out by means of analytical equations, whilst the ones obtained by Gasparini are by means of numerical eigenvalue calculations.

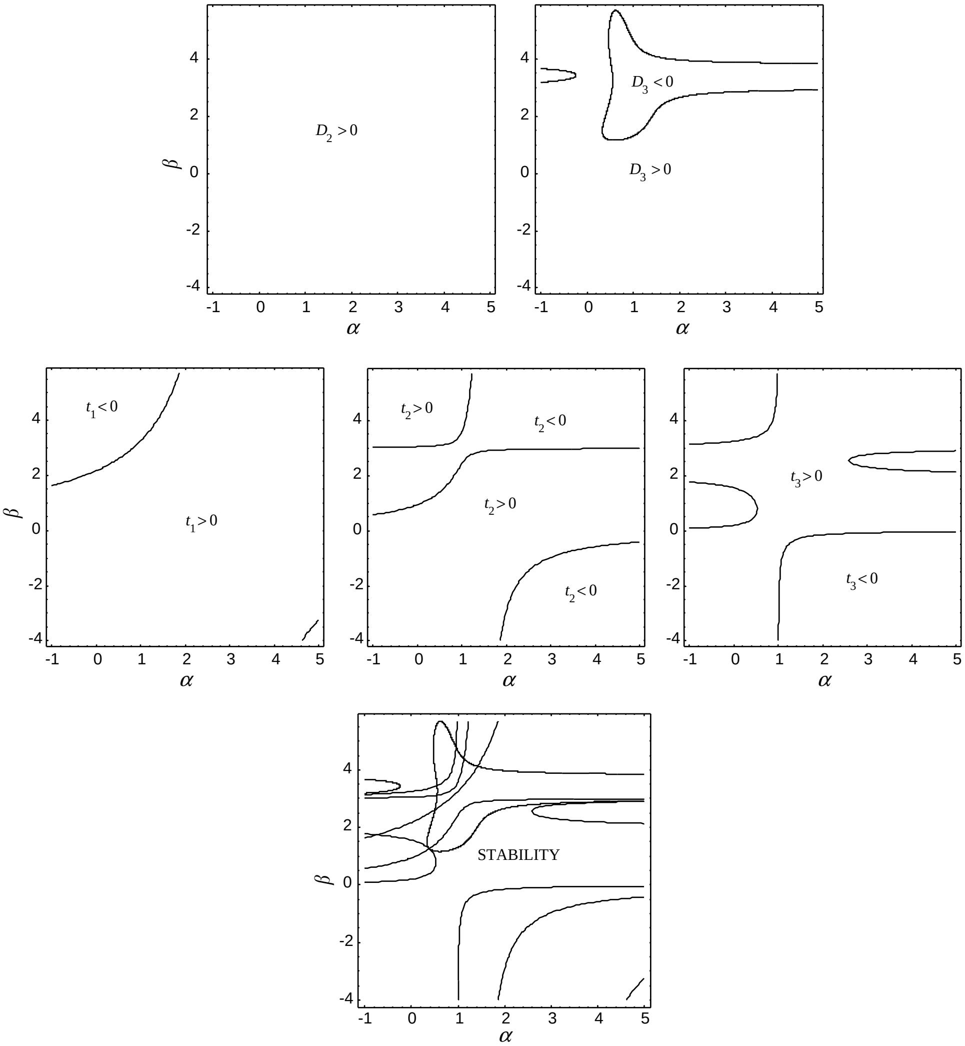

Fig. 2. Stability diagrams. Top plots: regions where sub-discriminants D_2 and D_3 are positive; top plot on the left: region where flutter instability is present ($D_3<0$); second row plots: regions where t_1, t_2 and t_3 are positive; bottom plot: stability region.

4. Conclusions

A new theorem for the weak stability analysis of linear undamped systems has been introduced. The theorem enables one to determine the weak stability of an undamped linear non-symmetric system without eigenvalue computation. In fact, this criterion does for undamped systems what Routh–Hurvitz criterion does for linear dynamic systems with damping. Moreover, this criterion enables one to classify the kind of instability: divergence or flutter. Eventually several examples show the effectiveness of the theorem.

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