Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 263 (2003) 1–20
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Vibration analysis of harmonically excited non-linear system using the method of multiple scales

Byung-Young Moon*, Beom-Soo Kang

Department of Aerospace Engineering, Pusan National University, Gumjung-ku, Pusan 609-735, South Korea

Received 18 October 2001; accepted 6 June 2002

Abstract

An analytical method is presented for evaluation of the steady state periodic behavior of non-linear systems. This method is based on the substructure synthesis formulation and a multiple scales procedure, which is applied to the analysis of non-linear responses. A complex non-linear system is divided into substructures, of which equations are approximately transformed to modal co-ordinates including non-linear term under the reasonable procedure. Then, the equations are synthesized into the overall system and the solution of the non-linear system can be obtained. Based on the method of multiple scales, the proposed procedure reduces the size of large-degree-of-freedom problem in solving the non-linear equations. Feasibility and advantages of the proposed method are illustrated by the application of the analytic procedure to the non-linear rotating machine system as a large mechanical structure system. Results obtained are reported to be an efficient approach with respect to non-linear response prediction when compared with other conventional methods.

1. Introduction

In recent years, machines of industrial field used for the gas turbine for propulsion of an aircraft, power-plant turbine, etc. tend toward the high speed and lightweight. These conditions may cause the trouble of non-linear vibration. In the present rotating machinery, non-linear vibration phenomena sometimes occur in the shrinkage fit rotor, in the assembly rotor and in the power-plant rotor with coil. Non-linear vibration phenomena also occur in a high polymer rotor, which is used for lightweight construction of an aircraft. Vibration analysis of such rotor systems is usually performed by the finite element method (FEM) with linear model. When a large amplitude vibration occurs, however, linearized spring and damping coefficients cannot model the

*Corresponding author.
E-mail address: moon_byung_young@hotmail.com (B.-Y. Moon).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(02)01016-7

complicated non-linear rotor system. It is important to consider the non-linear characteristics in vibration analysis and design of rotor systems. On the other hand, it is necessary that a high-speed rotor system used for the gas turbine for propulsion of an aircraft, power-plant turbine, etc. promptly pass a critical speed. Accordingly, the casing is often modelled elastically to decrease the critical speed. When such a rotor-bearing-casing system vibrates, the casing is excited to contact with the rotor and there is a danger that the bearing will be damaged. Therefore, the investigation of the response of a rotating machine is very important from the viewpoint of stable operation. To construct a real mathematical model in vibration analysis, dynamic characteristics of rotor, bearing and casing should be considered. In the analysis of a large complex degrees of freedom (d.o.f.) mechanical system, the substructure synthesis method (SSM) has been studied for efficient vibration analysis, Iwatsubo et al. [1] proposed an approximate analytical method to analyze the dynamic problems of a non-linear rotor-bearing-casing system using the SSM and a perturbation method. They applied the SSM technique to reduce the overall size of the problem and obtained approximate solutions by applying the perturbation method. Moon et al. [2,3] presented an analytical method to analyze the vibration of a non-linear rotor-bearing-casing system by applying the perturbation method. They considered the non-linearity in the shaft and bearing part and considered the effect of non-linear sensitivity in the subsystem. They derived the formulation of perturbation first order under the condition that the exciting force is near the first critical frequency of the system. Moon et al. [4] proposed an approximate analytical method to analyze the dynamic problems of a non-linear structure system using the SSM and a harmonic balance method.

However, a non-linear vibration problem needs more accurate analysis in some rotor system, which is used in the jet engine of an aircraft or some power-plant turbine. Such a high-speed and lightweight mechanical system shows more complex non-linear vibration. In the analysis of non-linear systems, there are a lot of analyzed research works using the method of multiple scales for the single d.o.f. of non-linear vibration system, and its application to the multi-d.o.f. system are reported [5,6]. However, the study, which applied the method of multiple scales to the non-linear vibration analysis of rotor system, has not been reported yet.

Therefore, this paper presents an analytical technique based on the method of multiple scales theory and the mode superposition principle for the dynamic analysis of non-linear mechanical systems. By applying the method of multiple scales, the governing equations of the complex non-linear system attains a compact form and can be solved. Furthermore, the proposed method enhanced the previous studies [1–3] such that it can be applied to more accurate analysis comparing with the perturbation method of the previous studies. Theoretical basis of the proposed method is presented in the derivation of the response of a non-linear system. The proposed method is then applied to a non-linear mechanical system in order to demonstrate the performance of the method in respect of the computational accuracy by comparing the results with those from other conventional methods.

2. Method of analysis

A structural system consists of a set of interconnected components that have segments with distributed mass and elasticity and non-linear parts. Non-linear structures can be divided into linear and non-linear substructures with assembling regions. The first stage in the analysis process, therefore, is the substructuring of the original non-linear system into some components that can be modelled separately with linear and non-linear sets. Small substructures may be easier to model and will eventually result in an economical analysis procedure.

2.1. Modelling of the system

In this paper, a rotor-bearing-casing system as shown in Fig. 1 is considered. The rotor is supported by bearings that are fixed on the casing. The casing and the foundation are elastically connected. The rotor has the material non-linearity. For dynamic analysis of this kind of complex system, the SSM can be applied. The whole system is divided into three components. The rotor has non-linear restoring force so that it is regarded as a non-linear component, while the casing is considered to be a linear component and the bearing is modelled as a linear assembling component.

The co-ordinate system of the rotor-bearing-casing system is shown in Fig. 1. The o–xyz co-ordinate system is fixed in space, where the x-axis is perpendicular to directions of shaft and casing, the y-axis is vertically upwards, and the z-axis is along the shaft and the casing for consistency of modelling. The acceleration of gravity is ignored for simplicity. The rotor gyroscopic effect caused by non-linear restoring force is not considered in this study for simplicity of the non-linear analysis. Instead, a proportional damping model is considered in the equations of motion. The shaft and casing components are modelled by using the FEM. A common form of excitation of a rotor system is the mass unbalance of the rotor. Then assumption of a steady state response is reasonable. The excitation forces at a given station by the imbalance mass m_i at a distance e_i from the rotor geometric center are given by

$$\{^1F_u(\Omega, t)\} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = \begin{Bmatrix} m_i e_i \Omega^2 \cos(\Omega t + \phi_i) \\ m_i e_i \Omega^2 \sin(\Omega t + \phi_i) \end{Bmatrix} \quad (i = 1, 2, 3, \ldots), \tag{1}$$

where ϕ, Ω are the phase quantity and the rotating frequency. The excitation force by the mass unbalance of the rotor can be treated as a harmonic excitation force. In general, the response shows well the non-linear characteristics around the natural frequency in the non-linear system, as well observed in the single-d.o.f. system. Especially in the rotor system, the dynamic behavior

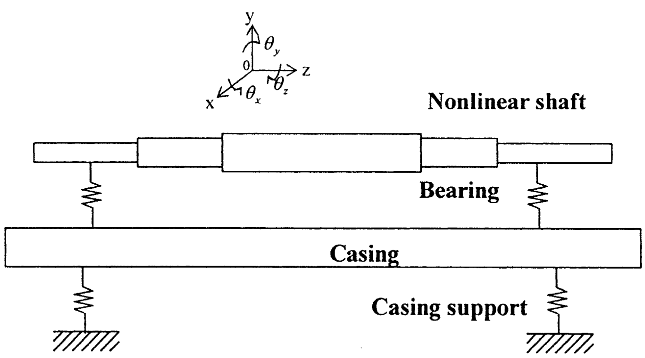


Fig. 1. Rotor-bearing-casing system.

around the critical speed is very important where most of the troubles occur. Therefore, it is needed to quickly pass the critical speed without troubles. Because of these reasons, the exciting frequency around the first natural frequency of the system is considered. The non-linear restoring force is transformed into modal co-ordinates under the condition that the excitation force is near the first natural frequency [2–4]. The non-linear term is handled with the MS method. The excitation force of the casing component is treated as the general force F_c.

2.2. Modelling of non-linear component

When the rotor is modelled by the FEM, the characteristic of non-linear restoring force for each element is based on the relation that the stress of the element by bending moment is represented by the sum of two terms, one that varies linearly with the strain plus another that varies with the third power of the strain. The internal force is considered because the non-linear component can be synthesized through the internal force with the other components. By considering the boundary conditions, the equation of motion for the non-linear component can be written as [2–4]

$$[{}^1M]\{{}^1\ddot{u}\} + [{}^1K]\{{}^1u\} + \varepsilon[K_N]\{{}^1u^3\} = \{{}^1F_u(\Omega, t)\} + \{{}^1F_b\}, \tag{2}$$

where $[{}^1M]$, $[{}^1K]$ are the mass and stiffness matrices, respectively, $\{{}^1F_u(t)\}$ is an external force vector by unbalance of rotor, $\{{}^1F_b\}$ is an internal force vector, $[K_N]$ $\{{}^1u^3\}$ is a non-linear term, and ε is a small parameter. Superscript denotes the non-linear component. The displacement vector can be written as

$$\{{}^1u\} = \{x_i, \theta_{xi}, y_i, \theta_{yi}\}^{\mathrm{T}} \quad (i = 1, 2, \ldots, n), \tag{3}$$

where x_i, θ_{xi} and y_i, θ_{yi} are the displacements and rotations for the x direction and y direction in the ith nodal point, and n is the number of nodes. Exactly to say, vibration modes of a non-linear system are slightly different from those of a linear system. But for simplicity of analysis, they are assumed to keep those of a linear one. Accordingly, the modal co-ordinate system can be obtained using the modal matrix $[{}^1\Phi]$ of the linear system. Then, the displacement can be transformed into the modal co-ordinate $\{{}^1\xi\}$ system as

$$\{{}^1u\} \equiv [{}^1\Phi]\{{}^1\xi\}. \tag{4}$$

Substituting Eq. (4) into Eq. (2) and pre-multiplying both sides of Eq. (2) by $[{}^1\Phi]^{\mathrm{T}}$, Eq. (2) is expanded to the non-linear modal equation as

$$\{{}^1\ddot{\xi}\} + [\diagdown {}^1\omega^2 \diagdown]\{{}^1\xi\} + \varepsilon[{}^1\Phi]^{\mathrm{T}}[K_N]\{{}^1u^3\} = \varepsilon\{{}^1f_u(\Omega, t)\} + \{{}^1f_b\}, \tag{5}$$

where $\{{}^1f_u\}(= [{}^1\Phi]^{\mathrm{T}}\{{}^1F_u(t)\})$, $\{{}^1f_b(t)\}(= [{}^1\Phi]^{\mathrm{T}}\{{}^1f_b\})$ are the external and internal forces in modal co-ordinates. Usually, $[{}^1\Phi]^{\mathrm{T}}[K_N]\{{}^1u^3\}$ is not a diagonal matrix. This term will be changed into modal co-ordinates in accordance with the reasonable procedure, as shown in Refs. [2–4]. Then, a non-linear term can be derived as $\varepsilon[\diagdown k'_N \diagdown]\{\xi_1^3\}$ where $[\diagdown k'_N \diagdown]$ is the diagonal matrix term when the system is excited around the first natural frequency. Accordingly, the non-linear term is approximated as a diagonal matrix resulting in an efficient analysis by adopting a small number of lower frequency modes [9]. Here, the perturbation method is introduced to solve the non-linear Eq. (5). The variant $\varepsilon[\diagdown k'_N \diagdown]$ can be regarded as the perturbation parameter term, because the term is relatively smaller than $[\diagdown {}^1\omega^2_\diagdown]$. Thus, $\{{}^1\xi\}$ can be expanded in terms of ε

$$\{{}^1\xi\} = \{{}^1\xi^{(0)}\} + \varepsilon\{{}^1\xi^{(1)}\} + \varepsilon^2\{{}^1\xi^{(2)}\} + \cdots, \tag{6}$$

where superscripts $(\cdot)$ denotes the perturbation order. Substituting Eq. (6) into Eq. (5), and arranging by ε, the perturbed equations are evaluated as

$$\begin{aligned}
&\{{}^1\ddot{\xi}^{(0)}\} + [\diagdown {}^1\omega^2_\diagdown]\{{}^1\xi^{(0)}\} = \{{}^1f_b^{(0)}\},\\
&\{{}^1\ddot{\xi}^{(1)}\} + [\diagdown {}^1\omega^2_\diagdown]\{{}^1\xi^{(1)}\} = \{{}^1f_u\} + \{{}^1f_{p1}({}^1\xi^{(0)})\} + \{{}^1f_b^{(1)}\},\\
&\{{}^1\ddot{\xi}^{(2)}\} + [\diagdown {}^1\omega^2_\diagdown]\{{}^1\xi^{(2)}\} = \{{}^1f_{p2}({}^1\xi^{(0)2}, {}^1\xi^{(1)})\} + \{{}^1f_{p1}({}^1\xi^{(0)})\} + \{{}^1f_b^{(2)}\},
\end{aligned} \tag{7}$$

where $\{{}^1f_b^{(0)}\}$, $\{{}^1f_b^{(1)}\}$ and $\{{}^1f_b^{(2)}\}$ are perturbed internal forces, $\{{}^1f_{p1}\}$, $\{{}^1f_{p2}\}$ include the non-linear stiffness term as $\{{}^1f_{p1}({}^1\xi^{(0)}\} = -[\diagdown {}^1k_N \diagdown]\{{}^1\xi^{(0)3}\}$, $\{{}^1f_{p2}({}^1\xi^{(0)2}, {}^1\xi^{(1)})\} = \{-3[\diagdown {}^1k_N \diagdown]\{{}^1\xi^{(0)2}\cdot{}^1\xi^{(1)}\}\}$. Here $\{{}^1\xi^{(0)2}\cdot{}^1\xi^{(1)}\}$ is a perturbed modal displacement term which comes from the perturbation zeroth order and perturbation first order.

2.3. Modelling of linear, assembling components and overall system

The casing is modelled as a linear substructure. After the eigenvalue analysis, the equation of motion in the modal co-ordinates is obtained as

$$[\diagdown I \diagdown]\{{}^2\ddot{\xi}\} + [\diagdown {}^2w^2_\diagdown]\{{}^2\xi\} = \varepsilon\{f_c\} + \{{}^1f_b\}, \tag{8}$$

where $[\diagdown {}^2w^2_\diagdown]$ and $[\diagdown I \diagdown]$ are eigenvalue of linear substructure and identity matrices, respectively, $[f_c]$ is the external force vector. The internal force is introduced in the equation because the linear substructure can be assembled through the internal force with the other substructures. Even the casing component is linear system, this component is perturbed as same as the non-linear component, because the higher order harmonic oscillation which is occurring in the non-linear component is translated through the higher order perturbed equation as

$$\begin{aligned}
&\{{}^2\ddot{\xi}^{(0)}\} + [\diagdown {}^2\omega^2_\diagdown]\{{}^2\xi^{(0)}\} = -\{{}^2f_b^{(0)}\},\\
&\{{}^2\ddot{\xi}^{(1)}\} + [\diagdown {}^2\omega^2_\diagdown]\{{}^2\xi^{(1)}\} = \{f_c\} - \{{}^2f_b^{(1)}\},\\
&\{{}^2\ddot{\xi}^{(2)}\} + [\diagdown {}^2\omega^2_\diagdown]\{{}^2\xi^{(2)}\} = -\{{}^2f_b^{(2)}\},
\end{aligned} \tag{9}$$

where $\{{}^2f_b^{(0)}\}, \{{}^2f_b^{(1)}\}$ and $\{{}^2f_b^{(2)}\}$ are the perturbed internal forces.

As an assembling component, ball bearings are considered. Generally, there is a damping term in the bearing, but it is ignored in this study for the simplified model of bearing in order to verify the effect of non-linear restoring force. The restoring force of the bearing is modelled as a linear force. In this case, the force and displacement are expressed as

$$[{}^1k_{b1}]\{{}^1x_b\} = \{{}^1f_b\}, \quad [{}^2k_{b2}]\{{}^2x_b\} = -\{{}^2f_b\}, \tag{10}$$

where $[{}^jk_{bj}]$ $(j = 1, 2)$ are the terms of bearing coefficient. $\{{}^1f_b\}$, $\{{}^2f_b\}$ are the internal force vectors of the non-linear component and linear component, respectively. $\{{}^jx_b\}$ is the relative displacements between the rotor and casing corresponding to the bearings. In order to solve the overall equation, the small parameter is set equal to the perturbation parameter of the non-linear component. Then, the displacement is expressed as

$$\{^j x_b\} = \{^j x_b^{(0)}\} + \varepsilon\{^j x_b^{(1)}\} + \varepsilon^2\{^j x_b^{(2)}\} \quad (j = 1, 2). \tag{11}$$

Accordingly, by using Eq. (11), the internal force vectors can be perturbed as

$$\begin{aligned} \{^1 f_b\} &= \{^1 f_b^{(0)}\} + \varepsilon \cdot \{^1 f_b^{(1)}\} + \varepsilon^2 \cdot \{^1 f_b^{(2)}\}, \\ \{^2 f_b\} &= \{^2 f_b^{(0)}\} + \varepsilon \cdot \{^2 f_b^{(1)}\} + \varepsilon^2 \cdot \{^2 f_b^{(2)}\}. \end{aligned} \tag{12}$$

To reduce the size of overall equation, the SSM can be applied. In order to synthesize the components, Eqs. (7), (9) and (12) are combined and rewritten according to the equation of order $\varepsilon^{(p)}$ $(p = 0, 1, 2)$

$$\{\ddot{\xi}^{(p)}\} + [\bar{K}^{(p)}]\{\xi^{(p)}\} = \{F^{(p)}(\Omega, t, \xi^{(0)}, \xi^{(1)})\}, \tag{13}$$

where $[\bar{K}^{(p)}]$ is the stiffness matrix of the overall system which is composed of all components

$$\{\xi^{(0)}\} = \{\{^1\xi^{(0)}\}^{\mathrm{T}}, \{^1 x_b^{(0)}\}^{\mathrm{T}}, \{^2 x_b^{(0)}\}^{\mathrm{T}}, \{^2\xi^{(0)}\}^{\mathrm{T}}\},$$

$$\{F^{(0)}\} = \{\{0\}^{\mathrm{T}}, \{-^1 f_b^{(0)}\}^{\mathrm{T}}, \{^2 f_b^{(0)}\}^{\mathrm{T}}, \{0\}^{\mathrm{T}}\}^{\mathrm{T}},$$

$$\{\xi^{(1)}\} = \{\{^1\xi^{(1)}\}^{\mathrm{T}}, \{^1 x_b^{(1)}\}^{\mathrm{T}}, \{^2 x_b^{(1)}\}^{\mathrm{T}}, \{^2\xi^{(1)}\}^{\mathrm{T}}\},$$

$$\{F^{(1)}\} = \{\{^1 f_u\}^{\mathrm{T}} + \{^1 f_{p1}\}^{\mathrm{T}}, \{-^1 f_b^{(1)}\}^{\mathrm{T}}, \{^2 f_b^{(1)}\}^{\mathrm{T}}, \{f_c^{(0)}\}^{\mathrm{T}}\},$$

$$\{\xi^{(2)}\} = \{\{^1\xi^{(1)}\}^{\mathrm{T}}, \{^1 x_b^{(1)}\}^{\mathrm{T}}, \{^2 x_b^{(1)}\}^{\mathrm{T}}, \{^2\xi^{(1)}\}^{\mathrm{T}}\},$$

$$\{F^{(2)}\} = \{\{^1 f_{p2}\}^{\mathrm{T}}, \{-^1 f_b^{(1)}\}^{\mathrm{T}}, \{^2 f_b^{(1)}\}^{\mathrm{T}}, \{0\}^{\mathrm{T}}\}.$$

In order to apply the SSM, the transformation matrix is introduced [2–4]. The transformation matrix is composed of $[\phi_{bi}]$ $(i = 1, 2)$, the eigenvector matrix of the assembling region, which is derived from the eigenvector of each component corresponding to the nodal point of bearing. By substituting the transformation matrix into Eq. (13) and pre-multiplying, the overall equation of order $\varepsilon^{(p)}$ can be expressed as

$$\begin{Bmatrix} ^1\ddot{\xi}_i^{(p)} \\ ^2\ddot{\xi}_i^{(p)} \end{Bmatrix} + \begin{bmatrix} [^1\omega_i^2] + [a_1] & [a_2] \\ [a_3] & [^2\omega_i^2] + [a_4] \end{bmatrix} \begin{Bmatrix} ^1\xi_i^{(p)} \\ ^2\xi_i^{(p)} \end{Bmatrix} = \begin{Bmatrix} ^1 f_\eta^{(p)} \\ ^2 f_\eta^{(p)} \end{Bmatrix} = \{f_\eta^{(p)}\}, \tag{14}$$

$$[a_1] = [\phi_{b1}]^{\mathrm{T}}[^1 k_{b1}][\phi_{b1}], \quad [a_2] = [\phi_{b1}]^{\mathrm{T}}[^2 k_{b1}][\phi_{b2}], \quad [a_3] = [\phi_{b2}]^{\mathrm{T}}[^1 k_{b2}][\phi_{b1}], \quad [a_4] = [\phi_{b2}]^{\mathrm{T}}[^2 k_{b2}][\phi_{b2}].$$

The external force term of order $\varepsilon^{(P)}$ is obtained as

$$\{f_\eta^{(0)}\} = \begin{Bmatrix} [\phi_{a1}]^{\mathrm{T}} \cdot \{0\}^{\mathrm{T}} \\ [\phi_{a2}]^{\mathrm{T}} \cdot \{0\}^{\mathrm{T}} \end{Bmatrix},$$

$$\{f_\eta^{(1)}\} = \begin{Bmatrix} [\phi_{a1}]^{\mathrm{T}} \cdot (\{^1 f_{p1}\}^{\mathrm{T}} + \{^1 f_u\}^{\mathrm{T}}) + [\phi_{b1}]^{\mathrm{T}} \cdot \{^1 f_b^{(1)}\} \\ [\phi_{a2}]^{\mathrm{T}} \cdot \{^2 f_c^{(0)}\}^{\mathrm{T}} + [\phi_{b2}]^{\mathrm{T}} \cdot \{^2 f_b^{(1)}\}^{\mathrm{T}} \end{Bmatrix},$$

$$\{f_\eta^{(2)}\} = \begin{Bmatrix} [\phi_{a1}]^{\mathrm{T}} \cdot \{^1 f_{p2}\}^{\mathrm{T}} + [\phi_{b1}]^{\mathrm{T}} \cdot \{^1 f_b^{(2)}\} \\ [\phi_{a2}]^{\mathrm{T}} \cdot \{0\} + [\phi_{b2}]^{\mathrm{T}} \cdot \{^2 f_b^{(2)}\} \end{Bmatrix}.$$

By applying the modal analysis technique $\{\xi\} = [\Phi_Z]\{\eta\}$, Eq. (14) can be solved, where $[\Phi_Z]$ is the modal matrix of the overall structure.

$$\begin{aligned}
&\{\ddot{\eta}^{(0)}\} + [\diagdown \omega_Z^2 \diagdown]\{\eta^{(0)}\} = \{0\},\\
&\{\ddot{\eta}^{(1)}\} + [\diagdown \omega_Z^2 \diagdown]\{\eta^{(1)}\} = -[Q]\{\dot{\eta}^{(0)}\} + \{G\} - [P]\{\eta^{(0)3}\},\\
&\{\ddot{\eta}^{(2)}\} + [\diagdown \omega_Z^2 \diagdown]\{\eta^{(2)}\} = -[Q]\{\dot{\eta}^{(1)}\} - 3[P]\{\eta^{(0)2} \cdot \eta^{(1)}\},
\end{aligned} \tag{15}$$

where $[Q] = [\Phi_Z][\diagdown C \diagdown][\Phi_Z]^{\mathrm{T}}$, $\{G\} = [\Phi_Z]\{^1 f_u\}$, $[P] = [\Phi_Z][\diagdown {}^1 k_N \diagdown][\Phi_Z]$. $[\diagdown \omega_Z^2 \diagdown]$ is the eigenvalue of the overall system. Here $\{\eta^{(0)2} \cdot \eta^{(1)}\}$ is a perturbed modal displacement term which comes from the perturbation zeroth order and perturbation first order.

In this study, damping term is considered in the overall system as a proportional damping that consists of the mass and stiffness matrices of the overall system as $[\diagdown C \diagdown] = \alpha[I] + \beta[\diagdown w_z^2 \diagdown]$, where α, β are the damping coefficients. The overall structure is analyzed by solving Eq. (15). The amplitude of rotor is large around the natural frequency of the system so that the non-linearity has much influence on the vibration of the rotor. Accordingly, the development of the accurate analytical method of non-linear dynamic response near the natural frequency is strongly desired in the turbines of aircraft.

3. Response analysis by applying the method of multiple scales

To obtain the equation to perturbation first order by the method of multiple scales, the following time scale is introduced:

$$T_n = \varepsilon^n t,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}T_0}{\mathrm{d}t}\frac{\partial}{\partial T_0} + \frac{\mathrm{d}T_1}{\mathrm{d}t}\frac{\partial}{\partial T_1} + \frac{\mathrm{d}T_2}{\mathrm{d}t}\frac{\partial}{\partial T_2} + \cdots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \tag{16}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) + \cdots.$$

By substituting Eq. (16) into Eq. (15), and by arranging with ε, the equations can be rewritten as

$$\begin{aligned}
&D_0^2\{\eta^{(0)}\} + [\diagdown \omega_Z^2 \diagdown]\{\eta^{(0)}\} = \{0\},\\
&D_0^2\{\eta^{(1)}\} + [\diagdown \omega_Z^2 \diagdown]\{\eta^{(1)}\} = -D_0 D_1\{\eta^{(0)}\} - 2[Q]D_0\{\eta^{(0)}\} - [P]\{\eta^{(0)3}\} + \{G\},\\
&D_0^2\{\eta^{(2)}\} + [\diagdown \omega_Z^2 \diagdown]\{\eta^{(2)}\} = -D_0 D_1\{\eta^{(1)}\} - (D_1^2 + 2D_0 D_2)\{\eta^{(0)}\}\\
&\qquad - 2[Q]D_1\{\eta^{(0)}\} - 2[Q]D_0\{\eta^{(1)}\} - [P]\{\eta^{(0)2} \cdot \eta^{(1)}\}.
\end{aligned} \tag{17}$$

The exciting frequency Ω is regarded around the first natural frequency ω_1. By noting the detuning parameter σ, the exciting frequency can be expressed as

$$\Omega = \omega_1 + \varepsilon\sigma. \tag{18}$$

Here, only the main resonance is considered by regarding there is no other resonance except the main resonance. The solution of the first equation of Eq. (17) can be expressed as

$$\{\eta^{(0)}\} = \{A\}\exp(\mathrm{i}\omega_0 T_0) + \{\bar{A}\}(-\mathrm{i}\omega_0 T_0). \tag{19}$$

According to the MS theory, by substituting Eqs. (18) and (19) into Eq. (17), the equation is expressed in the single-d.o.f. form as

$$\begin{aligned} D_0^2\eta_1^{(1)} + \omega_1^2\eta_1^{(1)} = {} & \left\{-2\mathrm{i}\omega_1(A_1' + Q_{11}A_1) - 3P_{11}A_1^2\bar{A}_1 + \tfrac{1}{2}G_1\exp(\mathrm{i}\sigma T_1)\right\}\exp(\mathrm{i}\omega_1 T_0) \\ & + \left(-2\mathrm{i}\sum_{k=2}^{n}\omega_k Q_{1k}A_k - 3\sum_{k=2}^{n}P_{1k}A_k^2\bar{A}_k\right)\exp(\mathrm{i}\omega_k T_0) \\ & - \sum_{k=2}^{n}P_{1k}A_k^3\exp(3\mathrm{i}\omega_k T_0) + c.c., \end{aligned} \tag{20}$$

where *c.c.* is conjugate complex term. The secular term is eliminated from the particular solution of Eq. (20) by choosing A as

$$-2\mathrm{i}\omega_1(D_1A_1 + Q_{11}A_1) - 3P_{11}A_1^2\bar{A}_1 + \tfrac{1}{2}G_1\exp(\mathrm{i}\sigma T_1) + c.c. = 0. \tag{21}$$

In a similar way, a condition to eliminate the secular term of the other component of equation to $m = 2, 3, \sim 2n$ is

$$\begin{aligned} & -2\mathrm{i}\omega_m(D_1A_m + Q_{mm}A_m) - 3P_{mm}A_m^2\bar{A}_m + c.c. = 0, \\ & \{A\} = \tfrac{1}{2}\{a\}\exp(\mathrm{i}\beta), \quad \{\bar{A}\} = \tfrac{1}{2}\{a\}\exp(-\mathrm{i}\beta). \end{aligned} \tag{22}$$

By dividing the equation into a real part and an imaginary part, the equation can be rewritten as

$$D_1a_1 = a_1' = -Q_{11}a_1 + \frac{1}{2\omega_1}G_1\sin\gamma, \quad a_1D_1\beta_1 = a_1\beta_1' = \frac{3}{8\omega_1}P_{11}a_1^3 - \frac{1}{2\omega_1}G_1\cos\gamma, \tag{23}$$

where $\gamma = \sigma T_1 - \beta_1$. Similarly, Eq. (22) can be rewritten as

$$a_m' = -Q_{mm}a_m, \quad a_m\beta_m' = \frac{3}{8\omega_m}P_{mm}a_m^3. \tag{24}$$

When the vibration is in the steady state ($a' = \gamma' = 0$), by considering Eq. (24)

$$a_m = 0, \quad m = 2, 3, \sim 2n. \tag{25}$$

This result corresponds to the relation $\eta_2, \eta_3, \sim\eta_{2n} = 0$ when the non-linear restoring force is transformed into modal co-ordinates. By squaring Eq. (23), adding each term, and by considering Eq. (25), the equation is arranged as

$$(Q_{11}a_1)^2 + \left(\sigma a_1 - \frac{3}{8\omega_1}P_{11}a_1^3\right)^2 = \frac{1}{4\omega_1^2}G_1^2, \quad 2_m = 0 \quad (m = 2, 3, \sim 2n). \tag{26}$$

The frequency response of the system to the perturbation first order is obtained by solving Eq. (26).

Next, a formulation procedure to obtain the equation to perturbation second order is introduced. According to the second equation of Eq. (17), the particular solution for the single d.o.f. is obtained by eliminating the secular term

$$\eta_1^{(1)} = \frac{1}{\omega_k^2 - \omega_1^2}\left(2\mathrm{i}\sum_{k=2}^{n} \omega_k Q_{1k} A_k + 3\sum_{k=2}^{n} P_{1k} A_k^2 \bar{A}_k\right) \exp(\mathrm{i}\omega_k T_0) + \frac{1}{9\omega_k^2 - \omega_1^2}\sum_{k=1}^{n} P_{1k} A_k^3 \exp(3\mathrm{i}\omega_k T_0) + c.c. \tag{27}$$

Similarly, the particular solution of equation to $m = 2 \sim 2n$ is obtained by eliminate the secular term as

$$\begin{aligned}\eta_m^{(1)} = {} & \frac{1}{\omega_k^2 - \omega_m^2}\left(2\mathrm{i}\sum_{k=1}^{n} \omega_k Q_{mk} A_k + 3\sum_{k=1}^{n} P_{mk} A_k^2 \bar{A}_k\right) \exp(\mathrm{i}\omega_k T_0) \\ & + \frac{1}{9\omega_k^2 - \omega_m^2}\sum_{k=1}^{n} P_{mk} A_k^3 \exp(3\mathrm{i}\omega_k T_0) \\ & + \frac{1}{2(\omega_m^2 - \omega_1^2)} G_m \exp(\mathrm{i}\sigma T_1) \exp(\mathrm{i}\omega_1 T_0) + c.c. \quad (k \neq m).\end{aligned} \tag{28}$$

By substituting Eqs. (19) and (28) into the third equation of Eq. (17), the equation can be solved. However, it is quite complex to solve all of the equations to the $m = 2, 3, \sim 2n$ component equation. Thus, the equation is arranged according to the condition to eliminate the secular term (the terms of $\mathrm{i}\omega_m T_0$):

$$-2\mathrm{i}\omega_m D_2 A_m + Q_{mm}^2 A_m + \frac{3}{\omega_m}\mathrm{i} Q_{mm} P_{mm} A_m^2 \bar{A}_m + \frac{9}{4\omega_m^2} P_{mm}^2 A_m^3 \bar{A}_m^2 + c.c. = 0. \tag{29}$$

When the vibration is in the steady state

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \varepsilon \frac{\mathrm{d}A}{\mathrm{d}T_1} + \varepsilon^2 \frac{\mathrm{d}A}{\mathrm{d}T_2} = 0. \tag{30}$$

After substituting Eqs. (22) and (29) into Eq. (30), the arranged equation can be obtained. The arranged equation can be rewritten in separated form as the real part and the imaginary part:

$$\begin{aligned} a_m \dot{\beta}_m &= \frac{3}{8\omega_m} \varepsilon P_{mm} a_m^3 - \frac{\varepsilon^2}{2\omega_m} Q_{mm}^2 a_m - \frac{9\varepsilon^2}{128\omega_m^2} P_{mm}^2 a_m^5, \\ \dot{a}_m &= -\varepsilon Q_{mm} a_m + \frac{3\varepsilon^2}{8\omega_m^2} Q_{mm} P_{mm} a_m^3. \end{aligned} \tag{31}$$

When the vibration is in the steady state ($a' = \gamma' = 0$), from Eq. (31)

$$a_m = 0 \quad (m = 2, 3, \sim 2n). \tag{32}$$

In accordance with the formulation to the perturbation first order, this result corresponds to the relation $\eta_2, \eta_3, \sim \eta_{2n} = 0$ when the non-linear restoring force is transformed into modal co-ordinates. From Eq. (32), the particular solution of the single d.o.f. $\eta_1^{(1)}$of Eq. (27) become

$$\eta_1^{(1)} = \frac{1}{8\omega_1^2} P_{11} A_1^3 \exp(3\mathrm{i}\omega_1 T_0) + c.c. \tag{33}$$

By substituting Eq. (19), Eq. (33) into the third equation of Eq. (17), the equation is arranged in a simple form. Using the relation of secular term (the terms of $i\omega_1 T_0$), the single-d.o.f. equation can be obtained as:

$$\begin{aligned} &-2\mathrm{i}\omega_1 D_2 A_1 + Q_{11}^2 A_1 + \frac{3}{\omega_1}\mathrm{i} P_{11} Q_{11} A_1^2 \bar{A}_1 + \frac{9}{4\omega_1^2} P_{11}^2 A_1^3 \bar{A}_1^2 \\ &+ \frac{G_1}{4\omega_1}\left(\mathrm{i}Q_{11} - \frac{3}{\omega_1} P_{11} A_1 \bar{A}_1 - \sigma\right)\exp(\mathrm{i}\sigma T_1) + \frac{3}{8\omega_1^2} P_{11} G_1 A_1^2 \exp(-\mathrm{i}\sigma T_1) + c.c. = 0, \end{aligned} \tag{34}$$

where $A_1 = \frac{1}{2} a_1 \exp(\mathrm{i}\beta_1)$. By substituting A_1 into Eq. (34), the equation can be rewritten in separated form as the real part and the imaginary part:

$$\begin{aligned} &\omega_1 a_1 D_2 \beta_1 + \tfrac{1}{2} a_1 Q_{11}^2 + \frac{9}{128\omega_1^2} P_{11}^2 a_1^5 + \frac{G_1}{4\omega_1}\left(-\sigma - \frac{3}{8\omega_1} P_{11} a_1^2\right)\cos\gamma - \frac{G_1}{4\omega_1} Q_{11} \sin\gamma = 0, \\ &\omega_1 D_2 a_1 - \frac{3}{8\omega_1} P_{11} Q_{11} a_1^3 + \frac{G_1}{4\omega_1}\left(\sigma + \frac{9}{8\omega_1} P_{11} a_1^2\right)\sin\gamma - \frac{G_1}{4\omega_1} Q_{11}\cos\gamma = 0. \end{aligned} \tag{35}$$

By substituting $A_1 = \frac{1}{2} a_1 \exp(\mathrm{i}\beta_1)$ into Eq. (21), the equation can be rewritten in separated form as the real part and the imaginary part

$$D_1 a_1 = -Q_{11} a_1 + \frac{1}{2\omega_1} G_1 \sin\gamma, \quad a_1 D_1 \beta_1 = \frac{3}{8\omega_1} P_{11} a_1^3 - \frac{1}{2\omega_1} G_1 \cos\gamma. \tag{36}$$

When the vibration is in the steady state and considering the relation ($\mathrm{d}A/\mathrm{d}t = \varepsilon D_1 A + \varepsilon^2 D_2 A = 0$),

$$D_1 A = -\varepsilon D_2 A, \quad D_1 \gamma = -\varepsilon D_2 \gamma, \quad D_2 \beta_1 = -D_2 \gamma. \tag{37}$$

Using these relations, Eqs. (35) and (36) become in a single d.o.f., which shows the relation between a and γ:

$$A_C \cos\gamma + B_C \sin\gamma = C_C, \quad D_C \cos\gamma + E_C \sin\gamma = F_C, \tag{38}$$

$$A_C = \left(\frac{1}{2} - \frac{\varepsilon\sigma}{4\omega_1} - \frac{3\varepsilon P_{11}}{32\omega_1^2} a_1^2\right) G_1, \quad B_C = -\frac{G_1 \varepsilon}{4\omega_1} Q_{11}, \quad D_C = -B_C,$$

where

$$-C_C = a_1\left(\frac{\varepsilon}{2} Q_{11}^2 - \omega_1 \sigma + \frac{3}{8} P_{11} a_1^2 - \frac{9\varepsilon}{128\omega_1^2} P_{11}^2 a_1^4\right),$$

$$E_C = \left(\frac{1}{2} - \frac{\varepsilon\sigma}{4\omega_1} - \frac{9\varepsilon P_{11}}{32\omega_1^2} a_1^2\right) G_1, \quad F_C = \omega_1 Q_{11} a_1 - \frac{3\varepsilon}{8\omega_1} Q_{11} P_{11} a_1^3.$$

By eliminating the term γ from Eq. (38), the following polynomial equation of a^2 is obtained:

$$\sum_{n=0}^{7} C_n(\varepsilon, \sigma, \omega_1, P_{11}, Q_{11}, G_1)a^{2n} = 0. \tag{39}$$

The frequency response of the system to the perturbation second order is obtained by solving Eq. (39). From Eq. (39), the solution of the equation of motion to the first order of ε is obtained. The response can be expressed for the single d.o.f. as

$$\eta_1 = a_1 \cos(\Omega t - \gamma) + \varepsilon \left\{ \frac{1}{32\omega_1^2} P_{11} a_1^3 \cos(3\Omega t - 3\gamma) \right\}, \tag{40}$$

where γ is obtained from Eq. (38). The time response of the equation of motion can be obtained by changing Eq. (40) into physical co-ordinates.

4. Results of the numerical examples

In this section, response analysis is presented to demonstrate the application of the proposed method. The frequency response is obtained by sweeping up the frequency gradually. The responses of the proposed method are compared with those obtained by the classical analysis technique for accuracy validation.

A non-linear rotor system, which is shown in Fig. 2, is considered. The rotor is considered to be a uniform beam and the casing is also considered to be a uniform beam approximately for the simplicity of calculation. Generally, there is a cross-coupling terms in the ball bearing. The cross-coupling terms in the bearing are ignored for the simplified model of bearing to verify the effect of non-linearity. The properties of the rotor system are tabulated in Table 1.

The rotor and casing are modelled by the eight beam elements. The modal damping ratios of the rotor system are given by α, $\beta = 0.05$.

Table 2 shows the natural frequency of rotor system to the 5th mode in rad/s by the proposed method. The second natural frequency is almost three times the first natural frequency of the system. And the other frequencies are apart from each other. A rotor is defined as circular when the second moment of area of its cross-section about any axis through the center of area is

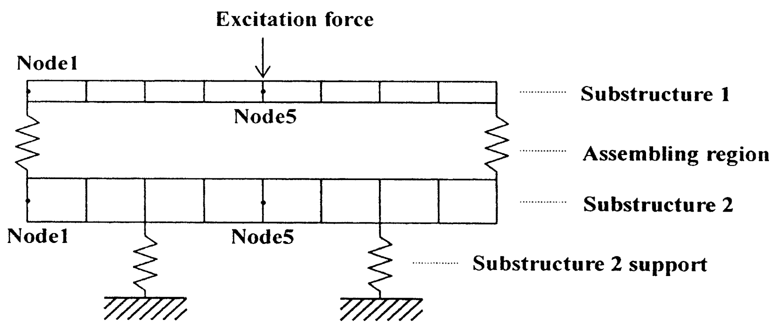


Fig. 2. Non-linear rotor model for analysis.

Table 1
Properties of the rotor system

Rotor, casing length (mm)	1.6×10^3
Rotor diameter (mm)	3.0×10^2
Casing diameter (mm)	1.0×10^2
Young's modules of rotor and casing (N/m^2)	2.1×10^{11}
Density of rotor and casing (kg/m^3)	7.81×10^4
Bearing coefficient (N/m)	6.69×10^4
Casing support coefficient (N/m)	1.0×10^{10}

Table 2
Natural frequencies of rotor system (rad/s)

Frequency no.	First	Second	Third	Forth	Fifth
Natural frequency	141.02	471.28	802.95	1149.40	1256.20

invariable. This means that the rotor is a circular type where the stiffness of the rotor is same in any direction. In this study as a case of analysis, the general circular-type rotor is adopted to verify the effect of the non-linear characteristics of the system. And the response is analyzed around the first natural frequency (= 141.02 rad/s).

4.1. Results of the non-linear response analysis

Responses of the non-linear rotor system are reviewed in frequency domain and time domain. In the analysis, the exiting force caused by unbalance of rotor is given the value 50 on the 5th nodal point of substructure 1. The perturbation parameter for the non-linearity is adopted as a small value $\varepsilon = 0.1$. First, as a numerical result of response analysis, the differences of the linear and non-linear responses in frequency domain and time domain are described. Linear and non-linear responses at the center of rotor in node 5, by the proposed and direct numerical methods, are presented in Fig. 3(a), (b) and (c). Responses are obtained using the perturbation first order approximation by the proposed method when the low 5 modes are adopted. To evaluate the proposed technique, the responses need to be compared with the other representative non-linear analyzing methods, such as the direct integration method. Using the FEM, the equation of motion of rotor system, which is composed of rotor-bearing casing, is obtained. Numerical integration is carried out conveniently in terms of first order equation, which is obtained from the equation of motion by FEM. Thus, the obtained non-linear equation is recast in state form. Then, the fourth order Runge–Kutta method is used to obtain the response against unbalance excitation. Fig. 3 reveals that there is a notable difference between the linear and non-linear response in time domain and frequency domain. Thus, it is important to figure out the non-linear response for the exact design and the diagnosis of the system. From Fig. 3(a), the non-linear response shows the typical response of hard spring-type non-linear restoring force (when $f = 50\,\mathrm{N}$, $\omega_1 = 141.02\,\mathrm{rad/s}$).

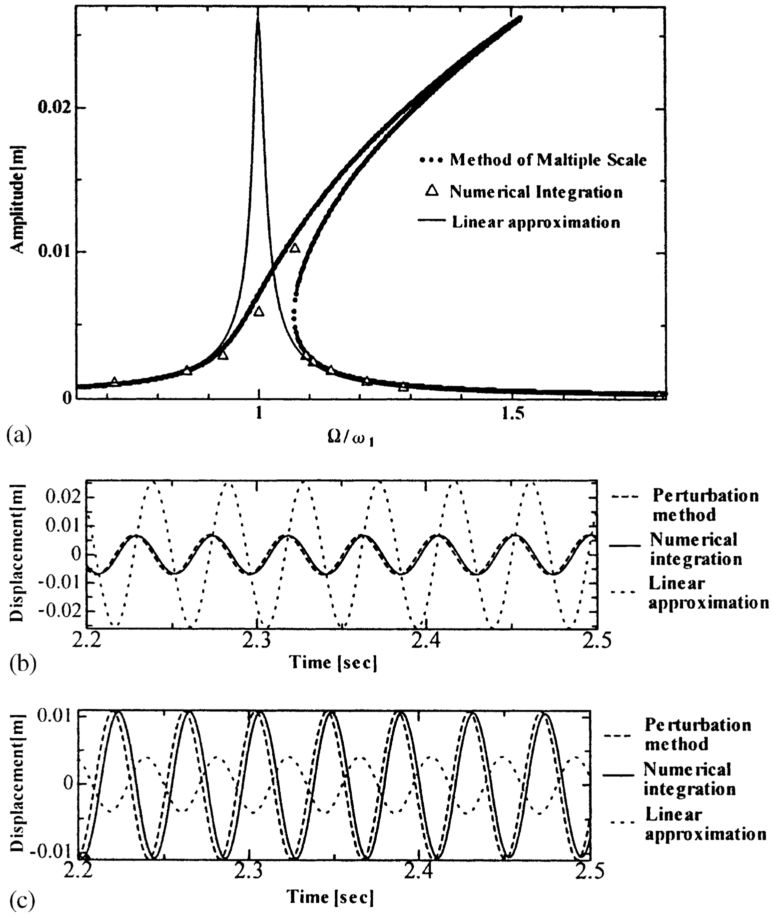


Fig. 3. Comparing the response of linear and non-linear approximation by the proposed method and direct numerical method: (a) linear and non-linear frequency responses by the proposed and numerical methods, (b) comparison of time history–displacement at node 5 in substructure 1 (when $\Omega = 140\,\text{rad/s}$), and (c) comparison of time history–displacement at node 5 in substructure 1 (when $\Omega = 150\,\text{rad/s}$).

The results of the time response by the proposed method, as shown in Fig. 3(b) and (c), are in very good agreement with those obtained by the direct integration method. However, the linear responses are larger than the non-linear responses, as shown in Fig. 3(b). As can be noticed in Fig. 3(a), the amplitude of the linear response is larger than the non-linear response near the natural frequency.

It can be understood that as the amplitude of the frequency response grows, even the exciting frequency exceeds the first natural frequency because of the effect of non-linearity, as can be observed in Fig. 3(a).

Time domain responses are compared, as shown in Fig. 4(a), (b) and (c), when the system is excited with an excitation frequency of 155 rad/s where the first natural frequency of the system is 141.02 rad/s, which is a little larger than the first natural frequency of the system. The time responses are obtained using the method of multiple scales by adopting 5 modes at nodes 1 and 5

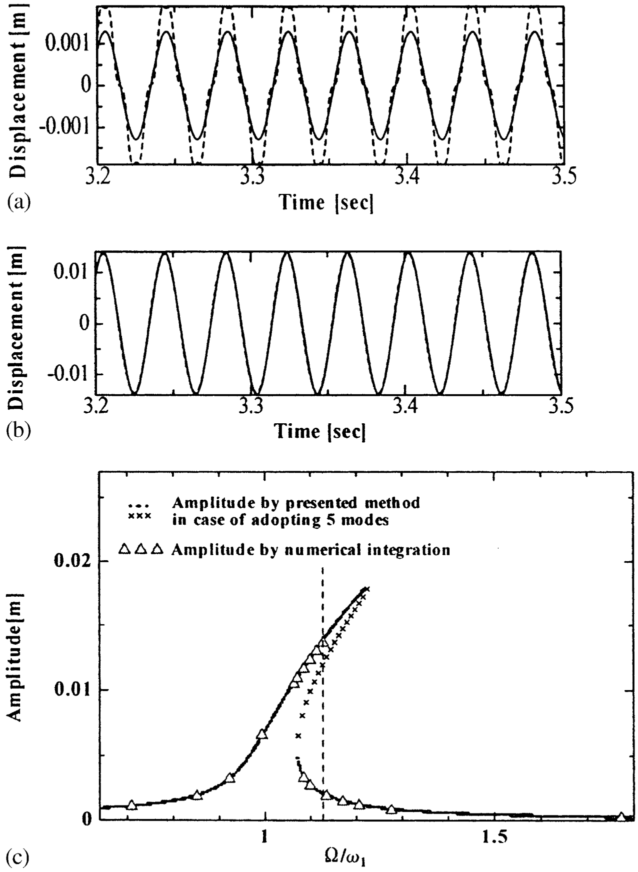


Fig. 4. Comparison of time responses (when $\Omega = 155\,\mathrm{rad/s}$): (a) displacement at node 1 in substructure 1, (b) displacement at node 5 in substructure 1 and (c) Frequency response at node 5 in substructure 1. (—) Presented method, (-----) integration method.

of substructure 1. And those responses are compared with the responses of the direct integration method. The response of the direct integration method is obtained from the overall equation of the system in physical co-ordinates, which is derived by FEM. Compared with the amplitude of the response by direct integration method, it can be observed at the selected point that comparatively accurate non-linear responses of the system are simulated with the corresponding phase. Nevertheless, there is a little difference of responses at node 1 of substructure 1, as shown in Fig. 4(a). Because of the bearing support, the response at node 1 of substructure 1 becomes relatively smaller than the response at the center of the rotor. Hence, it can be understood that the non-linear response shows good agreement with the response of the direct integral in the high amplitude range, which shows the non-linear characteristics well. This result can also be observed in Fig. 5(a) and (b), which shows the difference of the frequency component.

The corresponding Fast Fourier transform (FFT) analysis results of time response at nodes 1 and 5 of substructure 1 are shown in Fig. 5(a) and (b), which are calculated by the direct

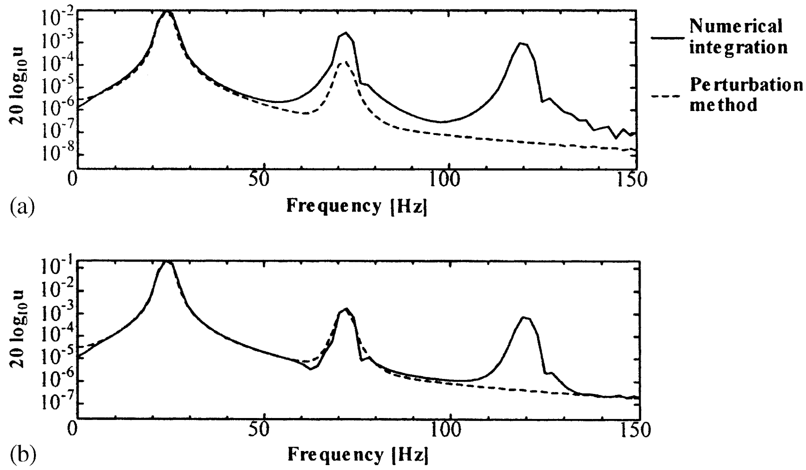


Fig. 5. Comparison of frequency components of the response by direct integration method and the proposed method: (a) response at node 1 in substructure 1 (when $\Omega = 155\,\text{rad/s}$) and (b) response at node 5 in substructure 1 (when $\Omega = 155\,\text{rad/s}$).

numerical integration and the proposed method. Each time response is obtained by the same simulation condition as in Fig. 4(a) and (b), and those are analyzed by FFT to observe the frequency component. The power spectrum is expressed in a logarithm display to confirm the non-linear frequency component easily in the diagram. Investigation of both results reveals comparatively good agreement. Relatively accurate responses can be simulated with 5 modes comparing with result of the direct integration method. The non-linear frequency element (3Ω) is observed in each spectrum where Ω is the exciting frequency. Nevertheless, it is observed that the spectrum of non-linear frequency element (3Ω) of the proposed method is smaller than the spectrum obtained by the direct numerical integration method at node 1 of substructure 1. It is estimated that the response of the proposed method shows good agreement in high amplitude range because non-linear restoring force is increased rapidly in large displacement area. Accordingly, there might arise a difference in low amplitude range compared with the direct integration method. There is no higher non-linear frequency element (5Ω) in the presented method where the result of the integration method shows one. Because the proposed method approximated the solution to the frequency (3Ω) element, there is no frequency element (5Ω).

The frequency responses, which are analyzed using the proposed method to the perturbation first order and the second order, are presented in Fig. 6. Those responses are obtained at the nodal point 5 of substructure 1. Five modes are adopted for each method. There is a little difference in high amplitude range between the response of the perturbation first order and the second order. It is estimated that the response of the perturbation second order is a more exact solution.

Those frequency responses are compared by changing the adopting modes according to the perturbation order, as shown in Fig. 7(a) and (b). The frequency responses that are calculated by perturbation first order approximation and perturbation second order approximation at the nodal point 5 of substructure 1 when adopting 5 modes and total modes (18 modes) are presented in Fig 7(a) and (b). It can be observed that the presented method simulated well the non-linear characteristic in comparatively good accuracy by adopting only low 5 modes compared with

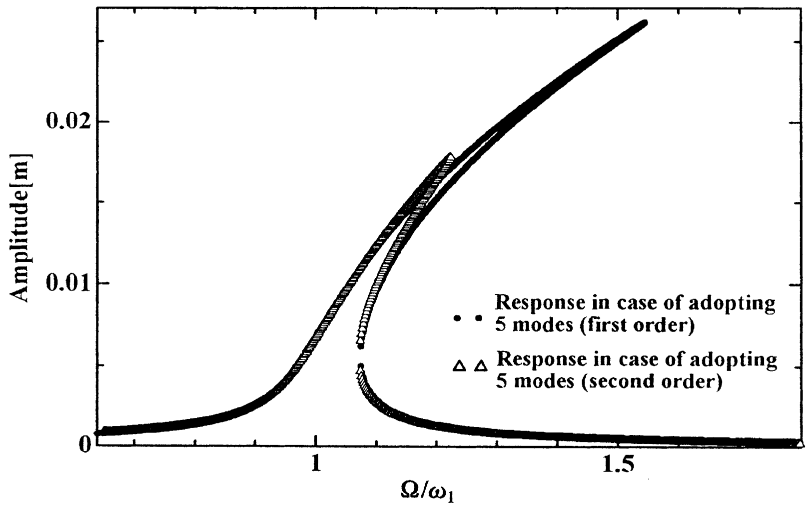


Fig. 6. Comparison of frequency response by the perturbation order.

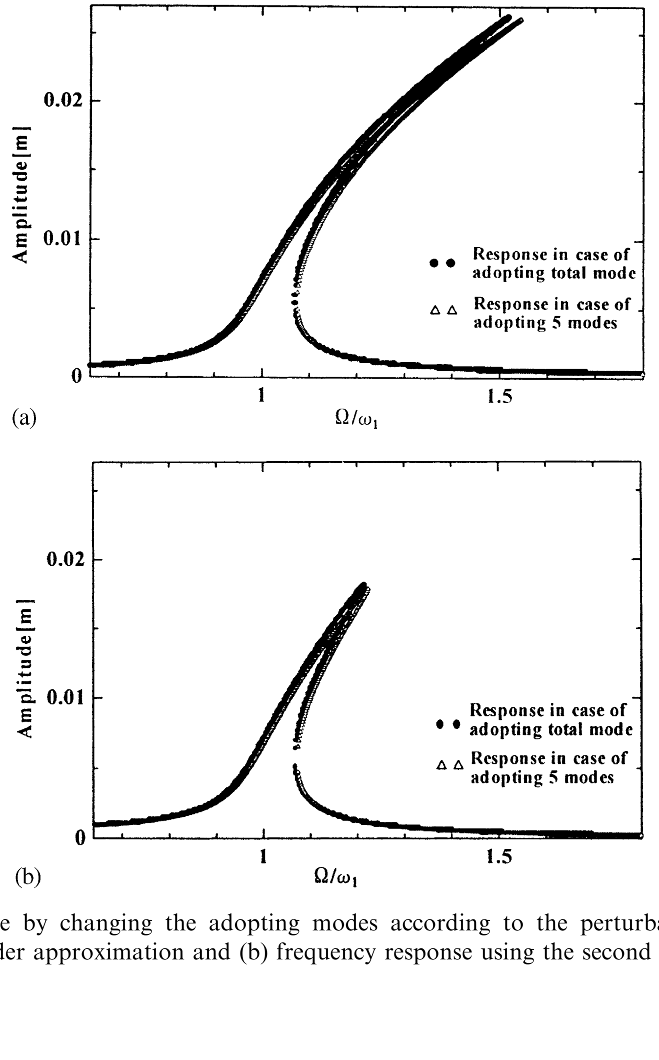
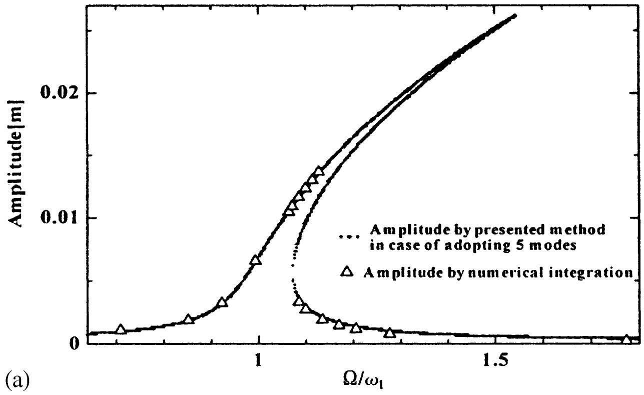
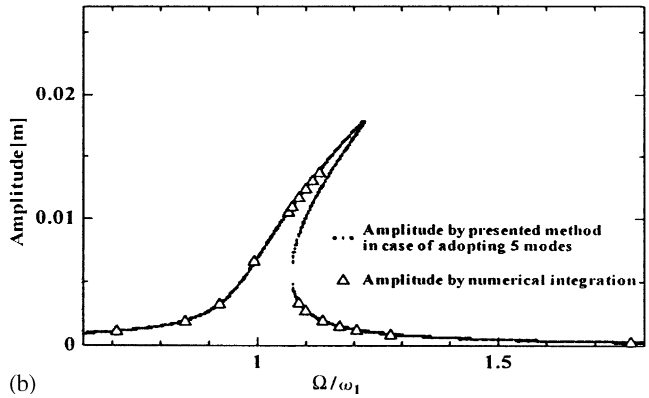


Fig. 7. Frequency response by changing the adopting modes according to the perturbation order: (a) frequency response using the first order approximation and (b) frequency response using the second order approximation.



(a)



(b)

Fig. 8. Comparison of frequency response by the perturbation order with direct integration method: (a) frequency response using the first order approximation and (b) frequency response using the second order approximation

responses of adopting all modes. From this result, it is believed that the non-linear restoring force term can be easily transformed into modal co-ordinates while retaining its accuracy with its lower modes according to the proposed procedure.

The frequency responses, which are analyzed at node 5 of substructure 1 using the proposed method and the direct integration method with the perturbation first order and the second order, are presented in Fig. 8(a) and (b). As in Fig. 7, this case also adopted only the low 5 modes in the method presented. It can be observed that the results of each method simulated well the non-linear characteristic in comparatively good accuracy. The response of the proposed method and the response of the direct integration method are in good agreement, keeping the accuracy, even though there is some error in the large amplitude range.

As shown in Refs. [7,8], "incorrect solutions", which do not exist in the direct numerical integration response, appear to be the solution when using the proposed method approximating to the second order. Though "incorrect solutions" appear in the large amplitude area around 0.03 m or more of the frequency response curve of Figs. 7(b) and 8(b), they are neglected because it is unrelated in substance with this study, so it is not shown on the graph.

It can be observed that the results of each method simulated well the non-linear characteristic in comparatively good agreement with the results of direct integration as shown in Figs. 7 and 8.

Especially, there is a good agreement with keeping the accuracy of the response between the proposed method of the perturbation second order and direct integration method, as shown in Fig. 8(b).

4.2. Computing efficiency of the proposed method

To evaluate the effectiveness of the proposed technique, the responses need to be compared with the other representative non-linear analyzing methods, such as the harmonic balance method.

The frequency responses, which are analyzed using the method of multiple scales with the first order, second order and harmonic balance method, are presented in Fig. 9. Those responses are obtained at the nodal point 5 of substructure 1. Five modes are adopted for each method. It can be observed from Fig. 9 that the results of each method simulated well the non-linear characteristic in comparatively good accuracy. Especially, the response of the perturbation second order in accordance with the multiple scales method and the response of the harmonic balance method are in good agreement by keeping the accuracy. From this result, it can be concluded that an analytical result obtained by the proposed method can secure the simulation accuracy.

Table 3 shows the maximum values of non-linear frequency response at the middle of rotor by changing its numbers of adopting modes. The values of the non-linear frequency response are investigated according to the analytical methods. To prove the computing efficiency, those values are compared with results of the direct integration method and the harmonic balance method, which are obtained by same calculation condition against the unbalanced excitation. Calculation accuracy of the proposed method is evaluated to show the effectiveness of the proposed method. The deviation of the calculation error of the analysis is defined as

$$\text{deviation} = \frac{\text{Value of direct integration method} - \text{Value of analytical method}}{\text{Value of direct integration method}} \times 100(\%). \quad (41)$$

The deviations of the result to the direct integration method are 71.7%, 72.3% in the case of proposed method with perturbation first order, where the number of adopting modes is 5, 18,

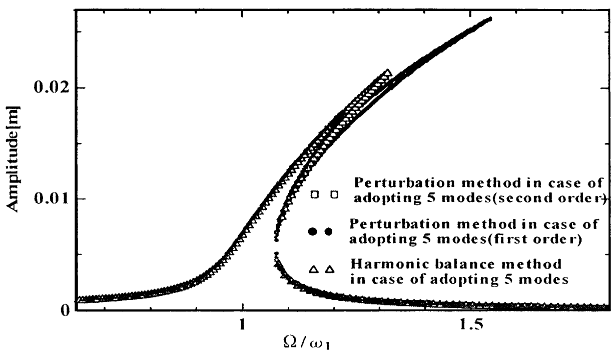


Fig. 9. Comparison of frequency responses with the harmonic balance method.

Table 3
Comparison of computing efficiency of the methods

Analysis method	Amplitude of response (m)	Deviation (%)	Calculation time (s)
Proposed method (perturbation first order) with 5 mode	0.0261	71.71	240
Proposed method (perturbation second order) with 5 mode	0.0175	15.13	312
Proposed method (perturbation first order) with 18 mode	0.0262	72.36	287
Proposed method (perturbation second order) with 18 mode	0.0176	15.78	368
Harmonic balance method	0.0225	48.02	353
Direct integration method (FEM)	0.0152	0	1370

respectively. The deviation of the result to the direct integration method are 15.1%, 15.7% in the case of proposed method with perturbation second order where the number of adopting modes is 5, 18, respectively. The deviation of the result to the direct integration method is 48.0% in the case of the harmonic balance method. The responses within 16% deviation error are obtained by the proposed analytical method with perturbation second order. It is believed that the accuracy of the response within 16% deviation error for the system is an effective analytical method.

Next, the calculation time is considered to verify the effectiveness of the proposed method. As a case, the calculation time for the responses of Table 3 is examined. The proposed method takes 312 and 368 s to calculate the frequency response by the perturbation second order within 16% deviation error, while the direct integration method takes 1370 s to compute the same response by using the personal computer *Logix IBM Co.*, the harmonic balance method takes 353 s to calculate the frequency response within 48% deviation error. As a result, it can be observed in this study that a drastic reduction in computational time can be obtained while retaining the accuracy of the solution. This is a critical factor in the analysis of the structural dynamics with a large number of d.o.f. systems. It is believed that the proposed method can analyze the response of the complex system by keeping the accuracy of the solution compared with the direct numerical integration.

5. Conclusions

In this paper, the vibration analysis of a non-linear mechanical system is theoretically formulated applying the method of multiple scales. The formulation is concerned with reducing the number of d.o.f. for each substructure by modal substitution. All the substructures are then re-assembled together and the non-linear response of the overall system is obtained against the harmonic excitation. This method is applied to a non-linear rotor system. The performance of the proposed method is compared with respect to the computational accuracy and time with the direct integral method. It is shown that non-linear responses can be efficiently calculated according to the selected number of vibration modes. And the non-linear characteristic of the non-linear restoring force is well simulated. As a result, the proposed method is proved to be an applicable technique for analyzing the dynamics of non-linear structure. Moreover, it is believed that those properties of the results can be utilized in the dynamic design of the non-linear system.

References

[1] T. Iwatsubo, S. Kawamura, B.-Y. Moon, Non-linear vibration analysis of rotor system using substructure synthesis method (analysis with consideration of non-linearity of rotor), JSME International Journal Series C 41 (4) (1998) 727–733.

[2] B.-Y. Moon, J.-W. Kim, B.-S. Yang, Non-linear vibration analysis of mechanical structure system using substructure synthesis method, KSME International Journal 13 (9) (1999) 620–629.

[3] B.-Y. Moon, B.-S. Kang, Non-linear vibration analysis of mechanical system (analysis with consideration of nonlinear sensitivity), JSME International Journal Series C 44 (1) (2001) 12–20.

[4] B.-Y. Moon, B.-S. Kang, B. Kim, Dynamic analysis of harmonically excited non-linear structure system using harmonic balance method, KSME International Journal Series C 15 (2001) 1207–1216.

[5] N. Haquang, D.T. Mook, Non-linear structural vibration under combined parametric and external excitation, Journal of Sound and Vibration 118 (2) (1987) 291–306.

[6] D.T. Mook, The influence of an internal resonance conditions, Journal of Sound and Vibration 102 (4) (1978) 473–492.

[7] A. Hassan, Use of transformations with the higher order method of multiple scales to determine the steady state periodic response of harmonically excited non-linear oscillators. Part I: transformation of derivative, Journal of Sound and Vibration 178 (1) (1994) 1–19.

[8] Z. Rahman, T.D. Burton, On high order methods of multiple scales in non-linear oscillations-periodic steady state response, Journal of Sound and Vibration 133 (3) (1989) 369–379.

[9] B.-Y. Moon, B.-S. Kang, Statistical seismic response analysis and reliability design of nonlinear structure system, Journal of Sound and Vibration 258 (2) (2002) 269–285.