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Forced response of FRP sandwich panels with viscoelastic materials

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Abstract

In this paper a new analytical model is presented that accurately predicts the forced response of fibre reinforced plastic (FRP) sandwich plates subjected to transverse applied loads. It is based on Reddy's refined high order shear deformation theory and offers the feasibility of accounting for the viscoelastic properties of the constitutive materials without restriction to the steady state motion. This is achieved by modelling the viscoelastic behaviour of the constitutive materials using the Golla Hughes McTavish mathematical tool. Validation of the new approach is achieved by comparing results under harmonic loading conditions against data obtained using the proposed new analytical model. Subsequently, predicted responses for a given FRP sandwich plate under various transverse applied loads are presented. The results outline the importance of being able to account for the viscoelastic properties of the constitutive materials when modelling the dynamic behaviour of sandwich structures.

1. Introduction

Fibre reinforced plastic (FRP) sandwich materials offer advantages in terms of high specific stiffness and strength values. Furthermore they are corrosion resistant, have good anti-vibration and anti-noise properties and this makes them suitable for a variety of structural engineering applications. Sandwich structures have been the subject of many investigations. A large amount of literature has been devoted to the development of theories for conventional sandwich structures and to the study of their static and dynamic behaviour. A detailed review of this work is given in books by Plantema [1] and Allen [2] and in the paper written by Noor et al. [3]. The pioneer worker, Reissner [4] suggested a simple model to describe a sandwich plate known as the classical

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sandwich theory (CST). This theory is based on the assumption that the facings are thin, stiff and heavy compared to the core. Most of the work, which was carried out up until the early 1990s, was based on these assumptions. Recent work has focused on improving the CST by including more "physical mechanics". The main issue, which has been investigated, is the feasibility of accounting for the shear effect in both the faces and the core since, as emphasized by Khdeir and Reddy [5], transverse shear deformation effects significantly alter the plate response. New analytical and finite element models [6,7] have therefore been developed based on the first order shear deformation theory (FSDT) to investigate the static and free vibration behaviour of FRP sandwich plates. Then, the authors developed an analytical model based on Reddy's refined high order shear deformation theory (HSDT), which allows the study of the dynamic behaviour of FRP sandwich plates [8]. This method has the advantage, when compared to the CST and FSDT, of taking into account both the bending and the shear effect in the core and in the faces. Besides, it represents the true parabolic variation of the shear stress through the thickness of the plate. This technique has been used with success to investigate the transient response of undamped sandwich panels [9]. However, sandwich structures made of fibre reinforced faces and PVC foam core exhibit viscoelastic damping. This mechanism influences the dynamic response of a structure and so needs to be accounted for when dynamic analyses are carried out.

The use of passive damping treatment in order to improve the damping properties of a given structure has led to numerous research works being carried out on plates made up of two elastic layers with a thin viscoelastic damping layer. Lu et al. [10] and He and Ma [11], for example, carried out free vibration analyses of such panels. However, as far as FRP sandwich plates made up of a relatively thick damped core are concerned, little work has been devoted to their analysis. The free vibration analysis of such panels was first investigated by Cupial and Niziol [6]. They modelled the laminated faces using the FSDT and the properties of the core material using the elastic–vicoelastic principle. Since they neglected the damping effect of the faces and the in-plane stresses within the core, only the knowledge of the complex shear modulus was required. More recently, the authors [12] have also used the elastic–viscoelastic approach to model FRP sandwich plates having viscoelastic material properties. In this case the sandwich plates were modelled using Reddy's HSDT. The proposed method allows the prediction of the dynamic properties, namely the natural frequencies and modal loss factors of FRP sandwich plates, offering the feasibility of accounting for the core and skins viscoelastic material properties. In this paper this method is extended in order to predict the harmonic responses of FRP sandwich plates. However, the use of the elastic–viscoelastic principle when applied to model the frequency-dependent dynamic material properties restricts the analysis to a single frequency, steady state analysis. Consequently, for transient analysis, a different viscoelastic model, which needs to be consistent across a broad range of frequencies, should be considered. The lack of such a model has, up until now, limited most of the analysis to steady state conditions. Nevertheless, with the increased use of active damping treatment in the aeronautic and astronautic industries, more attention has been devoted to the development of new mathematical models. New methods, based on the Laplace transform, have been investigated and various expressions for the Laplace transform function were suggested. One of the approaches to be considered was the fractional calculus method introduced by Bagley and Torvik [13]. More recently, Golla, Hughes and McTavish have proposed the so-called GHM method for the modelling of linear viscoelastic space structures [14]. This approach provides an alternative method which includes viscoelastic damping effects without restriction of the steady state motion by providing extra co-ordinates. This approach combined with finite element method has, for example, been used not long ago by Shi et al. [15] to model a cantilever beam with active constrained layer damping treatments. Until now, this method has not been used to model the viscoelastic behaviour of FRP or of PVC foam materials. Nevertheless, being able to use this new approach appears really attractive.

In this paper, both the theory related to the enhancement of the GHM method to model the viscoelastic behaviour of FRP sandwich constitutive materials and the theory related to its implementation into the analytical model previously developed by the authors to model FRP sandwich panels is presented. The responses obtained using this new method for a sandwich plate subjected to harmonic loading are validated against results predicted using the method based on the elastic–viscoelastic principle. The novel developed method, based on the GHM mathematical model, is then used to predict and to investigate the responses of FRP sandwich panels subjected to given transverse applied loads.

2. Theory

The sandwich panels considered in this paper are composed of two FRP composite laminated faces of thickness h_f and a rigid core of thickness h_c. The fibre orientation in each lamina of the faces is represented by an angle θ, which is measured from the x-axis. The studied plate is assumed to have a length a, width b and total thickness h as shown in Fig. 1.

2.1. Constitutive relation for viscoelastic material

FRP sandwich plates are made from viscoelastic materials. Viscoelastic materials possess a capacity to both store and dissipate energy under load. This dual viscous and elastic character of viscoelastic materials leads to a complicated behaviour, which cannot be described by either elasticity or viscosity theory but a combination of both. Consequently, whereas the response of an elastic material depends only on the total stress level at every instant of time, the response of a viscoelastic material is not only determined by the current state of stress, but also determined by all past states of stress. The fundamental mathematical characterization of viscoelasticity

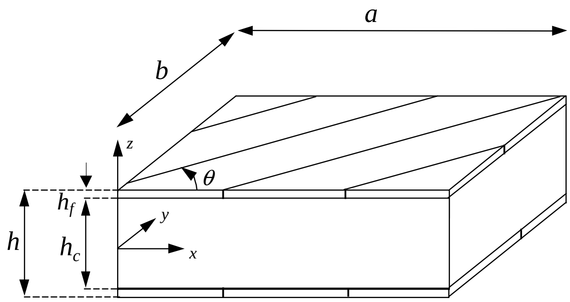

Fig 1. Sandwich plate geometry.

has been obtained from this latter observation and is given by the following stress constitutive relation [16]:

$$\sigma_{ij}(t) = \int_{-\infty}^{t} C_{ijkl}(t-\tau)\frac{\mathrm{d}\varepsilon_{kl}}{\mathrm{d}\tau}\mathrm{d}\tau, \tag{1}$$

where the variables are explained in Appendix C.

As explained by Christensen [16], the utility of the constitutive relation (1) is increased with little loss of generality if the strain history is required to vanish on the interval $t<0$. Further, the continuity requirement on ε_{kj} at $t=0$ can be relaxed to admit a non-zero value for $\varepsilon_{kj}(0)$. The constitutive relation then becomes

$$\sigma_{ij}(t) = C_{ijkl}(t)\varepsilon_{kj}(0) + \int_{0}^{t} C_{ijkl}(t-\tau)\frac{\mathrm{d}\varepsilon_{kl}(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau. \tag{2}$$

The Laplace transform of this equation yields

$$\tilde{\sigma}_{ij}(s) = s\tilde{C}_{ijkl}(s)\tilde{\varepsilon}_{kl}(s). \tag{3}$$

It should be noticed that in the specific case where the structure is subjected to such steady state oscillatory conditions, the viscoelastic stress–strain relation can be brought to the same form as the elastic stress–strain equation by applying the correspondence principle of linear viscoelasticity also called the elastic–viscoelastic principle [16]. Indeed when viscoelastic structures are subjected to steady state conditions, the strain can be specified as being a harmonic function of time and so expression (2) can be re-written in the form of relation (4) as explained in more detail in the Meunier and Shenoi paper [12]

$$\sigma_{ij}(t) = C^{*}_{ijkl}(\mathrm{j}\omega)\varepsilon_{kl}(t). \tag{4}$$

In relation (4), the complex material stiffness coefficients, $C^{*}_{ijkl}(\mathrm{j}\omega)$, are functions of the constitutive material frequency-dependent complex moduli, $E^{*}_{i}(\mathrm{j}\omega)$ and $G^{*}_{i}(\mathrm{j}\omega)$, which can be determined experimentally. This implies that under steady state condition the elastic constitutive equations can be easily updated to the viscoelastic case simply by replacing the real elastic moduli by their frequency-dependent complex viscoelastic counterparts.

2.2. *Equations of motion*

2.2.1. *Displacement field*

The proposed analytical model is based on Reddy's refined HSDT. This theory has initially been developed to model FRP laminated plates [17] and it offers the feasibility of taking into account the bending and the shear effects in both the faces and the core. Besides, it provides a parabolic distribution of the shear stresses through the thickness, by which the conditions of no shear stresses on the boundary planes are fulfilled. The displacement field associated with this theory is given by

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + z\phi_x(x,y,t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x}\right), \\ v(x,y,z,t) &= v_0(x,y,t) + z\phi_y(x,y,t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial y}\right), \\ w(x,y,z,t) &= w_0(x,y,t). \end{aligned} \tag{5}$$

Assuming this displacement field, the equations of motion expressed in terms of the displacement co-ordinates for FRP sandwich plates can be derived using the principle of virtual displacement and can then be solved exactly using, for example, a Navier-type solution. Following this procedure and assuming elastic material properties, an analytical model, which offers the feasibility to carry out free vibration and transient analyses of simply supported cross-ply and antisymmetric angle-ply FRP laminated and sandwich plates, has previously been developed by the authors [8,9]. However, this method, does not account for the viscoelastic properties of the constitutive material. In order to overcome this drawback and so to account for the material viscoelastic properties during dynamic analysis, a new analytical method based on the displacement field given by relation (5) is proposed in this paper.

2.2.2. Steady state analysis

In the case of steady state analysis, the viscoelastic behaviour of the constitutive materials can be modelled using the elastic–viscoelastic principle. With this method, simply by replacing the real elastic moduli by their frequency-dependent complex viscoelastic counterparts, the general vibration equations of motion can be written in the form of the following relation as explained in more detail in Meunier's thesis [18].

$$[\mathbf{M}]\left\{\ddot{\boldsymbol{\Delta}}^*(t)\right\} + \left(\left[\mathbf{K}'(\omega)\right] + \mathrm{j}\left[\mathbf{K}''(\omega)\right]\right)\left\{\boldsymbol{\Delta}^*(t)\right\} = \{\mathbf{F}(t)\}. \tag{6}$$

Assuming that the studied structure is subjected to harmonic excitation, that is to say $\{\mathbf{F}(t)\} = \{\mathbf{F}_0 \exp(\mathrm{j}\omega t)\}$, then the equations of motion for FRP sandwich plates having viscoelastic material properties can be written in the form

$$\left(-\omega^2[\mathbf{M}] + \mathrm{j}\left[\mathbf{K}''(\omega)\right] + \left[\mathbf{K}'(\omega)\right]\right)\left\{\boldsymbol{\delta}^*\right\} = \{\mathbf{F}_0\}. \tag{7}$$

Consequently, by assuming that for the excitation frequency and for this frequency only $[\mathbf{D}] = [\mathbf{K}''(\omega)]/\omega$ and $[\mathbf{K}] = \left[\mathbf{K}'(\omega)\right]$ the equations of motion for FRP sandwich plates subjected to steady state condition can be expressed as:

$$[\mathbf{M}]\left\{\ddot{\boldsymbol{\Delta}}(t)\right\} + [\mathbf{D}]\left\{\dot{\boldsymbol{\Delta}}(t)\right\} + [\mathbf{K}]\{\boldsymbol{\Delta}(t)\} = \{\mathbf{F}_0 \exp(\mathrm{j}\omega t)\}. \tag{8}$$

This equation can be solved using Newmark's integration technique. The results provide the coefficients of the column vector $\{\boldsymbol{\Delta}(t)\} = \{U_{mn}(t); V_{mn}(t); W_{mn}(t); X_{mn}(t); Y_{mn}(t)\}$ for each time step and for each mode number (m, n) considered. The transverse displacement at any point within the plate can then be determined using the following relation if a Navier solution type is assumed:

$$w(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}(t)\sin(\alpha x)\sin(\beta y). \tag{9}$$

2.2.3. Transient analysis

When transient analysis of structures having viscoelastic material properties needs to be carried out it is not possible to model the viscoelastic behaviour of the constitutive material using the elastic–viscoelastic principle. Indeed, when used this method restricts the analysis to a single frequency, steady state analysis. Consequently, when transient analysis of FRP sandwich plates made of viscoelastic materials needs to be carried out, a different viscoelastic model, which should be consistent across a broad range of frequencies, should be considered. As mentioned in the Introduction, the GHM method appears as an attractive alternative approach that offers the feasibility to model viscoelastic damping effects without restriction to the steady state motion by providing extra co-ordinates.

The equations of motion of the HSDT for FRP sandwich panels made of linear viscoelastic materials can be derived in the same way as the equations of motion of the HSDT for FRP sandwich panels made of elastic materials. This is achieved simply by using the linear hereditary stress–strain law (see relations (2)) instead of the generalized Hooke's law. The equations of motion obtained for FRP sandwich plates modelled using HSDT can then be expressed in the form

$$[\mathbf{M}]\{\ddot{\boldsymbol{\Delta}}(t)\}+\int_0^t[\mathbf{K}(t-\tau)]\{\dot{\boldsymbol{\Delta}}(t)\}\,\mathrm{d}\tau=\{\mathbf{F}(t)\}. \tag{10}$$

These equations can then be transformed in the Laplace domain. The result of this transformation leads to

$$s^2[\mathbf{M}]\{\tilde{\boldsymbol{\Delta}}(s)\}+\left[s\tilde{\mathbf{K}}(s)\right]\{\tilde{\boldsymbol{\Delta}}(s)\}=\{\tilde{\mathbf{F}}(s)\}. \tag{11}$$

In relations (10) and (11), the coefficients of the mass matrix $[\mathbf{M}]$, which are independent of the viscoelastic material properties, can be calculated using the relations derived in the authors' previous paper [8]. Having calculated the coefficients of the mass matrix, one faces the main task of deriving the coefficients of the stiffness matrix $\left[s\tilde{\mathbf{K}}(s)\right]$ in the Laplace domain. This has to be achieved such that when relation (11) is re-written in the time domain linear second order equations of motion are obtained.

The method developed to achieve this is based on the GHM approach. The GHM approach models viscoelastic damping by adding additional "dissipation co-ordinates" to the system in order to get a linear model providing the required damping properties. To achieve this the GHM method requires the representation of the material modulus functions as a series of damped mini-oscillator terms as shown below:

$$s\tilde{G}_i(s)=G_i^*(s)=\hat{G}_i^0\left(1+\sum_{p=1}^{k}\hat{\alpha}_i^p\frac{[s^2+2\hat{\zeta}_i^p\hat{\omega}_i^p s]}{[s^2+2\hat{\zeta}_i^p\hat{\omega}_i^p s+(\hat{\omega}_i^p)^2]}\right). \tag{12}$$

In relation (12), the number of terms kept in the expansion is determined by the high- or low-frequency dependence of the complex modulus.

In the equations of motion (11), developed for FRP sandwich panels having viscoelastic material properties, the stiffness matrix is composed of 15 different coefficients each of them being a function of at least one material modulus. In order to apply the GHM method to this multi-modulus problem, a factorization of the matrix $\left[s\tilde{\mathbf{K}}(s)\right]$ is necessary. Indeed, the matrix has to be expressed as a series of component matrices $\left[s\tilde{\mathbf{K}}(s)\right]_i$, each dependent upon a single modulus function $s\tilde{G}_i(s)$ as shown in the following relation:

$$\left[s\tilde{\mathbf{K}}(s)\right] = \sum_i \left[s\tilde{\mathbf{K}}(s)\right]_i = \sum_i s\tilde{G}_i(s)\left[\bar{\mathbf{K}}\right]_i. \tag{13}$$

Using relation (13) the equations of motion (11) in the Laplace domain can be re-written in the form

$$s^2[\mathbf{M}]\left\{\tilde{\boldsymbol{\Delta}}(s)\right\} + \sum_i \left(s\tilde{G}_i(s)\left[\bar{\mathbf{K}}\right]_i\right)\left\{\tilde{\boldsymbol{\Delta}}(s)\right\} = \left\{\tilde{\mathbf{F}}(s)\right\}. \tag{14}$$

Each of the ith modulus functions in the above equation can then be expressed in the mini-oscillator form of Eq. (12). Consequently, by introducing a column vector of dissipation co-ordinates, as expressed by relation (15), one can re-write the equations of motion in the Laplace domain (14) in the form of relation (16)

$$\left\{\tilde{\bar{\mathbf{z}}}^p(s)\right\}_i = \frac{(\hat{\omega}_i^p)^2}{[s^2 + 2\hat{\zeta}_i^p\hat{\omega}_i^k s + (\hat{\omega}_i^p)^2]}\left\{\tilde{\boldsymbol{\Delta}}(s)\right\}, \tag{15}$$

$$\left\{ s^2 \begin{bmatrix} [\mathbf{M}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{M}_{\bar{z}}] \end{bmatrix} + s \begin{bmatrix} [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{D}_{\bar{z}}] \end{bmatrix} + \begin{bmatrix} \left[\mathbf{K}_q\right] & \left[\mathbf{K}_{q\bar{z}}\right] \\ \left[\mathbf{K}_{q\bar{z}}\right]^{\mathrm{T}} & [\mathbf{K}_{\bar{z}}] \end{bmatrix} \right\} \left\{ \begin{matrix} \{\tilde{\boldsymbol{\Delta}}\} \\ \{\tilde{\bar{\mathbf{z}}}\} \end{matrix} \right\} = \left\{ \begin{matrix} \{\tilde{\mathbf{F}}\} \\ \{\mathbf{0}\} \end{matrix} \right\}. \tag{16}$$

The expression of the constitutive sub-matrices of the mass, stiffness and damping matrices in Eq. (16) are provided in Appendix A.

The coefficients of the sub-matrix $[\mathbf{M}_{\bar{z}}]$ are a function of the coefficients of the matrices $\left[\bar{\mathbf{K}}\right]_i$ as can be seen by looking at the relations provided in Appendix A. Since the matrices $\left[\bar{\mathbf{K}}\right]_i$ admit rigid motion, the mass matrix in relation (16) is not guaranteed to be a definite positive. To remedy this situation, spectral decomposition of the stiffness matrices $\left[\bar{\mathbf{K}}\right]_i$ is achieved and it leads to the derivation, as shown in relation (17), of the diagonal matrices $\left[\bar{\boldsymbol{\Lambda}}\right]_i$, of the non-zero eigenvalues of the matrices $\left[\bar{\mathbf{K}}\right]_i$ and of the matrices $\left[\bar{\mathbf{R}}\right]_i$ of which each column corresponds to the associated orthogonalized eigenvectors. It should be noticed that this decomposition separates the eigenvalues into elastic (non-zero) and rigid-body (or zero) modes

$$\left[\bar{\mathbf{K}}\right]_i = \left[\bar{\mathbf{R}}\right]_i\left[\bar{\boldsymbol{\Lambda}}\right]_i\left[\bar{\mathbf{R}}\right]_i^{\mathrm{T}}. \tag{17}$$

Finally, to achieve the objective of having fewer dissipative co-ordinates and a positive definite viscoelastic matrix, each diagonal matrix $\left[\bar{\boldsymbol{\Lambda}}\right]_i$ is multiplied by the associated equilibrium modulus $\hat{G}_i^0$, such that $[\boldsymbol{\Lambda}]_i = \hat{G}_i^0\left[\bar{\boldsymbol{\Lambda}}\right]_i$. Then the substitutions provided in relations (18) and (19) are carried out

$$\{\mathbf{z}\}_i = \left[\bar{\mathbf{R}}\right]_i^{\mathrm{T}}\{\bar{\mathbf{z}}\}_i, \tag{18}$$

$$[\mathbf{R}]_i = \left[\bar{\mathbf{R}}\right]_i[\boldsymbol{\Lambda}]_i. \tag{19}$$

Applying these substitutions to Eqs. (16) leads to the final form of the viscoelastic equations of motion written in the Laplace domain as expressed by relation (20). This relation is then re-written in the simplified notation form in relation (21):

$$\left\{ s^2 \begin{bmatrix} [\mathbf{M}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{M}_z] \end{bmatrix} + s \begin{bmatrix} [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{D}_z] \end{bmatrix} + \begin{bmatrix} [\mathbf{K}_q] & [\mathbf{K}_{qz}] \\ [\mathbf{K}_{qz}]^{\mathrm{T}} & [\mathbf{K}_z] \end{bmatrix} \right\} \left\{ \begin{matrix} \{\tilde{\boldsymbol{\Delta}}\} \\ \{\tilde{\bar{\mathbf{z}}}\} \end{matrix} \right\} = \left\{ \begin{matrix} \{\tilde{\mathbf{F}}\} \\ \{\mathbf{0}\} \end{matrix} \right\}, \tag{20}$$

$$\left(s^2[\mathbf{M}_v] + s[\mathbf{D}_v] + [\mathbf{K}_v]\right)\left\{\tilde{\boldsymbol{\Delta}}_v\right\} = \left\{\tilde{\mathbf{F}}_v\right\}. \tag{21}$$

In relation (21), the matrices $[\mathbf{M}_v]$, $[\mathbf{D}_v]$ and $[\mathbf{K}_v]$ are, respectively, the GHM viscoelastic mass, damping and stiffness matrices. Their coefficients, which are amongst other things dependent on the mini-oscillator parameters and on the coefficients of the matrices $[\mathbf{R}]_i$ and $[\boldsymbol{\Lambda}]_i$, are derived using the expressions provided in Appendix B.

In the Laplace domain the stiffness coefficients have been expressed using the GHM analytical model involving a sum of rational functions of second degree. The algebraic form of the mini-oscillator terms allows one to write the equations of motion in the Laplace domain in the form of relation (21). This then enables one to derive linear second order equations of motion in the time domain. The derived second order equations of motion in the time domain are expressed by

$$[\mathbf{M}_v]\left\{\ddot{\boldsymbol{\Delta}}_v(t)\right\} + [\mathbf{D}_v]\left\{\dot{\boldsymbol{\Delta}}_v(t)\right\} + [\mathbf{K}_v]\{\boldsymbol{\Delta}_v(t)\} = \{\mathbf{F}_v(t)\}. \tag{22}$$

After deriving the equations of motion in the time domain, the response of the studied structure subjected to a given applied load can be obtained by solving Eq. (22) using Newmark's numerical integration technique. Different Newmark integration schemes exist. In this paper the constant acceleration method is used. This particular scheme is well known for its properties of being numerically stable. Using Newmark method, one can determine the coefficients of the column vector $\{\boldsymbol{\Delta}_v(t)\} = \{\{\boldsymbol{\Delta}(t)\}; \{\bar{\mathbf{z}}(t)\}\}$ for each time step and for each mode number (m, n) considered. Then, the transverse displacement at any point within the plate can be derived as for the harmonic case using relation (9).

2.3. Transverse applied loads

A correct mathematical representation of the applied load is essential in order to accurately predict, using the presented method, the response of the studied panel. If a Navier solution type is assumed the load needs to be expanded in a double trigonometric series. That is to say, the transverse applied load can be expressed in the form

$$q(x, y, t) = \sum \sum Q_{mn}(t) \sin \alpha x \sin \beta y. \tag{23}$$

The load coefficient $Q_{mn}(t)$ is a function of the type of applied loads, For example the Navier solution for a sinusoidally distributed transverse load is given by

$$q(x, y, t) = q_0(t) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}. \tag{24}$$

This implies that for a sinusoidally distributed load, the load coefficient, $Q_{mn}(t)$, is equal to $q_0(t)$ with $m = n = 1$.

For other types of load, the Navier solution is a series solution, which can be evaluated for a sufficient number of terms in the series. In particular, for a uniformly distributed load,

$q(x, y, t) = q_0(t)$, the load coefficient is expressed as shown by Eq. (25) and for an hydrostatic load, $q(x, y, t) = -q_0(t)y/b$, it is expressed as shown by relation (26)

$$Q_{mn}(t) = -\frac{16q_0(t)}{\pi^2 mn} \quad \text{for } m,\ n \text{ odd numbers,} \tag{25}$$

$$Q_{mn}(t) = -\frac{8q_0(t)}{\pi^2 mn} \quad \text{for } m,\ n \text{ odd numbers.} \tag{26}$$

3. GHM method applied to model FRP sandwich constitutive materials

The presented new method is used in the following to predict the dynamic response of simply supported rectangular sandwich plates ($a{=}b{=}1\,\text{m}$) composed of two FRP composite laminated faces and a rigid PVC core as shown in Fig. 1. To achieve this, a program based on the theory presented above has been developed using the software MATLAB. The sandwich plates that are considered in this paper are assumed to have a length to thickness ratio, a/h, equal to 10 and a relative core thickness to plate ratio, h_c/h, equal to 0.94.

3.1. Dynamic material properties

In order to predict the transient response of simply supported FRP sandwich panels the knowledge of the viscoelastic constitutive material properties (e.g., complex moduli) is required. As far as FRP materials are concerned, a reasonable amount of data can be found in the literature. In the present case it is assumed that each face is constituted of three layers of Eglass fibres in epoxy resin oriented at 0°, 90° and 0° having the elastic material properties provided in Table 1. The core on the other hand is assumed to be made of closed cell polymer foam, namely the HEREX C70.130 PVC foam, of which the viscoelastic properties have been determined experimentally [12]. The measured material properties of the HEREX C70.130 PVC foam are summarized in Table 2.

Since the faces are assumed to have elastic material properties only the complex Young's modulus and complex shear modulus of the core material need to be modelled using the GHM mini-oscillator technique. The equations of motion in the Laplace domain of the studied FRP sandwich plate can be derived from relation (14) and so can be expressed in the form

$$s^2[\mathbf{M}]\{\tilde{\mathbf{\Delta}}(s)\} + ([\mathbf{K}]_s + s\tilde{\mathbf{G}}_c(s)[\bar{\mathbf{K}}]_{G_c} + s\tilde{\mathbf{E}}_c(s)[\bar{\mathbf{K}}]_{E_c})\{\tilde{\mathbf{\Delta}}(s)\} = \{\tilde{\mathbf{F}}(s)\}. \tag{27}$$

In this relation (27), $[\mathbf{K}]_s$ is the stiffness due only to the faces of the sandwich panels, $s\tilde{\mathbf{G}}_c(s)$ is the shear modulus function of the core material, $s\tilde{\mathbf{E}}_c(s)$ is the Young's modulus function of the core material and $[\bar{\mathbf{K}}]_{G_c}$ and $[\bar{\mathbf{K}}]_{E_c}$ are the stiffness matrices with modulus factored out respectively associated with the core shear and Young's modulus functions. The first step of the analysis consists of deriving the expressions for the modulus functions $s\tilde{\mathbf{G}}_c(s)$ and $s\tilde{\mathbf{E}}_c(s)$. This task is described in the following section.

Table 1
FRP laminated faces dynamic material properties [19]

Material properties	E'_1 (GPa)	E'_2 (GPa)	$G'_{12} = G'_{13} = G'_{23}$ (GPa)	v_{12}	ρ (kg/m^3)
Eglass/DX-210	37.78	10.90	4.91	0.3	1813.9

Table 2
Measured dynamic material properties of HEREX C70.130 PVC foam at 30°C [12]

Material properties	E' (MPa)	η_E	G' (MPa)	η_G	v	ρ (kg/m^3)
HEREX C70.130@30°C	113.5	0.0288	18.86	0.067	0.32	130

3.2. Representation of material properties using GHM mini-oscillator parameters

As mentioned before, the first task consists of deriving the parameters of each mini-oscillator mathematical tool used to model each material complex modulus. Since in the present case only the core constitutive material is assumed to exhibit viscoelastic damping, only two series of mini-oscillator expression need to be determined namely one for the complex Young's modulus and one for the complex shear modulus of the HEREX C70.130 measured at 30°C. To achieve this, a least square fit has been applied to both of the complex moduli spanning a defined range of frequencies. The real and imaginary parts of each complex modulus are considered simultaneously with equal weighting using the following relations which are derived from relation (7)

$$G'_i(\omega) = \hat{G}_i^0 \left(1 + \omega^2 \sum_{p=1}^{k} \hat{\alpha}_i^p \frac{\left(\omega^2 - \left(\hat{\omega}_i^p\right)^2\right) + \left(2\hat{\zeta}_i^p \hat{\omega}_i^p\right)^2}{\left(\omega^2 - \left(\hat{\omega}_i^p\right)^2\right)^2 + \omega^2 \left(2\hat{\zeta}_i^p \hat{\omega}_i^p\right)^2} \right), \tag{28}$$

$$G''_i(\omega) = \hat{G}_i^0 \left(\omega \sum_{p=1}^{k} \hat{\alpha}_i^p \frac{\left(\hat{\omega}_i^p\right)^2 \left(2\hat{\zeta}_i^p \hat{\omega}_i^p\right)}{\left(\omega^2 - \left(\hat{\omega}_i^p\right)^2\right)^2 + \omega^2 \left(2\hat{\zeta}_i^p \hat{\omega}_i^p\right)^2} \right). \tag{29}$$

The program used has been realized using the software MATLAB. The function "lscurvefit" has been used to achieve the task. "lscurvefit" function allows one to solve non-linear curve-fitting problems in the least-squares sense. This function requires:

- *The input data*: that is to say the frequencies for which GHM approximation of the material data is required.
- *The output data*: that is to say the real and imaginary values of the complex modulus for the corresponding input frequency.
- The equations containing the parameters that need to be derived.

It should be noticed that first of all the program has been run assuming a varying number k of mini-oscillator terms with the objective of finding the best approximation over the interval

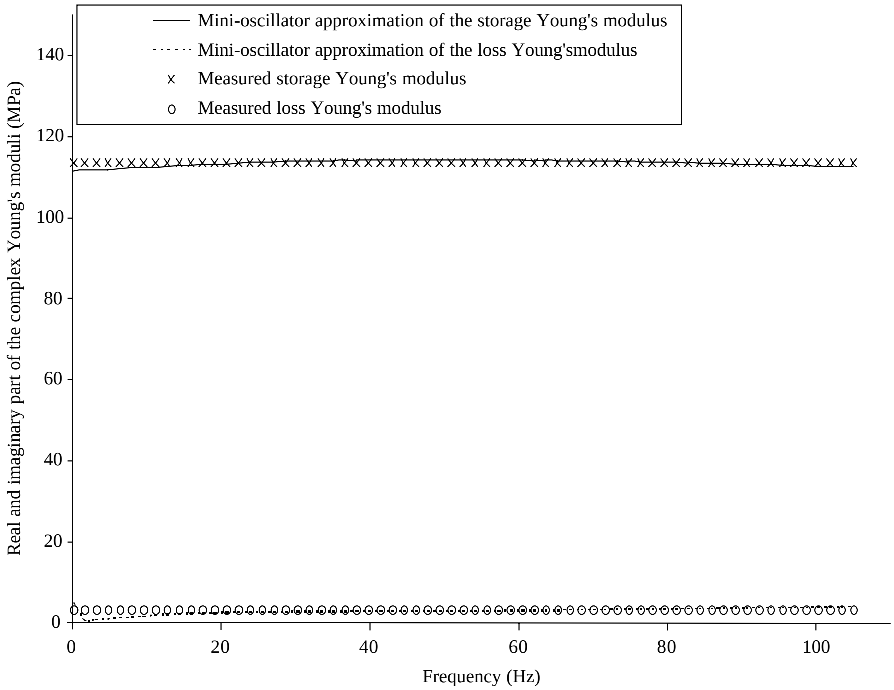

Fig 2. Comparison of the calculated values with the approximated values using the GHM approach for the real and imaginary parts of the complex Young's modulus.

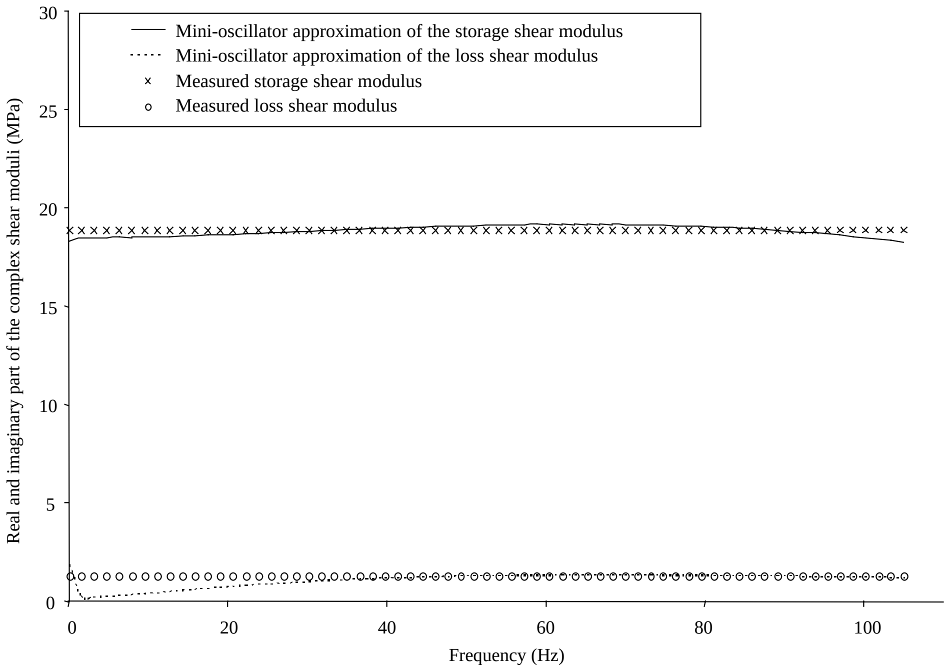

Fig. 3. Comparison of the calculated values with the approximated values using the GHM approach for the real and imaginary parts of the complex shear modulus.

$f \in [1\,\text{Hz}, 100\,\text{Hz}]$. The computational time has been found to increase drastically when k increases. The optimum number of mini-oscillator terms for the studied examples has been found to be 3. However, depending on the frequency range of interest a different number of terms might be required. Indeed, if the frequency range of interest is increased, in order to keep the same accuracy more mini-oscillator terms would be required for the modelling.

In Figs. 2 and 3, respectively, the real part and imaginary part of the complex Young's modulus and the complex shear modulus respectively are presented in the frequency range from 1 to 100 Hz. In these figures the approximated real and imaginary values of the complex moduli over the defined frequency range are compared with the calculated values using the equations of the complex stiffness matrix [12]. The GHM mini-oscillator approximation form, of which coefficients have been determined by best-fit curve, appears to accurately model, over the considered frequency range, the complex material moduli.

4. Validation of the method

In order to validate the new developed method for transient analysis of FRP sandwich plates having viscoelastic material properties the FRP sandwich plate, Defined in Part 3, is firstly assumed to be subjected to a transverse sinusoidally distributed load harmonic in the time domain. That is to say the transverse applied load is expressed by

$$q(x, y, t) = q_0 \cos(\omega t) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}. \tag{30}$$

In the example presented, it is assumed that q_0 is equal to $10^6\,\text{N/m}^2$ and $\omega = 80\pi\,\text{rad/s}$.

The transverse displacement at the middle of the considered sandwich plates has been predicted as a function of time using two different techniques, namely the novel developed analytical model

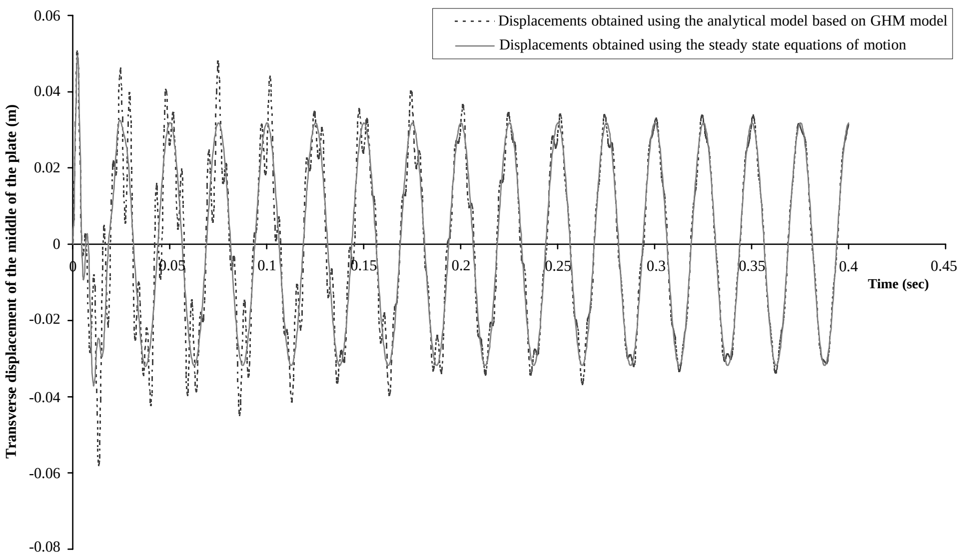

Fig. 4. Response of a square FRP sandwich plate subjected to harmonic excitation.

based on the GHM mathematical representation and the analytical method based on the elastic–viscoelastic principle and summarized in Section 2.2.2 of this paper. The results obtained over a period of time of 0.4 s are presented in Fig. 4. The plain line represents the displacement of the plate computed using the analytical model based on the GHM model. The dotted line shows the results predicted using the steady state equations of motion (6). It can be observed from Fig. 4 that when the steady state conditions are reached, the responses obtained with both methods are identical. This leads to the conclusion that the proposed new analytical model, based on GHM mathematical model, provides an accurate prediction of the dynamic responses of FRP sandwich plates having viscoelastic material properties. This novel method can therefore be used with confidence to predict the force response of composite plates having viscoelastic material properties.

5. Transient response of FRP sandwich plates

Two examples are presented in this part. In the first example the square sandwich plate is assumed to be subjected to a uniformly distributed load, $q(x, y, t) = q_0(t)$, and in the second example the plate is assumed to be subjected to a hydrostatic load, $q(x, y, t) = -q_0(t)(y/b)$. Both loads are applied over two periods of time each of 0.06 s duration with an interval of 0.04 s as represented in Fig. 5. The predicted transverse displacements as a function of time for the middle of the FRP sandwich plates for the two types of applied loads are presented in Figs. 6 and 7, respectively. As expected, the amplitude of the displacement of the sandwich plate subjected to a hydrostatic load is smaller than when subjected to a uniformly distributed load. In each presented example, after the load has been removed, the plate is subjected to free vibration. If the plate was undamped the oscillation, once set up, would continue indefinitely. In the present situation, since the viscoelastic damping of the core has been modelled and so accounted for, the response of the FRP sandwich plate dies away with time. The rate at which the response of the plate dies away gives us information on the amount of damping. This rate is known as the logarithmic decrement and is related to ratio of the nth to the $(n+N)$th cycle amplitude as shown below:

$$\delta = \frac{1}{N} \ln \frac{w_{(n+N)}}{w_n}. \tag{31}$$

The logarithmic decrement, δ, for the considered examples has been found to be equal to 0.028. Knowing this value it has been possible to derive the envelope of the decaying oscillation.

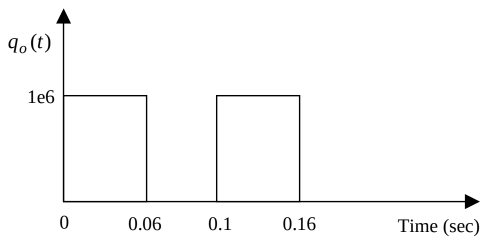

Fig. 5. Transverse applied load intensity.

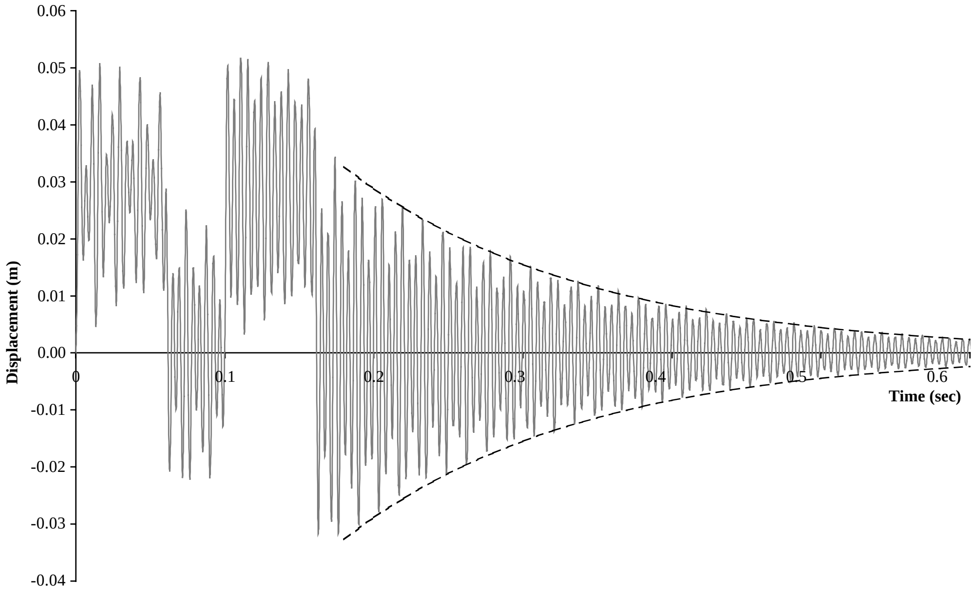

Fig. 6. Response of a square FRP sandwich plate subjected to a uniformly distributed transverse load.

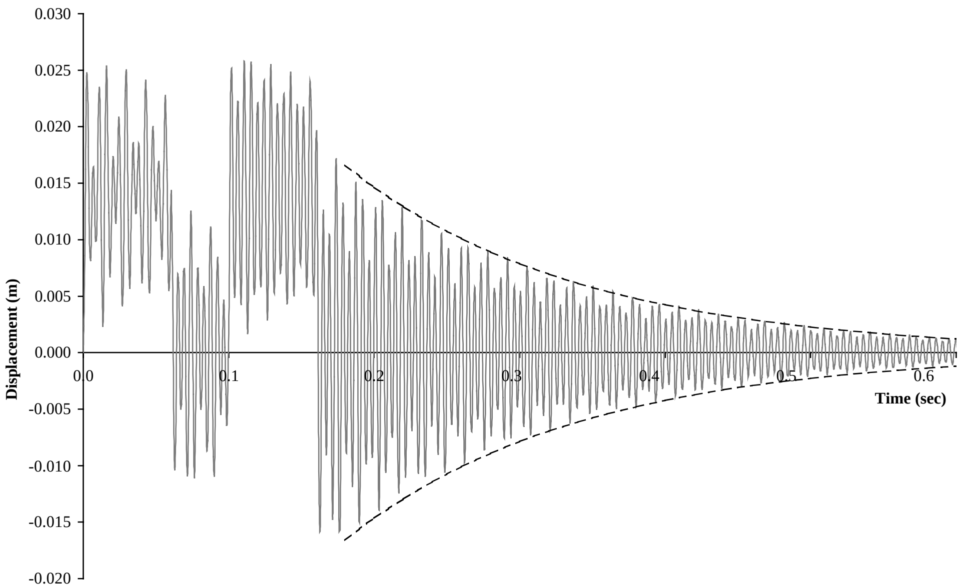

Fig. 7. Response of a square FRP sandwich plate subjected to a hydrostatic load.

This envelope is defined by an exponential curves, which has been drawn using dotted lines in both Figs. 5 and 6. The higher the effect of damping the higher the value of the logarithmic decrement and so the quicker the response will die away. It can be observed that

0.3 s after the load has been removed the amplitude of displacement of the studied plate has been reduced by more than 60%. So, by modelling the viscoelastic properties of the PVC foam core material it was possible to quantify the effect of viscoelastic damping. As demonstrated the effect of the damping due to the viscoelastic behaviour of the constitutive materials has a significant effect and so should not be neglected. This outlines the necessity of using the developed new method when the responses of FRP sandwich plates need to be accurately predicted and analyzed.

Finally, it should be emphasized that in the presented examples only the viscoelastic properties of the core material have been modelled. However, the program allows the possibility to model the viscoelastic properties of the constitutive materials of the skins as well as the core if these should also prove to be significant. Besides it should be noticed that this method can also be applied with a low number of mini-oscillator terms, as seen in a previous publication [14], to model materials that present a more pronounced frequency dependence for their storage and loss moduli than the HEREX C70.130 PVC foam material.

6. Conclusion

A new analytical model based on Reddy's refined HSDT has been developed. It has been found that this new analytical model provides an accurate prediction of the dynamic responses of simply supported FRP sandwich plates. The main advantage of this new analytical model is that it offers the possibility using the GHM mathematical tool to predict accurately the transient responses as well as the steady state behaviour of FRP sandwich plates accounting for their viscoelastic material properties. Consequently, it is possible using the developed analytical model to gain a good understanding of the dynamic behaviour of FRP sandwich panels likely to be subjected to dynamic loading. This can be of particular importance when designing FRP sandwich structures likely to be subjected during their service life to dynamic environments.

Appendix A

A.1. GHM viscoelastic mass matrix coefficients

The GHM viscoelastic mass matrix before decomposition is a function of the mass matrix $[\mathbf{M}]$, and of the diagonal mass matrix $[\mathbf{M}_{\bar{z}}]$, whose coefficients are a function of the parameters of the GHM mini-oscillator forms. The diagonal sub-mass matrix $[\mathbf{M}_{\bar{z}}]$ is defined as follows:

$$[\mathbf{M}_{\bar{z}}] = \begin{bmatrix} [\mathbf{M}_{\bar{z}}]_1 & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & \ddots & & \vdots \\ \vdots & & \ddots & [\mathbf{0}] \\ [\mathbf{0}] & \cdots & [\mathbf{0}] & [\mathbf{M}_{\bar{z}}]_n \end{bmatrix}.$$

Each of the diagonal matrices $[\mathbf{M}_{\bar{z}}]_i$ are defined by

$$[\mathbf{M}_{\bar{z}}]_i = \begin{bmatrix} \hat{\alpha}_i^1 \dfrac{1}{\left(\hat{\omega}_i^1\right)^2}\left[\bar{\mathbf{K}}\right]_i & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & \ddots & & \vdots \\ \vdots & & \ddots & [\mathbf{0}] \\ [\mathbf{0}] & \cdots & [\mathbf{0}] & \hat{\alpha}_i^k \dfrac{1}{\left(\hat{\omega}_i^k\right)^2}\left[\bar{\mathbf{K}}\right]_i \end{bmatrix}.$$

A.2. GHM viscoelastic damping matrix coefficients

The GHM viscoelastic damping matrix before decomposition is a function of the diagonal damping matrix $[\mathbf{D}_{\bar{z}}]$, whose coefficients are a function of the parameters of the GHM mini-oscillator forms. This diagonal matrix is defined as follows:

$$[\mathbf{D}_{\bar{z}}] = \begin{bmatrix} [\mathbf{D}_{\bar{z}}]_1 & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & \ddots & & \vdots \\ \vdots & & \ddots & [\mathbf{0}] \\ [\mathbf{0}] & \cdots & [\mathbf{0}] & [\mathbf{D}_{\bar{z}}]_n \end{bmatrix}.$$

Each of the diagonal matrices $[\mathbf{D}_{\bar{z}}]_i$ are defined by

$$[\mathbf{D}_{\bar{z}}]_i = \begin{bmatrix} \hat{\alpha}_i^1 \dfrac{2\hat{\zeta}_i^1}{\hat{\omega}_i^1}\left[\bar{\mathbf{K}}\right]_i & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & \ddots & & \vdots \\ \vdots & & \ddots & [\mathbf{0}] \\ [\mathbf{0}] & \cdots & [\mathbf{0}] & \hat{\alpha}_i^k \dfrac{2\hat{\zeta}_i^k}{\hat{\omega}_i^k}\left[\bar{\mathbf{K}}\right]_i \end{bmatrix}.$$

A.3. GHM viscoelastic stiffness matrix coefficients

The GHM viscoelastic stiffness matrix before decomposition is a function of the stiffness matrix $[\mathbf{K}_z]$, of the stiffness matrix $\left[\mathbf{K}_{q\bar{z}}\right]$ and its matrix transpose and of the diagonal stiffness matrix $[\mathbf{K}_{\bar{z}}]$. These sub-matrices are defined as follows:

$$\left[\mathbf{K}_q\right] = [\mathbf{K}] + \sum_{i=1}^{n}\sum_{j=1}^{k} \alpha_i^j [\mathbf{K}]_i,$$

$$\left[\mathbf{K}_{q\bar{z}}\right] = \left[-\hat{\alpha}_1^1 \left[\bar{\mathbf{K}}\right]_1 \quad \cdots \quad -\alpha_n^k \left[\bar{\mathbf{K}}\right]_n \right],$$

$$\left[\mathbf{K}_{\bar{z}q}\right] = \left[\mathbf{K}_{q\bar{z}}\right]^{\mathrm{T}},$$

$$[\mathbf{K}_{\bar{z}}] = \begin{bmatrix} [\mathbf{K}_{\bar{z}}]_1 & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & \ddots & & \vdots \\ \vdots & & \ddots & [\mathbf{0}] \\ [\mathbf{0}] & \cdots & [\mathbf{0}] & [\mathbf{K}_{\bar{z}}]_n \end{bmatrix},$$

where the matrix $[\mathbf{K}]$ is defined by

$$[\mathbf{K}] = \sum_{i=1}^{n} [\mathbf{K}]_i = \sum_{i=1}^{n} \hat{G}_i^0 \left[\bar{\mathbf{K}}\right]_i$$

and, where each of the diagonal matrices $[\mathbf{K}_{\bar{z}}]_i$ are given by

$$[\mathbf{K}_{\bar{z}}]_i = \begin{bmatrix} \alpha_i^1 \left[\bar{\mathbf{K}}\right]_i & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & \ddots & & \vdots \\ \vdots & & \ddots & [\mathbf{0}] \\ [\mathbf{0}] & \cdots & [\mathbf{0}] & \alpha_i^k \left[\bar{\mathbf{K}}\right]_k \end{bmatrix}.$$

Appendix B

B.1. GHM viscoelastic mass matrix coefficients

The GHM viscoelastic mass matrix is a function of the mass matrix $[\mathbf{M}]$, and of the diagonal mass matrix $[\mathbf{M}_z]$, whose coefficients are a function of the parameters of the GHM mini-oscillator forms

$$[\mathbf{M}_v] = \begin{bmatrix} [\mathbf{M}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{M}_z] \end{bmatrix}.$$

The diagonal mass matrix $[\mathbf{M}_z]$ is defined as follows:

$$[\mathbf{M}_z] = \begin{bmatrix} [\mathbf{M}_z]_1 & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & \ddots & & \vdots \\ \vdots & & \ddots & [\mathbf{0}] \\ [\mathbf{0}] & \cdots & [\mathbf{0}] & [\mathbf{M}_z]_n \end{bmatrix}.$$

Each of the diagonal matrices $[\mathbf{M}_z]_i$ are defined by

$$[\mathbf{M}_z]_i = \begin{bmatrix} \hat{\alpha}_i^1 \dfrac{1}{\left(\hat{\omega}_i^1\right)^2} [\boldsymbol{\Lambda}]_i & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & \ddots & & \vdots \\ \vdots & & \ddots & [\mathbf{0}] \\ [\mathbf{0}] & \cdots & [\mathbf{0}] & \hat{\alpha}_i^k \dfrac{1}{\left(\hat{\omega}_i^k\right)^2} [\boldsymbol{\Lambda}]_i \end{bmatrix}.$$

B.2. GHM viscoelastic damping matrix coefficients

The GHM viscoelastic damping matrix $[\mathbf{D}_v]$ is a function of the diagonal damping matrix $[\mathbf{D}_z]$, whose coefficients are a function of the parameters of the GHM mini-oscillator forms

$$[\mathbf{D}_v] = \begin{bmatrix} [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{D}_z] \end{bmatrix}.$$

The diagonal damping matrix $[\mathbf{D}_z]$ is defined as follows:

$$[\mathbf{D}_z] = \begin{bmatrix} [\mathbf{D}_z]_1 & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & \ddots & & \vdots \\ \vdots & & \ddots & [\mathbf{0}] \\ [\mathbf{0}] & \cdots & [\mathbf{0}] & [\mathbf{D}_z]_n \end{bmatrix}.$$

Each of the diagonal matrices $[\mathbf{D}_z]_i$ are defined by

$$[\mathbf{D}_z]_i = \begin{bmatrix} \hat{\alpha}_i^1 \dfrac{2\hat{\zeta}_i^1}{\hat{\omega}_i^1}[\mathbf{\Lambda}]_i & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & \ddots & & \vdots \\ \vdots & & \ddots & [\mathbf{0}] \\ [\mathbf{0}] & \cdots & [\mathbf{0}] & \hat{\alpha}_i^k \dfrac{2\hat{\zeta}_i^k}{\hat{\omega}_i^k}[\mathbf{\Lambda}]_i \end{bmatrix}.$$

B.3. GHM viscoelastic stiffness matrix coefficients

The GHM viscoelastic stiffness matrix $[\mathbf{K}_v]$ is a function of the diagonal stiffness matrix $[\mathbf{K}_z]$, of the stiffness matrix $\left[\mathbf{K}_{qz}\right]$ and of the stiffness matrix $[\mathbf{K}_z]$ whose coefficients are a function of the parameters of the GHM mini-oscillator forms

$$[\mathbf{K}_v] = \begin{bmatrix} \left[\mathbf{K}_q\right] & \left[\mathbf{K}_{qz}\right] \\ \left[\mathbf{K}_{qz}\right]^T & [\mathbf{K}_z] \end{bmatrix}.$$

The sub-matrices of the viscoelastic stiffness matrix $[\mathbf{K}_v]$ are defined as follows:

$$\left[\mathbf{K}_q\right] = [\mathbf{K}] + \sum_{i=1}^{n} \sum_{j=1}^{k} \hat{\alpha}_i^j [\mathbf{K}]_i,$$

$$\left[\mathbf{K}_{qz}\right] = \left[-\hat{\alpha}_1^1 [\mathbf{R}]_1 \quad \cdots \quad -\alpha_n^k [\mathbf{R}]_n \right],$$

$$\left[\mathbf{K}_{zq}\right] = \left[\mathbf{K}_{qz}\right]^{\mathrm{T}},$$

$$[\mathbf{K}_z] = \begin{bmatrix} [\mathbf{K}_z]_1 & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & \ddots & & \vdots \\ \vdots & & \ddots & [\mathbf{0}] \\ [\mathbf{0}] & \cdots & [\mathbf{0}] & [\mathbf{K}_z]_n \end{bmatrix}.$$

Each of the diagonal matrices $[\mathbf{K}_z]_i$ are defined by

$$[\mathbf{K}_z]_i = \begin{bmatrix} \alpha_i^1[\mathbf{\Lambda}]_i & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & \ddots & & \vdots \\ \vdots & & \ddots & [\mathbf{0}] \\ [\mathbf{0}] & \cdots & [\mathbf{0}] & \alpha_i^k[\mathbf{\Lambda}]_i \end{bmatrix}.$$

Appendix C. Nomenclature

a, b	length and width of the plate
f	frequency in Hertz
h	thickness of the studied plate
h_c, h_f	thickness of the core and of the faces
m, n	number of vibration half-waves in the x and y directions
q, q_0	transverse applied load and maximum amplitude of the transverse applied load
s	Laplace domain operator
t	time variable
u, v, w	displacement components along the x, y, z co-ordinate direction
u_0, v_0, w_0	displacement components along the x, y, z co-ordinate direction of a point on the mid-plane
z	thickness co-ordinate measured from the mid-plane of the plate
$C_{ijkl}(t)$	relaxation modulus components
C^*_{ijkl}	complex material stiffness coefficients
E_c, G_c	Young's and shear modulus of the core material
E', G'	storage Young's and shear moduli
E^*, G^*	complex Young's and shear moduli
E'_1, E'_2	storage Young's modulus of FRP material in the direction along and normal to fibres
G'_i, G''_i	storage and loss moduli
G'_{21},	FRP in-plane storage shear modulus
G'_{13}, G'_{23}	FRP transverse storage shear moduli
G^0	equilibrium value of modulus
$\hat{G}^0$	GHM approximation to G^0
$s\hat{G}(s)$	Laplace transform function of the material modulus
Q_{mn}	transverse load coefficients

$U_{mn}, V_{mn}, W_{mn}, X_{mn}, Y_{mn}$	generalized spatial variation amplitudes
α, β	$= m\pi/a,\ n\pi/b$
$\hat{\alpha}, \hat{\omega}, \hat{\zeta}$	GHM mini-oscillator parameters
δ	logarithmic decrement
ε_{kl}	strain components
ϕ_x, ϕ_y	rotation of the transverse normal about the y- and x-axis, respectively
θ	fibre orientation relative to the x-axis in a lamina
ν, ν_{12}	Poisson ratios
ρ	density
σ_{ij}	stress components
τ	input time
ω	frequency in rad/s
$\{\bar{\mathbf{z}}\}, \{\mathbf{z}\}$	vector of the dissipative co-ordinates
$\{\mathbf{F}\}, \{\mathbf{F}_v\}$	vector of the applied force and viscoelastic vector of the applied force
$\{\mathbf{F}_0\}$	column vector of the amplitude components of the harmonic applied forces
$\{\boldsymbol{\delta}^*\}$	complex eigenvectors
$\{\boldsymbol{\Delta}\}, \{\boldsymbol{\Delta}_v\}$	vector and viscoelastic vector of the generalized spatial variation amplitudes
$\{\boldsymbol{\Delta}^*\}$	column vector of the complex values of the generalized displacements
$[\mathbf{D}], [\mathbf{D}_v]$	damping and viscoelastic damping matrices
$[\mathbf{D}_z]$	sub-matrix of the viscoelastic damping matrix
$[\mathbf{D}_{\bar{z}}]$	sub-matrix of the viscoelastic damping matrix before decomposition
$[\mathbf{K}], [\boldsymbol{K}_v]$	stiffness and viscoelastic stiffness matrices
$[\mathbf{K}'], [\mathbf{K}'']$	storage and loss stiffness matrices
$[\mathbf{K}_q]$	sub-matrix of the viscoelastic stiffness matrix
$[\mathbf{K}_{qz}], [\mathbf{K}_z]$	sub-matrices of the viscoelastic stiffness matrix
$[\mathbf{K}_{q\bar{z}}], [\mathbf{K}_{\bar{z}}]$	sub-matrices of the viscoelastic stiffness matrix before decomposition
$[s\tilde{\mathbf{K}}(s)]$	matrix of the Laplace transform function of the complex stiffness coefficients
$[\bar{\mathbf{K}}]$	modulus factored out stiffness matrix
$[\mathbf{K}]_s$	stiffness matrix due to the faces of the sandwich plate
$[\bar{\mathbf{K}}]_{G_c}, [\bar{\mathbf{K}}]_{E_c}$	modulus factored out stiffness matrix associated with the core shear and Young's modulus
$[\mathbf{M}], [\mathbf{M}_v]$	mass matrix and viscoelastic mass matrix
$[\mathbf{M}_z]$	sub-matrix of the viscoelastic matrix
$[\mathbf{M}_{\bar{z}}]$	sub-matrix of the viscoelastic matrix before decomposition
$[\bar{\mathbf{R}}]$	matrix of the mode eignevectors of the stiffness matrix $[\bar{\mathbf{K}}]$
$[\mathbf{R}]$	$= [\bar{\mathbf{R}}][\boldsymbol{\Lambda}]$
$[\bar{\boldsymbol{\Lambda}}]$	diagonal matrix of the positive eigenvalues of $[\bar{\mathbf{K}}]$
$[\boldsymbol{\Lambda}]$	$= \hat{G}^0[\bar{\boldsymbol{\Lambda}}]$
$(\tilde{\ })$	Laplace transform
$(\dot{\ })$	first time derivative
$(\ddot{\ })$	second time derivative
$[\ ]^{\mathrm{T}}$	matrix transpose

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