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Exact analysis of localized modes in bi-periodic mono-coupled mass–spring systems with a single disorder

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Abstract

The frequency bands of perfect bi-periodic mass–spring systems and the localized modes in the same systems with one disordered subsystem are exactly analyzed using the U-transformation method.

The linear bi-periodic system with an infinite number of subsystems may be considered as an equivalent cyclic bi-periodic system having infinite subsystems. The governing equation for such an equivalent system with cyclic bi-periodicity can be uncoupled by applying the U-transformation twice to form a set of single-degree-of-freedom equations. These equations can be used to analyze the pass bands and localized modes corresponding to the considered system with and without disorder, respectively.

Some specific systems are taken as examples to demonstrate how to apply the formulas obtained in this paper and to find the localized modes and frequencies.

1. Introduction

Analyses of bi-periodic systems have been presented by a number of researchers using various methods, including transfer matrix method [1], wave approach [2–4], standard stiffness and transmission methods [5] and U-transformation method [6–10]. Vibration analyses of free wave motion and response in mono-coupled periodic systems with a single or multiple disorders have been investigated by Bansal [11] and Mead et al. [12–14] using receptance method.

Localized phenomenon was first predicted by Anderson [15] in the field of solid-state physics. It was shown that the electron eigenstates in a disordered solid may become localized. The localized problems in solid-state physics have been an active area of research for the past over 40 years. In

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structural dynamics, Hodges [16] was the earliest to show that in one-dimensional periodic structures all modes are localized for arbitrarily small extended disorders. There is a great volume of literature on localization. A detailed discussion of that literature is contained in the special issue of Chaos, Solitons and Fractals on localization problems [17].

Mode localization phenomenon in an infinite periodic mass–spring system with a single disorder was investigated by Cai et al. [18] using the U-transformation method. Localized modes in mono-coupled periodic mass–spring systems having one or two non-linear disorders were studied by means of the U-transformation method and L-P method [19]. Recently, the same method was applied to analyze the localized modes in a periodic mass–spring system with two coupling co-ordinates between adjacent elements and a non-linear disordered subsystem [20]. Meanwhile, the method was also used to analyze the forced vibration of damped periodic systems having one non-linear disorder [21].

Generally, a bi-periodic mass–spring system can be regarded as a single periodic one but there could be many degrees of freedom in a typical subsystem. If the U-transformation method is applied to this single periodic system as illustrated in the book [22], each uncoupled equation still contains many unknown variables, the number of which is equal to the number of degrees of freedom for each subsystem. Therefore, it may be impossible to obtain the solution in explicit form. Though the subsystem or the uncoupled equation possesses periodicity, it is not cyclic periodic. Hence, it is not possible to go any further to apply the U-transformation technique directly to uncouple these equations. In order to find explicit solution for the bi-periodic mass–spring system, it is necessary to apply the U-transformation twice to the equivalent cyclic bi-periodic system with one-degree-of-freedom subsystems as illustrated in Ref. [6].

The considered system consists of two kinds of periodic mass–spring subsystems and has infinite number of subsystems with a single disorder. The aim of the present paper is to study the application of the U-transformation to the localized vibration analysis in bi-periodic systems. The explicit form of the frequency equation for localized modes is derived and the attenuation constants of localized modes are also found.

It should be emphasized that the expressions of the obtained results (e.g., frequency equation and attenuation constant) are very simple and concise although the derivation is somewhat lengthy by using the U-transformation method. On the other hand, the analysis of the considered problem can also be conducted if the receptance methods or the transfer matrix methods are used. The two kinds of methods are very successful in the analysis of one-dimensional periodic systems. For two-dimensional bi-periodic systems (e.g., the dynamic analysis of rectangular networks with periodically distributed supports and a single disorder); however, the receptance methods and the transfer matrix methods may cease to be effective. But the U-transformation method can be very easily extended to analyze the two-dimensional bi-periodic systems so long as the double U-transformation is used to replace the U-transformation [7,8].

2. Governing equation

Consider an infinite bi-periodic mass–spring system with one disorder, as shown in Fig. 1. This system consists of two different kinds of subsystems, say M- and M'-subsystems, where only one

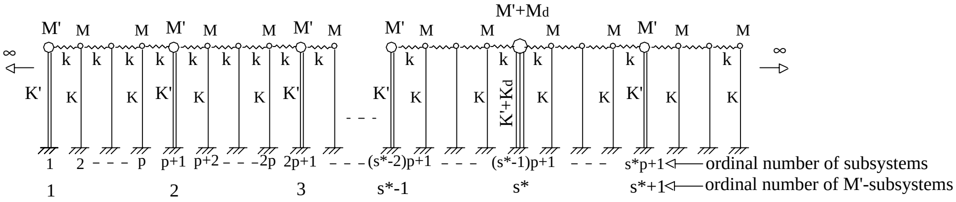

Fig. 1. Infinite bi-periodic mass–spring system with one disorder.

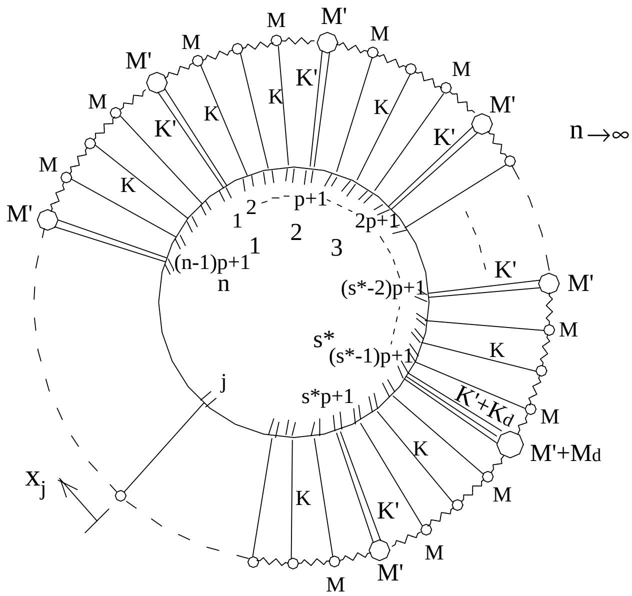

Fig. 2. Rotationally bi-periodic mass–spring system with one disorder.

subsystem departs from the regularity in both stiffness and mass. In Fig. 1, K (K') and M (M') denote the stiffness and mass for M (M')-subsystems, k denotes the coupling stiffness and K_d, M_d denote the magnitudes of the disorders in stiffness and mass, respectively. Without loss of generality, $(s-1)p+1$ ($s = 1, 2, \dots$) denotes the ordinal number of the sth M'-subsystem and $(s^*-1)p+1$ denotes the ordinal number of the disordered subsystem (see Fig. 1). If the considered bi-periodic system is regarded as a single periodic one, p denotes the number of degrees of freedom for a typical subsystem.

The localized modes in an infinite periodic mass–spring system are negligibly affected by the conditions at infinity. Consequently, the system under consideration may be regarded as a cyclic bi-periodic one, as shown in Fig. 2.

At the outset a cyclic bi-periodic system with n M'-subsystems (see Fig. 2) is considered. Then by adopting a limiting process with n approaching infinity, the governing equation and its solution will be applicable for a cyclic bi-periodic system with infinite subsystems.

Applying Newton's second law to every M' and M, the natural vibration equations can be expressed as

$$(K + 2k - M\omega^2)x_j - k(x_{j+1} + x_{j-1}) = -(\Delta K - \Delta M\omega^2)x_j + F_j,$$
$$j = 1 + (s-1)p, \quad s = 1, 2, \ldots, n \tag{1a}$$

and

$$(K + 2k - M\omega^2)x_j - k(x_{j+1} + x_{j-1}) = 0,$$
$$j \neq 1, 1+p, \ldots, 1+(n-1)p, \quad j = 1, 2, \ldots, np, \tag{1b}$$

respectively, where x_j denotes the displacement for the jth subsystem; ω represents the natural frequency; and

$$\Delta K = K' - K, \quad \Delta M = M' - M, \tag{2}$$

$$F_{j^*} = (M_d\omega^2 - K_d)x_{j^*}, \quad j^* \equiv 1 + (s^* - 1)p \quad (j^* \text{ is the fixed number}) \tag{3}$$

with the other F_j vanishing and $x_{np+1} \equiv x_1$, $x_0 \equiv x_{np}$ due to the cyclic periodicity.

Formally, the terms $-(\Delta K - \Delta M\omega^2)x_j$ and F_j on the right side of Eq. (1a) act as the loads.

3. The first application of the U-transformation

The left sides of Eqs. (1a) and (1b) possess cyclic periodicity. In order to uncouple the left sides of the simultaneous Eqs. (1a) and (1b), one can now apply the U-transformation to Eqs. (1a) and (1b). Let

$$q_m = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-i(j-1)m\psi} x_j, \quad m = 1, 2, \ldots, N \tag{4a}$$

and its inverse transformation is

$$x_j = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} e^{i(j-1)m\psi} q_m, \quad j = 1, 2, \ldots, N, \tag{4b}$$

where

$$N = np, \quad \psi = \frac{2\pi}{N}, \quad i = \sqrt{-1}. \tag{5}$$

The right side of Eq. (4b) can be regarded as a series of rotating modes for the cyclic periodic system with N M-subsystems and q_m is the coefficient of the mth rotating mode having phase difference $m\psi$ between two adjacent subsystems. $q_1, q_2, \ldots, q_N$ can be regarded as a set of generalized displacements.

The natural vibration equations (1a) and (1b) can be expressed in terms of the generalized displacements q_m $(m = 1, 2, \cdots, N)$ as

$$(K + 2k - M\omega^2)q_m - 2k\cos m\psi q_m = f_m^0 + f_m', \quad m = 1, 2, \ldots, N, \tag{6}$$

in which

$$f_m^0 = -\frac{(\Delta K - \Delta M\omega^2)}{\sqrt{N}} \sum_{u=1}^{n} \mathrm{e}^{-\mathrm{i}(u-1)mp\psi} x_{1+(u-1)p}, \tag{7a}$$

$$f_m' = \frac{M_d\omega^2 - K_d}{\sqrt{N}} \mathrm{e}^{-\mathrm{i}(j^*-1)m\psi} x_{j^*}. \tag{7b}$$

Both f_m^0 and f_m' are dependent on the M' and disordered subsystems, respectively.

The generalized displacement q_m may be formally expressed as

$$q_m = \frac{f_m^0 + f_m'}{K + 2k - M\omega^2 - 2k\cos m\psi}, \quad m = 1, 2, \ldots, N. \tag{8}$$

Substituting Eqs. (8), (7a) and (7b) into Eq. (4b) yields

$$x_j = -\frac{(\Delta K - \Delta M\omega^2)}{N} \sum_{u=1}^{n} \sum_{m=1}^{N} \frac{\mathrm{e}^{\mathrm{i}[j-1-(u-1)p]m\psi}}{K + 2k - M\omega^2 - 2k\cos m\psi} x_{1+(u-1)p} + x_j' \tag{9a}$$

and

$$x_j' = \frac{M_d\omega^2 - K_d}{N} x_{j^*} \sum_{m=1}^{N} \frac{\mathrm{e}^{\mathrm{i}(j-j^*)m\psi}}{K + 2k - M\omega^2 - 2k\cos m\psi}. \tag{9b}$$

Inserting $j = 1 + (s-1)p$ $(s = 1, 2, \ldots, n)$ and $j^* = 1 + (s^* - 1)p$ in Eqs. (9a) and (9b) gives

$$X_s + (\Delta K - \Delta M\omega^2) \sum_{u=1}^{n} \beta_{s,u}^0 X_u = X_s', \quad s = 1, 2, \ldots, n, \tag{10}$$

where

$$X_s \equiv x_{1+(s-1)p}, \quad s = 1, 2, \ldots, n, \tag{11}$$

$$\beta_{s,u}^0 \equiv \frac{1}{N} \sum_{m=1}^{N} \frac{\mathrm{e}^{\mathrm{i}(s-u)mp\psi}}{K + 2k - M\omega^2 - 2k\cos m\psi}, \quad s, u = 1, 2, \ldots, n, \tag{12}$$

$$X_s' \equiv x_{1+(s-1)p}' = (M_d\omega^2 - K_d) X_{s^*} \beta_{s,s^*}^0, \tag{13}$$

X_s and X_{s^*} indicate the displacements for the sth M' and disordered subsystems, respectively. $\beta_{s,u}^0$ denotes the harmonic influence coefficient (i.e., receptance) for the perfectly single periodic system with the parameters ΔK, ΔM, K_d and M_d vanishing.

By using the U-transformation once, the natural vibration equations (1a) and (1b) with $N(= pn)$ unknowns become Eq. (10) with n unknowns.

4. The second application of the U-transformation

Note that the harmonic influence coefficients for the cyclic periodic system possess cyclic periodicity, namely

$$\beta^0_{1,1} = \beta^0_{2,2} = \cdots = \beta^0_{n,n}, \tag{14a}$$

$$\beta^0_{s,1} = \beta^0_{s+1,2} = \cdots = \beta^0_{n,n-s+1} = \beta^0_{1,n-s+2} = \cdots = \beta^0_{s-1,n}, \quad s = 2, 3, \ldots, n. \tag{14b}$$

The left side terms of the simultaneous equations (10) possess cyclic periodicity. The U-transformation can be performed for the second time. Introducing the U- and inverse U-transformation

$$Q_r = \frac{1}{\sqrt{n}} \sum_{s=1}^{n} e^{-i(s-1)r\varphi} X_s, \quad r = 1, 2, \ldots, n \tag{15a}$$

and

$$X_s = \frac{1}{\sqrt{n}} \sum_{r=1}^{n} e^{i(s-1)r\varphi} Q_r, \quad s = 1, 2, \ldots, n \tag{15b}$$

with $\varphi = 2\pi/n = p\psi$, into Eq. (10) results in

$$\left[1 + (\Delta K - \Delta M\omega^2) \sum_{s=1}^{n} \beta^0_{s,1} e^{-i(s-1)r\varphi}\right] Q_r = b_r, \quad r = 1, 2, \ldots, n, \tag{16}$$

where

$$b_r = \frac{1}{\sqrt{n}} \sum_{s=1}^{n} e^{-i(s-1)r\varphi} X'_s. \tag{17}$$

For the second U-transformations (15a) and (15b), Eq. (15b) shows that the displacements of M'-subsystems is expressed by the series of rotating modes for the cyclic periodic system having n M'-subsystems. $Q_1, Q_2, \ldots, Q_n$ are a set of generalized displacements for M'-subsystems.

Substituting Eqs. (12) and (13) into Eqs. (16) and (17), respectively, and using the identical relation

$$\frac{1}{n} \sum_{s=1}^{n} e^{i(s-1)(m-r)\varphi} = \begin{cases} 1, & m = r, r+n, \ldots, r+(p-1)n, \\ 0, & m \neq r, r+n, \ldots, r+(p-1)n, \end{cases}$$

$$m = 1, 2, \ldots, N(=pn), \quad r = 1, 2, \ldots, n \tag{18}$$

we have

$$a_r(\omega) Q_r = b_r(\omega), \quad r = 1, 2, \ldots, n, \tag{19a}$$

in which

$$a_r(\omega) = 1 + (\Delta K - \Delta M\omega^2)\sum_r(\omega), \tag{19b}$$

$$b_r(\omega) = \frac{(M_d\omega^2 - K_d)}{\sqrt{n}} X_{s^*} e^{i(1-s^*)r\varphi} \sum_r(\omega) \tag{19c}$$

and

$$\sum_r(\omega) = \frac{1}{p}\sum_{u=1}^{p}\left\{K + 2k - M\omega^2 - 2k\cos\left[\frac{r\varphi}{p} + (u-1)\frac{2\pi}{p}\right]\right\}^{-1}, \tag{19d}$$

where Q_r can be formally expressed as

$$Q_r = \frac{b_r(\omega)}{a_r(\omega)} = \frac{M_d\omega^2 - K_d}{\sqrt{n}} X_{s^*} e^{i(1-s^*)r\varphi} \frac{\sum_r(\omega)}{1 + (\Delta K - \Delta M\omega^2)\sum_r(\omega)}, \quad r = 1, 2, \ldots, n, \tag{20}$$

where the displacement X_{s^*} for the disordered subsystem is unknown.

Introducing Eq. (20) into the second inverse U-transformation (15b) yields

$$X_s = (M_d\omega^2 - K_d)X_{s^*}\beta_{s,s^*}, \quad s = 1, 2, \ldots, n \tag{21}$$

and

$$\beta_{s,s^*} \equiv \frac{1}{n}\sum_{r=1}^{n} e^{i(s-s^*)r\varphi} \frac{\sum_r(\omega)}{1 + (\Delta K - \Delta M\omega^2)\sum_r(\omega)}, \tag{22}$$

β_{s,s^*} represents the harmonic influence coefficient (i.e., receptance) for the cyclic bi-periodic system without disorder and means the amplitude of the sth M'-subsystem caused by unit harmonic force acting at the s^*th M'-subsystem.

By letting $s = s^*$ and $X_{s^*} \neq 0$ in Eq. (21), the frequency equation can be found as

$$M_d\omega^2 - K_d = \frac{1}{\beta_{s^*,s^*}}, \tag{23}$$

where

$$\beta_{s^*,s^*} = \frac{1}{n}\sum_{r=1}^{n} \frac{\sum_r(\omega)}{1 + (\Delta K - \Delta M\omega^2)\sum_r(\omega)}. \tag{24}$$

The frequency equations (23) and (24) are applicable to the cyclic bi-periodic system with n subsystems and a single disorder.

We can now consider the limiting case of n approaching infinity. By letting n approach infinity, the limit of the series summation on the right side of Eq. (22) becomes the definite integral, namely

$$\beta_{s,s^*} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(s-s^*)\theta \sum_\theta(\omega)}{1 + (\Delta K - \Delta M\omega^2)\sum_\theta(\omega)} d\theta,$$
$$s = s^*, s^* \pm 1, s^* \pm 2, \ldots, s^* \pm \infty, \tag{25}$$

where

$$\sum_{\theta}(\omega) \equiv \frac{1}{p}\sum_{u=1}^{p}\left\{K + 2k - M\omega^2 - 2k\cos\left[\frac{\theta}{p} + (u-1)\frac{2\pi}{p}\right]\right\}^{-1}. \tag{26}$$

Eq. (25) leads to

$$\beta_{s*,s*} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sum_\theta(\omega)}{1 + (\Delta K - \Delta M\omega^2)\sum_\theta(\omega)}\,\mathrm{d}\theta. \tag{27}$$

The frequency equations (23) and (27) are applicable to the original system shown in Fig. 1. It is well known that if the natural frequency ω lies in the pass band of the ordered system, the corresponding mode must extend through to the whole system and if ω lies in the stop band, the corresponding mode is localized. We are interested in the latter case.

5. Pass bands for the ordered bi-periodic systems

One can now consider the case of $M_d = K_d = 0$, leading to $b_r(\omega) = 0$ $(r = 1, 2, \ldots)$. Eq. (19a) becomes

$$a_r(\omega)Q_r = 0, \quad r = 1, 2, \ldots, n. \tag{28}$$

When $Q_r \neq 0$, i.e., $X_s \neq 0$, the corresponding frequency equation can be expressed as

$$a_r(\omega) \equiv 1 + (\Delta K - \Delta M\omega^2)\sum_r(\omega) = 0, \quad r = 1, 2, \ldots, n, \tag{29}$$

which is applicable to the ordered cyclic bi-periodic system. By letting n approach infinity, the frequency equation (29) becomes

$$1 + (\Delta K - \Delta M\omega^2)\sum_\theta(\omega) = 0, \quad 0 < \theta \leqslant 2\pi, \tag{30}$$

which is the frequency band equation for the infinite bi-periodic system without disorder. The solutions for ω of Eq. (30) will be continuously distributed in each pass band. In general if θ is given, there are p roots for ω of Eq. (30). Let $\omega_r(\theta)$ $(r = 1, 2, \ldots, p)$ denote the rth root of Eq. (30). $\omega_r(\theta)$ is the function (may be implicit form) of θ, $\omega_r(\theta)$ $(\theta \in [0, \pi])$ represents the rth pass band. The parameter θ in Eq. (30) indicates the rotating mode phase difference between two adjacent M'-subsystems. We can show readily that if θ is replaced by $2\pi - \theta$, the frequency equation (30) does not change. Its physical meaning is that the two rotating modes with phase differences θ and $2\pi - \theta$ between two adjacent M'-subsystems correspond to a same frequency. Hence, we consider only the case of $\theta \in [0, \pi]$ or $[\pi, 2\pi]$ in Eq. (30).

One can now consider the upper and lower limits (say ω_U and ω_L) of the pass bands. By introducing $\theta = 0$ (or 2π), $\theta = \pi$ and Eq. (26) in Eq. (30), the equations for ω_L and ω_U can be obtained as

$$1 + (\Delta K - \Delta M\omega^2)\frac{1}{p}\sum_{u=1}^{p}\left\{K + 2k - M\omega^2 - 2k\cos(u-1)\frac{2\pi}{p}\right\}^{-1} = 0$$

$$\text{for } \theta = 0 \text{ (or } 2\pi) \tag{31a}$$

and

$$1 + (\Delta K - \Delta M\omega^2)\frac{1}{p}\sum_{u=1}^{p}\left\{K + 2k - M\omega^2 - 2k\cos\left[\frac{\pi}{p} + (u-1)\frac{2\pi}{p}\right]\right\}^{-1} = 0$$
$$\text{for } \theta = \pi. \tag{31b}$$

Eqs. (31a) and (31b) correspond to two kinds of modes having the same and opposite phases for two adjacent M'-subsystems.

When $Q_r \equiv 0$, i.e., $X_s \equiv 0$, which means every mass M' is located at the nodal point of the mode, the corresponding frequency equation for $n \to \infty$ can be obtained, from Eq. (6) with f_m^0 and f_m' vanishing, as

$$K + 2k - M\omega^2 - 2k\cos\frac{r\pi}{p} = 0, \quad r = 1, 2, \ldots, p-1. \tag{32}$$

In Eq. (6), $m\psi$ denotes the mode phase difference between two adjacent subsystems, meaning that $pm\psi$ represents the mode phase difference between two adjacent M'-subsystems. In order to have a node at every mass point M', $pm\psi$ must be equal to $r\pi$ ($r = 1, 2, \ldots, p-1$), i.e., $m\psi = r\pi/p$, where r denotes a mode number.

The solution for ω^2 of Eq. (32) can be expressed as

$$\omega^2 = \frac{K + 2k - 2k\cos\dfrac{r\pi}{p}}{M}, \quad \begin{array}{l} r = \text{odd number } (r \leqslant p-1) \text{ for } \theta = \pi, \\ r = \text{even number } (r \leqslant p-1) \text{ for } \theta = 0, \end{array} \tag{33}$$

in which θ denotes the mode phase difference between two adjacent M'-subsystems.

The bounds of pass bands for the infinite bi-periodic system are made up of all roots for ω of Eqs. (31a), (31b) and (33), which are dependent on the parameter p besides the other system parameters. Let us consider some specific values of p as follows:

(a) $p = 2$: There is one M-subsystem between adjacent pairs of M'-subsystems for the system under consideration.

Eqs. (31a) and (31b) can be simplified, respectively, as

$$(K' + 2k - M'\omega^2)(K + 2k - M\omega^2) - 4k^2 = 0 \quad \text{for } \theta = 0 \tag{34a}$$

and

$$K' + 2k - M'\omega^2 = 0 \quad \text{for } \theta = \pi. \tag{34b}$$

The solutions for ω^2 of Eqs. (34a) and (34b) are

$$\omega_{1,2}^2 = \frac{1}{2}\left(\frac{K' + 2k}{M'} + \frac{K + 2k}{M}\right) \mp \frac{1}{2}\left[\left(\frac{K' + 2k}{M'} - \frac{K + 2k}{M}\right)^2 + \frac{16k^2}{MM'}\right]^{1/2}$$
$$\text{for} \quad \theta = 0 \tag{35a}$$

and

$$\omega^2 = \frac{K' + 2k}{M'} \quad \text{for } \theta = \pi. \tag{35b}$$

The latter frequency corresponds to the mode in which nodes occur at all of the masses M. For $\theta = \pi$, the other frequency limit can be obtained from Eq. (33) with $p = 2$ and $r = 1$, as

$$\omega^2 = \frac{K+2k}{M} \quad \text{for } \theta = \pi. \tag{35c}$$

The nodes of the corresponding mode lie in the M'-subsystems.

In general, the bounds of the sth pass band are made up of both the sth frequencies for $\theta = 0$ and π ($s = 1, 2, \ldots, p$). For the present case of $p = 2$, there are two pass bands which can be denoted by $[\omega_{1L}\ \omega_{1U}]$ and $[\omega_{2L}\ \omega_{2U}]$.

Here ω_{sL} and ω_{sU} ($s = 1, 2, \ldots, p$) represent the lower and upper bounds of the sth pass band, respectively. They can be expressed as

$$\omega_{1L}^2 = \frac{1}{2}\left(\frac{K'+2k}{M'} + \frac{K+2k}{M}\right) - \frac{1}{2}\left[\left(\frac{K'+2k}{M'} - \frac{K+2k}{M}\right)^2 + \frac{16k^2}{MM'}\right]^{1/2}, \tag{36a}$$

$$\omega_{1U}^2 = \min\left(\frac{K'+2k}{M'}, \frac{K+2k}{M}\right), \tag{36b}$$

$$\omega_{2L}^2 = \max\left(\frac{K'+2k}{M'}, \frac{K+2k}{M}\right), \tag{36c}$$

$$\omega_{2U}^2 = \frac{1}{2}\left(\frac{K'+2k}{M'} + \frac{K+2k}{M}\right) + \frac{1}{2}\left[\left(\frac{K'+2k}{M'} - \frac{K+2k}{M}\right)^2 + \frac{16k^2}{MM'}\right]^{1/2}. \tag{36d}$$

If $(K'+2k)/M' = (K+2k)/M$, the two pass bands will degenerate into one pass band, with

$$\omega_L^2 = \frac{K+2k}{M} - \frac{2k}{\sqrt{MM'}} \quad \text{and} \quad \omega_U^2 = \frac{K+2k}{M} + \frac{2k}{\sqrt{MM'}}. \tag{37}$$

When $K = K'$ and $M = M'$, the considered system is an uni-periodic one. Eq. (37) becomes

$$\omega_L^2 = \frac{K}{M}, \quad \omega_U^2 = \frac{K+4k}{M}. \tag{38}$$

(b) $p = 3$: There are two M-subsystems between adjacent pairs of M'-subsystems for the considered one. Eqs. (31a) and (31b) can be simplified as

$$(K' - M'\omega^2)(K + k - M\omega^2) + 2k(K - M\omega^2) = 0 \quad \text{for } \theta = 0 \tag{39a}$$

and

$$(K' + k - M'\omega^2)(K + 3k - M\omega^2) + k(K + k - M\omega^2) = 0 \quad \text{for } \theta = \pi. \tag{39b}$$

The solutions for ω^2 of Eqs. (39a) and (39b) can be found, respectively, as

$$\omega_{01,02}^2 = \frac{1}{2}\left(\frac{K+k}{M} + \frac{K'+2k}{M'}\right) \mp \frac{1}{2}\left[\left(\frac{K+k}{M} - \frac{K'+2k}{M'}\right)^2 + \frac{8k^2}{MM'}\right]^{1/2}$$
$$\text{for } \theta = 0 \tag{40a}$$

and

$$\omega^2_{\pi 1,\pi 2} = \frac{1}{2}\left(\frac{K+3k}{M} + \frac{K'+2k}{M'}\right) \mp \frac{1}{2}\left[\left(\frac{K+3k}{M} - \frac{K'+2k}{M'}\right)^2 + \frac{8k^2}{MM'}\right]^{1/2}. \quad \text{for } \theta = \pi. \tag{40b}$$

In addition, Eq. (33) with $p = 3$ leads to

$$\omega_0^2 = \frac{K+3k}{M} \quad \text{for} \quad \theta = 0 \tag{41a}$$

and

$$\omega_\pi^2 = \frac{K+k}{M} \quad \text{for } \theta = \pi. \tag{41b}$$

If $\omega_{1a}, \omega_{2a}, \omega_{3a}$ $(\omega_{1a} \leqslant \omega_{2a} \leqslant \omega_{3a})$ represent the three frequencies shown in Eqs. (40a) and (41a) and $\omega_{1b}, \omega_{2b}, \omega_{3b}$ $(\omega_{1b} \leqslant \omega_{2b} \leqslant \omega_{3b})$ denote the other three frequencies shown in Eqs. (40b) and (41b), the three pass bands can be expressed as

$$[\omega_{1a}\ \omega_{1b}], \quad [\omega_{2b}\ \omega_{2a}], \quad [\omega_{3a}\ \omega_{3b}]. \tag{42}$$

It can be verified that

$$\omega_{1L} = \omega_{1a} = \omega_{01}, \tag{43a}$$

$$\omega_{2U} = \omega_{2a} = \min(\omega_{02}, \omega_0), \tag{43b}$$

$$\omega_{3L} = \omega_{3a} = \max(\omega_{02}, \omega_0), \tag{43c}$$

$$\omega_{1U} = \omega_{1b} = \min(\omega_{\pi 1}, \omega_\pi), \tag{43d}$$

$$\omega_{2L} = \omega_{2b} = \max(\omega_{\pi 1}, \omega_\pi), \tag{43e}$$

$$\omega_{3U} = \omega_{3b} = \omega_{\pi 2}. \tag{43f}$$

One can show readily that if $(K+k)/M = (K'+k)/M'$, $\omega_{\pi 1} = \omega_\pi$, i.e., $\omega_{1b} = \omega_{2b}$ and if $(K + 3k)/M = (K'+3k)/M'$, $\omega_{02} = \omega_0$, i.e., $\omega_{2a} = \omega_{3a}$. Therefore, for the particular case of $(K + k)/M = (K'+k)/M'$ or $(K+3k)/M = (K'+3k)/M'$, the three pass bands will degenerate into two pass bands.

6. Frequency equation for localized modes

For the original system shown in Fig. 1, the frequency equation for localized modes has been found as shown in Eqs. (23), (26) and (27), which are dependent on the parameter p.

(a) $p = 1$: Substituting Eq. (26) with $p = 1$ into Eq. (27) results in

$$\beta_{s^*,s^*} = \frac{1}{\pi}\int_0^\pi \frac{1}{K'+2k-M'\omega^2-2k\cos\theta}\,\mathrm{d}\theta. \tag{44a}$$

Consider the frequency range given by

$$|K' + 2k - M'\omega^2| > 2k \tag{44b}$$

so that either

$$\omega^2 < \frac{K'}{M'}(= \omega_L^2) \quad \text{or} \quad \omega^2 > \frac{K' + 4k}{M'}(= \omega_U^2). \tag{44c}$$

The definite integral can then be evaluated to yield

$$\beta_{s^*,s^*} = \frac{1}{2k\eta\sqrt{1 - \eta^{-2}}}, \quad |\eta| > 1, \tag{44d}$$

where

$$\eta = \frac{K' + 2k - M'\omega^2}{2k}. \tag{44e}$$

The condition $|\eta| > 1$ is equivalent to Eq. (44c), i.e., ω lies in the stop band. Because the frequency of localized modes must lie in the stop band, Eq. (44d) holds true.

For localized modes, the frequency equation shown in Eqs. (23) and (44d) is in agreement with that given in Ref. [18].

(b) $p = 2$: For this case Eqs. (26) and (27) become

$$\sum_{\theta}(\omega) = \frac{K + 2k - M\omega^2}{(K + 2k - M\omega^2)^2 - 2k^2 - 2k^2\cos\theta} \tag{45a}$$

and

$$\begin{aligned} \beta_{s^*,s^*} &= \frac{1}{\pi}\int_0^{\pi} \frac{K + 2k - M\omega^2}{(K + 2k - M\omega^2)(K' + 2k^2 - M'\omega^2) - 2k^2 - 2k^2\cos\theta}\,\mathrm{d}\theta \\ &= \frac{K + 2k - M\omega^2}{2k^2}\frac{1}{\eta\sqrt{1 - \eta^{-2}}}, \quad |\eta| > 1, \end{aligned} \tag{45b}$$

where

$$\eta = \frac{(K + 2k - M\omega^2)(K' + 2k - M'\omega^2)}{2k^2} - 1. \tag{45c}$$

One can show that the condition $|\eta| > 1$ is equivalent to

$$(\omega^2 - \omega_{1L}^2)(\omega^2 - \omega_{2U}^2) > 0 \tag{45d}$$

or

$$(\omega^2 - \omega_{1U}^2)(\omega^2 - \omega_{2L}^2) < 0, \tag{45e}$$

in which ω_{1L}, ω_{1U}, ω_{2L} and ω_{2U} are given in Eqs. (36a)–(36d).

Eqs. (45d) and (45e) lead to $\omega < \omega_{1L}$ or $\omega > \omega_{2U}$ and $\omega_{1U} < \omega < \omega_{2L}$, respectively. Therefore if and only if ω lies in the stop bands, implying that $|\eta| > 1$, the definite integral for $\beta_{s*,s*}$ exists. This conclusion is the same as that for $p = 1$.

When ω lies in the pass band, $\beta_{s*,s*}$ approaches infinity. This phenomenon can be explained from the physical meaning of $\beta_{s*,s*}$. Because $\beta_{s*,s*}$ is harmonic influence coefficient, if ω approaches a natural frequency, the harmonic influence coefficient must approach infinity. But when ω is equal to a frequency given in Eq. (33), i.e., each M'-subsystem is located at the nodal point of the corresponding mode, $\beta_{s*,s*} = 0$. For the present case, when $\omega = \sqrt{(K+2k)/M}$, $\beta_{s*,s*} = 0$.

Substituting Eq. (45b) into Eq. (23) yields

$$M_d\omega^2 - K_d = \frac{2k^2}{K + 2k - M\omega^2}\eta\sqrt{1-\eta^{-2}}, \quad |\eta| > 1. \tag{46}$$

Eqs. (46) and (45c) represent the frequency equation for localized modes in the considered system with $p = 2$.

Introducing the non-dimensional parameters

$$\Omega^2 = \omega^2\frac{M}{K}, \quad \varepsilon_M = \frac{M_d}{M}, \quad \varepsilon_K = \frac{K_d}{K}, \quad \varepsilon_C = \frac{k}{K}, \quad \alpha = \frac{K'}{K}, \quad \gamma = \frac{M'}{M} \tag{47}$$

into Eqs. (46) and (45c), we have

$$F(\Omega^2) = D(\Omega^2), \tag{48a}$$

where

$$F(\Omega^2) = \varepsilon_M\Omega^2 - \varepsilon_K, \tag{48b}$$

$$D(\Omega^2) = \frac{2\varepsilon_C^2}{1 + 2\varepsilon_C - \Omega^2}\eta\sqrt{1-\eta^{-2}} \tag{48c}$$

and

$$\eta = \frac{(1 + 2\varepsilon_C - \Omega^2)(\alpha + 2\varepsilon_C - \gamma\Omega^2)}{2\varepsilon_C^2} - 1. \tag{48d}$$

The number and magnitude of the frequencies for localized modes can be explained qualitatively by using the graphic representation.

When all the parameters are given, the functions, $y = D(\Omega^2)$ and $F(\Omega^2)$, can be plotted against Ω. The number of the points of intersection between the two curves is equal to the number of localized modes, and the transverse co-ordinates of the intersection points represent the magnitudes of the corresponding frequencies.

Let us consider a specific case, namely

$$M' = 2M, \quad K' = 3K, \quad k = 0.1K \tag{49a}$$

leading to

$$\gamma = 2, \quad \alpha = 3, \quad \varepsilon_C = 0.1. \tag{49b}$$

Introducing Eq. (49a) into Eqs. (36a)–(36d) gives

$$\Omega_{1L}^2 = \omega_{1L}^2\frac{M}{K} = 1.155051026, \tag{50a}$$

$$\Omega_{1U}^2 = \omega_{1U}^2\frac{M}{K} = 1.2, \tag{50b}$$

$$\Omega_{2L}^2 = \omega_{2L}^2\frac{M}{K} = 1.6, \tag{50c}$$

$$\Omega_{2U}^2 = \omega_{2U}^2\frac{M}{K} = 1.644948974. \tag{50d}$$

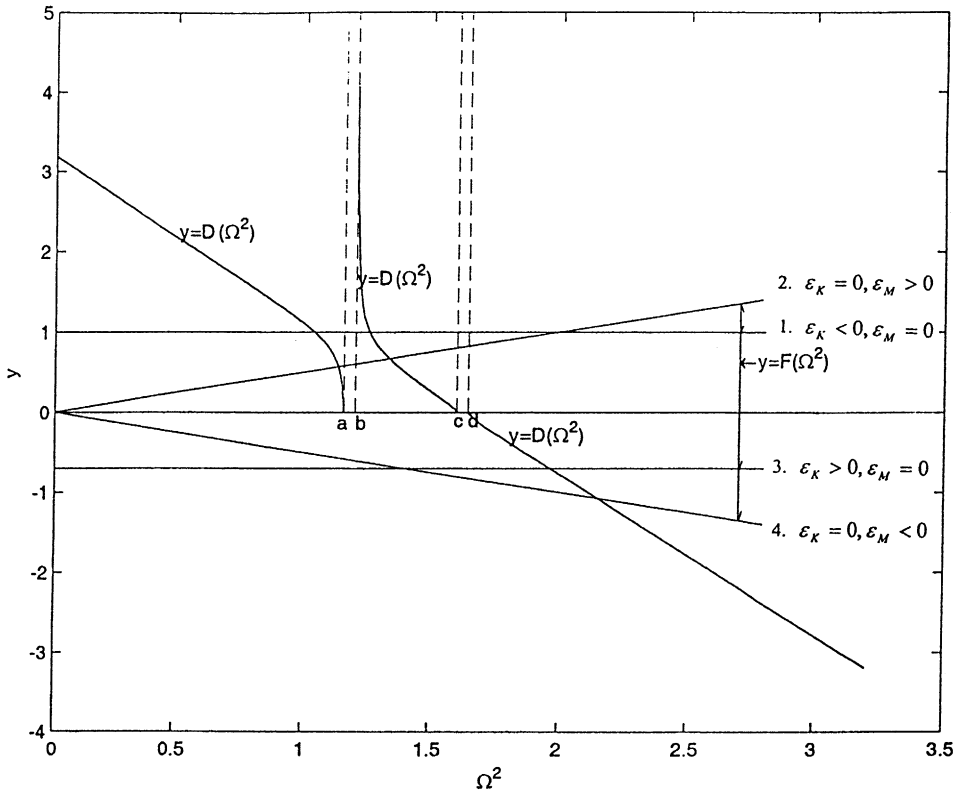

Fig. 3. Functions $D(\Omega^2)$ and $F(\Omega^2)$ for $p=2$: (a) $\Omega_{1L}^2 = 1.155051026$, (b) $\Omega_{1U}^2 = 1.2$, (c) $\Omega_{2L}^2 = 1.6$, (d) $\Omega_{2U}^2 = 1.644948974$.

The function $D(\Omega^2)$ can be defined when Ω lies in the stop bands, namely $\Omega < \Omega_{1L}, \Omega_{1U} < \Omega < \Omega_{2L}, \Omega > \Omega_{2U}$.

The curve, D versus Ω^2, is plotted in Fig. 3, which is made up of three continuous and monotonically decreasing curves, corresponding to the three stop bands. The straight lines, $y = F(\Omega^2)$, corresponding to various disordered parameters, are also plotted in Fig. 3. By observation from Fig. 3, it is clear that in either case of $\varepsilon_K < 0, \quad \varepsilon_M = 0$ or $\varepsilon_K = 0, \quad \varepsilon_M > 0$, two localized modes with $\Omega < \Omega_{1L}$ and $\Omega_{1U} < \Omega < \Omega_{2L}$ will occur, and in either case of $\varepsilon_K > 0, \quad \varepsilon_M = 0$ or $\varepsilon_K = 0, \quad \varepsilon_M < 0$, one localized mode with $\Omega > \Omega_{2U}$ will occur.

(c) $p = 3$: For this case, one can show that Eqs. (26) and (27) lead to

$$\sum_\theta(\omega) = \frac{(K+2k-M\omega^2)^2 - k^2}{(K+2k-M\omega^2)^3 - 3k^2(K+2k-M\omega^2) - 2k^3\cos\theta} \tag{51a}$$

and

$$\begin{aligned} \beta_{s^*,s^*} &= \frac{1}{\pi}\int_0^\pi \frac{(K+2k-M\omega^2)^2 - k^2}{2k^3(\eta - \cos\theta)}\,\mathrm{d}\theta \\ &= \frac{(K+2k-M\omega^2)^2 - k^2}{2k^3}\frac{1}{\eta\sqrt{1-\eta^{-2}}}, \quad |\eta| > 1, \end{aligned} \tag{51b}$$

where

$$\eta = \{(K + 2k - M\omega^2)^2(K' + 2k - M'\omega^2) \\ - k^2[2(K + 2k - M\omega^2) + K' + 2k - M'\omega^2]\}/2k^3. \tag{51c}$$

It can be proved that the condition $|\eta| > 1$ is again equivalent to ω lying in the stop bands, namely $\omega < \omega_{1L}$, $\omega_{1U} < \omega < \omega_{2L}$, $\omega_{2U} < \omega < \omega_{3L}$ and $\omega > \omega_{3U}$.

The frequency equation (23) can be expressed as

$$M_d\omega^2 - K_d = \frac{2k^3}{(K + 2k - M\omega^2)^2 - k^2}\eta\sqrt{1 - \eta^{-2}}, \quad |\eta| > 1. \tag{52}$$

By introducing the non-dimensional parameters shown in Eq. (47) into Eqs. (52) and (51c), we have

$$F(\Omega^2) = D(\Omega^2), \quad |\eta| > 1, \tag{53a}$$

where

$$F(\Omega^2) \equiv \varepsilon_M\Omega^2 - \varepsilon_K, \tag{53b}$$

$$D(\Omega^2) \equiv \frac{2\varepsilon_C^3}{(1 + 2\varepsilon_C - \Omega^2)^2 - \varepsilon_C^2}\eta\sqrt{1 - \eta^{-2}} \tag{53c}$$

and

$$\eta = \{(1 + 2\varepsilon_C - \Omega^2)^2(\alpha + 2\varepsilon_C - \gamma\Omega^2) - \varepsilon_C^2[2(1 + 2\varepsilon_C - \Omega^2) + \alpha + 2\varepsilon_C - \gamma\Omega^2]\}/2\varepsilon_C^3. \tag{53d}$$

Consider the same system parameters as those shown in Eq. (49a). Introducing Eqs. (40a), (40b), (41a), (41b) and (49) into Eqs. (43a)–(43f) results in

$$\Omega_{1L}^2 \equiv \omega_{1L}^2\frac{M}{K} = 1.08074176, \tag{54a}$$

$$\Omega_{1U}^2 \equiv \omega_{1U}^2\frac{M}{K} = 1.1, \tag{54b}$$

$$\Omega_{2L}^2 \equiv \omega_{2L}^2\frac{M}{K} = 1.269722437, \tag{54c}$$

$$\Omega_{2U}^2 \equiv \omega_{2U}^2\frac{M}{K} = 1.3, \tag{54d}$$

$$\Omega_{3L}^2 \equiv \omega_{3L}^2\frac{M}{K} = 1.61925824, \tag{54e}$$

$$\Omega_{3U}^2 \equiv \omega_{3U}^2\frac{M}{K} = 1.630277563. \tag{54f}$$

In the same way as before, the curve, $y = D(\Omega^2)$ with $\alpha = 3, \gamma = 2, \varepsilon_C = 0.1$, is plotted in Fig. 4. This curve is made up of four sectional curves corresponding to the four stop bands: $\Omega < \Omega_{1L}$, $\Omega_{1U} < \Omega < \Omega_{2L}$, $\Omega_{2U} < \Omega < \Omega_{3L}$ and $\Omega > \Omega_{3U}$. Four straight lines, $y = F(\Omega^2)$, corresponding to four sets of ε_K and ε_M are also plotted in Fig. 4. From Fig. 4, it can be observed that for either case of $\varepsilon_K < 0$, $\varepsilon_M = 0$ or $\varepsilon_K = 0$, $\varepsilon_M > 0$ three localized modes will occur and the

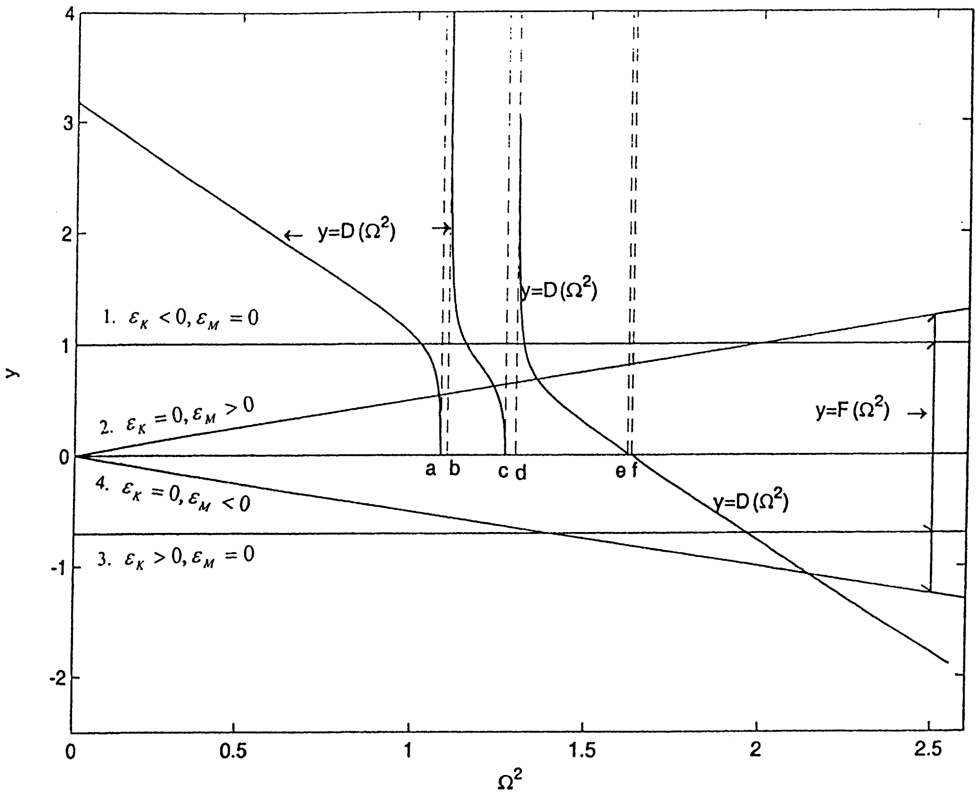

Fig. 4. Functions $D(\Omega^2)$ and $F(\Omega^2)$ for $p=3$: (a) $\Omega_{1L}^2=1.08074176$, (b) $\Omega_{1U}^2=1.1$, (c) $\Omega_{2L}^2=1.269722437$, (d) $\Omega_{2U}^2=1.3$, (e) $\Omega_{3L}^2=1.61925824$, (f) $\Omega_{3U}^2=1.630277563$.

corresponding frequencies lie in the first three stop bands and for either case of $\varepsilon_K>0,\ \varepsilon_M=0$ or $\varepsilon_K=0,\ \varepsilon_M<0$, one localized mode with $\omega>\omega_{3U}$ will occur.

7. Localized modes

When the natural frequency is found by solving Eq. (23), the corresponding mode can be obtained from Eqs. (21) and (23). Introducing Eq. (23) into Eq. (21), yields

$$X_s=X_{s*}\frac{\beta_{s,s*}}{\beta_{s*,s*}},\quad s=s^*,s^*\pm1,s^*\pm2,\ldots, \tag{55a}$$

where X_s and X_{s^*} denote the amplitudes of the sth M' and disordered subsystems, respectively. The above equation can be rewritten as

$$X_{s*+m}=X_{s*}\frac{\beta_{s*+m,s*}}{\beta_{s*,s*}},\quad m=0,\pm1,\pm2,\ldots \tag{55b}$$

and

$$\beta_{s^*+m,s^*} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos m\theta \sum_\theta(\omega)}{1+(\Delta K - \Delta M\omega^2)\sum_\theta(\omega)}\mathrm{d}\theta, \quad m = 0, \pm 1, \pm 2, \ldots. \tag{55c}$$

Eq. (55b) indicates the localized mode for M'-subsystems, where X_{s^*} acts as an arbitrary constant factor.

(a) $p = 1$: Substituting Eq. (26) with $p = 1$ into Eq. (55c) gives

$$\beta_{s^*+m,s^*} = \frac{1}{\pi}\int_0^{\pi} \frac{\cos m\theta}{K' + 2k - M'\omega^2 - 2k\cos\theta}\,\mathrm{d}\theta = \frac{1}{2k} I(m,\eta), \tag{56}$$

where

$$I(m,\eta) \equiv \frac{1}{\pi}\int_0^{\pi} \frac{\cos m\theta}{\eta - \cos\theta}\mathrm{d}\theta, \quad |\eta| > 1, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{57a}$$

$$\eta = \frac{K' + 2k - M'\omega^2}{2k}. \tag{57b}$$

It can be proved that $I(m,\eta)$ possesses the following property:

$$\frac{I(1,\eta)}{I(0,\eta)} = \frac{I(2,\eta)}{I(1,\eta)} = \cdots = \eta(1 - \sqrt{1-\eta^{-2}}). \tag{58}$$

Introducing Eqs. (56) and (58) into Eq. (55b), we have

$$X_{s^*-1} = X_{s^*+1} = X_{s^*}\frac{I(1,\eta)}{I(0,\eta)} = X_{s^*}\xi, \tag{59a}$$

$$X_{s^*-2} = X_{s^*+2} = X_{s^*}\frac{I(2,\eta)}{I(0,\eta)} = X_{s^*}\xi^2, \tag{59b}$$

$$X_{s^*-m} = X_{s^*+m} = X_{s^*}\xi^m, \quad m = 0, 1, 2, \ldots, \tag{59c}$$

where

$$\xi = \eta(1 - \sqrt{1-\eta^{-2}}), \quad |\eta| > 1 \tag{60}$$

is the attenuation constant of the localized mode.

It is obvious that ξ and η are of the same sign, and $|\xi|$ always less than one. ξ is an odd function of η, as shown in Fig. 5.

These results for $p = 1$ are in agreement with those given in Ref. [18].

(b) $p = 2$: Introducing Eq. (45a) into Eq. (55c) results in

$$\begin{aligned}\beta_{s^*+m,s^*} &= \frac{1}{\pi}\int_0^{\pi} \frac{(K+2k-M\omega^2)\cos m\theta}{(K+2k-M\omega^2)(K'+2k-M'\omega^2) - 2k^2 - 2k^2\cos\theta}\mathrm{d}\theta \\ &= \frac{K+2k-M\omega^2}{2k^2}\frac{1}{\pi}\int_0^{\pi}\frac{\cos m\theta}{\eta - \cos\theta}\mathrm{d}\theta = \frac{K+2k-M\omega^2}{2k^2} I(m,\eta),\end{aligned} \tag{61a}$$

where

$$\eta = \frac{(K+2k-M\omega^2)(K'+2k-M'\omega^2)}{2k^2} - 1. \tag{61b}$$

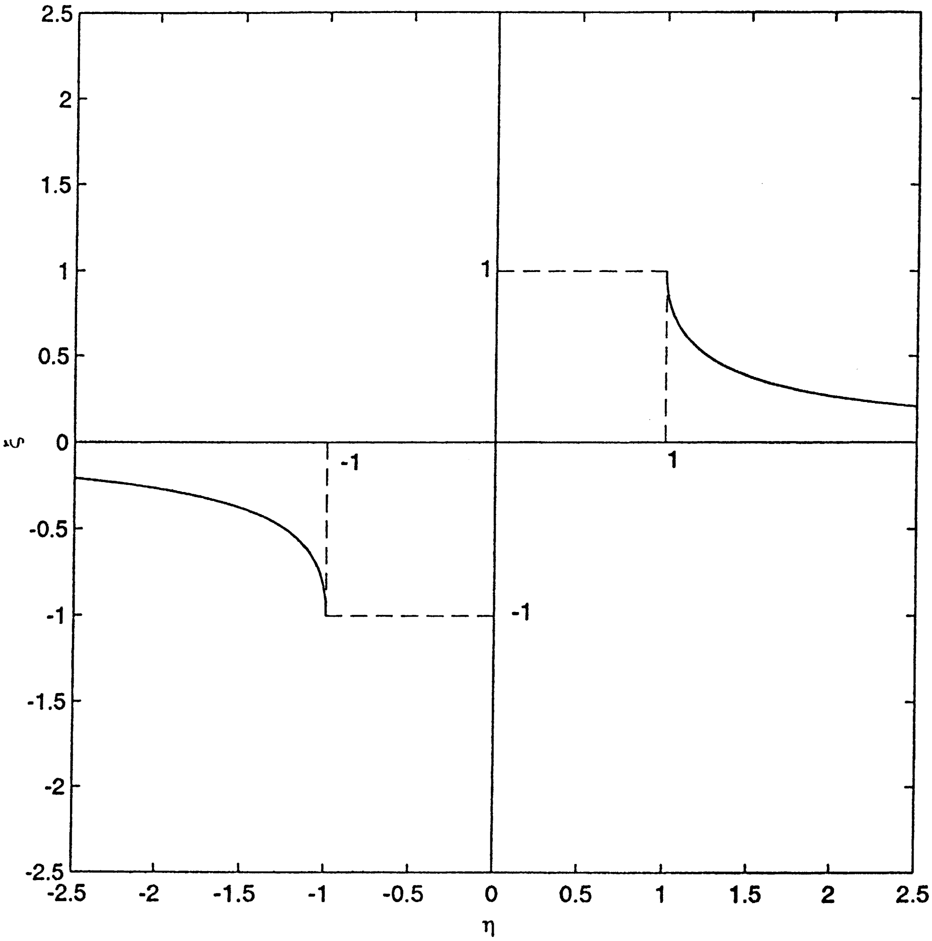

Fig. 5. Attenuation constant ξ versus η, $\xi = \eta(1 - \sqrt{1-\eta^{-2}})$, $|\eta| > 1$.

The definition of $I(m, \eta)$ is the same as that shown in Eq. (57a) except that the definition of η must be Eq. (61b) instead of Eq. (57b).

The localized mode and its attenuation constant can be also expressed as Eqs. (59a)–(59c) and (60), respectively, in which η is defined as that shown in Eq. (61b).

When the frequency corresponding to the localized mode has been found the attenuation constant can be calculated from Eqs. (60) and (61b).

(c) $p = 3$: Inserting Eq. (51a) into Eq. (55c) yields

$$\begin{aligned}\beta_{s^*+m,s^*} &= \frac{(K+2k-M\omega^2)^2 - k^2}{2k^3}\frac{1}{\pi}\int_0^{\pi}\frac{\cos m\theta}{\eta - \cos\theta}\,\mathrm{d}\theta \\ &= \frac{(K+2k-M\omega^2)^2 - k^2}{2k^3} I(m,\eta),\end{aligned} \tag{62a}$$

where

$$\eta = \{(K + 2k - M\omega^2)^2(K' + 2k - M'\omega^2) - k^2[2(K + 2k - M\omega^2) + K' + 2k - M'\omega^2]\}/2k^3. \tag{62b}$$

The localized mode can be also expressed as that shown in Eqs. (59a)–(59c) in which the attenuation constant ξ is the same function of η as that shown in Eq. (60). But the functions η of ω^2 for $p = 1, 2, 3$ have different expressions, as shown in Eqs. (57b), (61b) and (62b) corresponding to $p = 1, 2$ and 3.

When the natural frequency for localized mode is found, its attenuation constant can be calculated by applying formulas (60) and (62b).

8. Conclusions

In the present study, the application of the U-transformation method has been extended from the localized mode analysis of nearly periodic mass–spring systems to the same analysis of bi-periodic mono-coupled mass–spring systems with a single disorder. In order to utilize completely the property of bi-periodicity, the proposed analysis method requires the application of U-transformation twice. The governing equation of natural vibration is uncoupled to form a set of single-degree-of-freedom equations in terms of the harmonic influence coefficients. Then the frequency equation and localized modes can be obtained.

Some specific bi-periodic systems with $p = 1, 2, 3$ are taken as examples. In the two examples, with $p = 2$ and 3, the degenerate phenomenon of pass bands is discovered, i.e., two pass bands have merged into one. The conditions under what the degenerate phenomenon occurs are found in explicit form.

It should be pointed out that the U-transformation method used in the present paper is also applicable to the dynamic analysis of some two-dimensional bi-periodic systems with and without disorders.

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