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Numerical solutions of the acoustic eigenvalue equation in the rectangular room with arbitrary (uniform) wall impedances

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Abstract

Two numerical procedures for finding the acoustic eigenvalues in the rectangular room with arbitrary (uniform) wall impedances are developed. One numerical procedure applies Newton's method. Here, starting with soft walls, the eigenvalues are found by increasing the impedances of each wall pair in small increments up to the *terminal impedances*. Another procedure poses the eigenvalue problem as one of homotopic continuation from a non-physical reference configuration in which all eigenvalues are known and obvious. The continuation is performed by the numerical integration of two differential equations. The latter procedure was found to be faster and finds all possible solutions. The set of eigenvalues allowed the room modal natural frequencies and damping constants to be obtained. From sound decays measured in a hard-walled rectangular room, and from the collective-modal-decay curve, the impedances of the hard walls are estimated. These are then used to find the reverberation times of the modes in the room with the floor lined with sound absorbing material of known acoustic impedance. It was found that a single reverberation time, for all modes, is only supported in the rectangular room with hard walls and at the higher frequency bands, consistent with Sabine's theory, which assumes a diffuse sound field. In the rectangular room with hard walls and at the lower frequency bands, and in the rectangular room with the floor lined with sound absorbing material and for all frequency bands, modes with rather distinctive reverberation times may produce sound decays not always consistent with Sabine's prediction.

1. Introduction

The main goal of the present work is to numerically solve the acoustic eigenvalue equation in the rectangular room for the general case where each wall has an arbitrary (uniform) impedance.

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This would ultimately allow the prediction of sound decays from which the sound quality of the room can be assessed.

Since the earliest discoveries of Sabine [1] and until today, the reverberation time is considered to be one of the most important factors in determining the sound quality of a room. When the sound decay is linear, on a decibel scale, the slope is related to the reverberation time, which is then the only parameter necessary to describe the sound decay process. This is very convenient, but it is also an idealized situation only found in rooms with a diffuse sound field, a characteristic of reverberation chambers at high frequencies. In practical rooms, such as concert halls, theatres, classrooms, meeting rooms and industrial rooms, considering the process of sound decay to be linear is quite often only a crude approximation to the actual conditions. In such rooms, the sound decay is usually non-linear and peculiarities such as double slopes, stepped decays and curvature are often reported.

When Sabine [1] proposed the reverberation time to characterize the sound decay process, and his famous reverberation formula to calculate it, he considered the room as a lumped system, where the sound energy was assumed spatially uniform at any instance during the decay process. The transient response during the decay was then determined by formulating an overall energy balance on the room, in which the sound power loss at the surfaces was related to the rate of change of the internal sound energy. Experience shows that this model is only consistent for rooms where the sound field is diffuse.

The exact treatment of the transient and stationary response of a room is possible theoretically by means of the wave theory. Morse and Bolt [2] established a complete formulation for sound waves in the rectangular room in terms of the normal modes. Using the acoustic impedance to describe the boundary condition at the walls, an eigenvalue equation was established. The steady state and transient solutions were then obtained for large wall impedances (hard walls). These solutions became classical and are often used to study low-modal-density acoustical phenomena in small rooms and/or low frequencies.

A complete implementation of the method of Morse and Bolt has not been possible yet because there is no known general procedure for finding the eigenvalues. Therefore, the existence of such a procedure would allow the normal modes to be obtained in cases where the walls in the rectangular room are not hard. In cases where there is a non-uniform impedance distribution over each wall, the separation of variables is not possible and the method cannot be applied. Although one might still consider that the applicability of this method is still very limited (restricted to rectangular rooms with uniform impedances over each wall), a procedure for finding the eigenvalues is a necessary complement for the application of the method of Morse and Bolt to its full extent. The method can also provide a reference point for checking some of the more versatile approximate methods using finite or boundary elements, which are more fitted to non-uniform impedance distributions.

Upon writing the velocity potential for the standing waves and applying the boundary conditions at the room walls, an equation that gives the set of eigenvalues for each wall pair is established. In the present work, the original eigenvalue equation was re-written as an entire function, which greatly simplified the development of the numerical procedures presented here. One procedure applies Newton's method and the other the homotopic continuation.

From the set of acoustic eigenvalues numerically found in a rectangular room with two sound absorbing configurations, the natural frequencies and damping constants of the room acoustic modes are calculated. Applications are given to obtain the collective-modal-decay and the modal reverberation time in octave frequency bands. The results are then discussed in the light of Sabine's theory.

2. Theory

Morse and Bolt [2] wrote the velocity potential for a standing wave in a rectangular room with dimensions L_x, L_y and L_z as

$$\psi(\omega; x, y, z) = D(x)E(y)F(z)\mathrm{e}^{\mathrm{i}\omega t}, \tag{1}$$

where ω is the driving angular frequency and

$$D(x) = \cosh[(\pi \mathrm{i}x/L_x)\chi_x - \phi_x] \tag{2}$$

with similar expressions for E and F. The complex wave number in the x direction is $\pi\chi_x/L_x$; and χ_x, in terms of its real and imaginary parts can be written as $\chi_x(\omega) = \mu_x + \mathrm{i}\kappa_x$, where μ_x was called the *wave number parameter* and κ_x was called the *attenuation parameter*. In the case of damped standing waves, the wave number ends up being generalized to a complex quantity whose imaginary part is the spatial attenuation factor given by $\pi\kappa_x/L_x$.

The ratio of the sound pressure p to normal particle velocity v_n into the wall surface equals the impedance of the surface Z. For the walls normal to the x direction this ratio is given by

$$Z_x = \frac{p}{v_x} = \frac{\rho(\partial\psi/\partial t)}{-(\partial\psi/\partial x)}. \tag{3}$$

Substituting Eq. (1) into Eq. (3), and recognizing that the specific acoustic impedance is $\zeta_x = Z_x/\rho c$, results

$$\zeta_x = -\frac{\eta_x}{\chi_x}\coth[(\pi \mathrm{i}x/L_x)\chi_x - \phi_x], \tag{4}$$

where $\eta_x = (\omega L_x/\pi c)$, which was called the *frequency parameter*, and gives the x-dimension of the room in half wavelengths.

The specific acoustic impedance for the wall at $x = 0$ is $\zeta_{x=0} = -\zeta_{x_1}$, which substituted into Eq. (4) gives

$$\phi_x = -\coth^{-1}\left[\frac{\zeta_{x_1}}{\eta_x}\chi_x\right]. \tag{5}$$

The specific acoustic impedance for the wall at $x = L_x$ is $\zeta_{x=L_x} = \zeta_{x_2}$, and Eq. (4) together with Eq. (5) results in

$$\pi \mathrm{i}\chi_x + \coth^{-1}\left[\left(\frac{\zeta_{x_1}}{\eta_x}\right)\chi_x\right] + \coth^{-1}\left[\left(\frac{\zeta_{x_2}}{\eta_x}\right)\chi_x\right] = 0 \tag{6}$$

with two other equations for χ_y and χ_z.

As discussed by Morse and Bolt [2], Eq. (6) has an infinite number of roots χ. There is at least one root (and not more than two roots) with μ between zero and 1, another root with μ between 2 and 3, and so on. The different roots are distinguished by assigning to them different values of the subscript n, with $n = 0$ for the value of χ with the smallest value of μ, $n = 1$ for the next smallest value of μ, and so on.

When both parallel walls have arbitrary, however, equal impedances $\zeta_{x_1} = \zeta_{x_2} = \zeta_x$ and dropping the subscripts x, Eq. (6) reduces to

$$\frac{\coth(-\pi i\chi/2)}{\chi/2} = 2\frac{\zeta}{\eta}. \tag{7}$$

Morse [3] and Morse and Bolt [2] gave plots of the real and imaginary parts of the roots of Eq. (7) as functions of the magnitude and phase angle of the specific acoustic impedance ζ. These plots, known as Morse charts, correspond to the conformal transformation between χ^2 and $\ln(\zeta/\eta)$ corresponding to Eq. (7). The transformation is multi-valued, with an infinite number of sheets, corresponding to the infinite number of roots χ. The Morse charts were given for only a few cycles of the transformation [2].

A numerical procedure may in principle be developed for finding the roots of Eq. (7). However, there are difficulties in solving this transcendental equation in the complex plane. The infinity of roots gives rise to numerical problems when solving Eq. (7), because the numerical method may jump without control from one solution to the other. This poses a problem when it is necessary to orderly find the roots, $n = 0$, 1, 2, n_{max}.

The complete Riemann surface of the transformation given by Eq. (7) is not known. Therefore is not clear where the branch cuts shall be placed, to subdivide the Riemann surface into unambiguous Riemann sheets. The basic rule for the positions of the branch cuts is that they must go through the branch points. Branch points are places where the adjacent roots come together in different Riemann sheets. Therefore, the knowledge of the branch points is of fundamental importance.

An alternative form of writing Eq. (7) is

$$\frac{\chi}{2}\tan\left(\pi\frac{\chi}{2}\right) = i\frac{\eta}{2\zeta}. \tag{8}$$

By selecting appropriate branch cuts, Mechel [4] presents a procedure to find the modal solutions in rectangular ducts based on the direct determination of the branch points of an equation with the same basic form of Eq. (8). Lippold [5], based on the identification of the boundary lines between modes, presents a numerical method for the solution of Eq. (8) in the rectangular room to find unambiguously the parameters μ and κ of the nth mode for an arbitrary ζ. In both works, the numerical solutions are valid when two parallel (duct or room) walls have the same (arbitrary) impedance.

No numerical procedure seems to exist for the case where any of two parallel walls in the room have different and arbitrary impedances. This is the situation of most practical rectangular rooms. In this case, Eq. (6) is the applicable eigenvalue equation for each wall pair. The present work chooses to attack this problem by re-writing the defining equation, avoiding working with inverse functions. Making use of the identity

$$\coth(Z_1 + Z_2) = \frac{\coth Z_1 \coth Z_2 + 1}{\coth Z_1 + \coth Z_2} \tag{9}$$

with the aid of Eqs. (5) and (4), when dropping the subscripts x, Eq. (9) can be written as

$$\coth(\pi i\chi) = \frac{-1 - \left(\frac{\zeta_1}{\eta}\chi\right)\left(\frac{\zeta_2}{\eta}\chi\right)}{\left(\frac{\zeta_1}{\eta}\chi\right) + \left(\frac{\zeta_2}{\eta}\chi\right)}. \tag{10}$$

Applying to Eq. (10) the identity

$$\coth Z = \frac{1 + e^{-2Z}}{1 - e^{-2Z}} \tag{11}$$

with $Z = \pi i\chi$ results in

$$e^{i\pi\chi}\left(1 + \frac{\zeta_1}{\eta}\chi\right)\left(1 + \frac{\zeta_2}{\eta}\chi\right) - e^{-i\pi\chi}\left(1 - \frac{\zeta_1}{\eta}\chi\right)\left(1 - \frac{\zeta_2}{\eta}\chi\right) = 0. \tag{12}$$

Eq. (12) is an entire function. It has no branch points or branch cuts whatsoever. This form of the acoustic eigenvalue equation greatly simplifies the development of numerical solutions.

3. Numerical procedures

3.1. Increment of impedances procedure

In Eq. (12) $\chi_n = n$, with $n = 0, 1, 2, 3, \ldots$, are roots for $\zeta_1 = \zeta_2 = \zeta_0 = 0$. That is, for any wall pair, for which both walls are soft, n is a root of Eq. (12). The nth root χ_n is found by increasing the ζ's of both walls, in small increments, from $\zeta_0 = 0$ to $\zeta = \zeta_1$ for the first wall of the pair, and from $\zeta_0 = 0$ to $\zeta = \zeta_2$ for the second wall of the pair. Here ζ_1 and ζ_2 will be called the *specific terminal impedances*.

By defining ε as a fraction of the terminal impedances, $\varepsilon = 0, \ldots, 1$, χ_{n_ε} is the root of Eq. (12) when the first wall of the pair has the impedance $\zeta_\varepsilon = \varepsilon\zeta_1$ and the second wall of the pair has the impedance $\zeta_\varepsilon = \varepsilon\zeta_2$. After each small increment in the impedances of both walls, the root χ_{n_ε} is found by Newton's method, having as initial approximation the root found in the previous step. The root χ_{n_1}, corresponding to $\varepsilon = 1$, is the sought solution.

An example of the results produced by such a procedure is shown in Fig. 1. Fig. 1(b) shows the real and the imaginary parts of the impedances of two parallel room walls as they vary according to $\varepsilon\zeta$, with the specific terminal impedance of the first wall of the pair $\zeta_1 = 5 - i2$ and the specific terminal impedance of the second wall of the pair $\zeta_2 = 82.8$. Fig. 1(a) shows, in the complex plane and for $\eta = 20.7$, the loci of the first 10 non-trivial roots of Eq. (12), as the specific impedances of both walls vary according to Fig. 1(b).

Fig. 2 shows another representation of the results of Fig. 1(a) in terms of the variation with ε of the wave number parameter μ and the attenuation parameter κ.

Table 1 shows, for the first 10 non-trivial roots of Eq. (12), values of the wave number parameters μ and the attenuation parameters κ, which are obtained from Fig. 2 for $\varepsilon = 1$.

It was observed, however, that the final solution for $\varepsilon = 1$ depends on the step size adopted to increment the impedances. For the solutions presented above, the adopted step size was 10^{-4}. For step sizes greater than this value, 10^{-3} for instance, the procedure keeps finding roots already

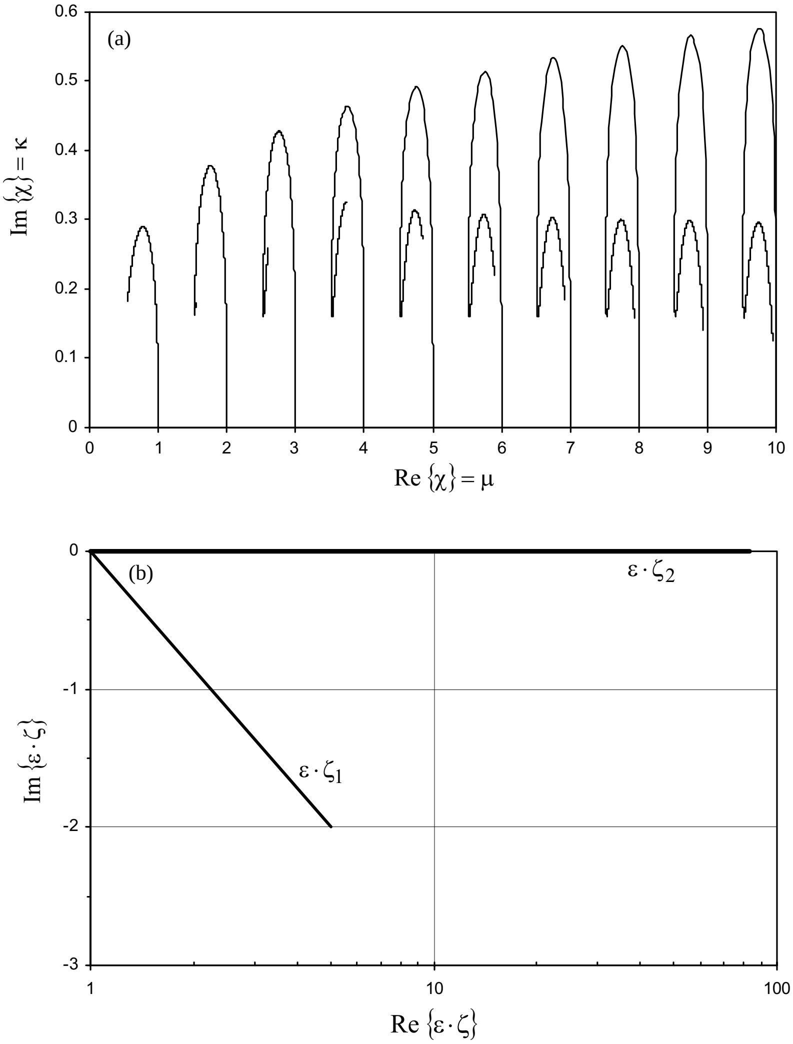

Fig. 1. (a) Loci of the first 10 non-trivial roots of Eq. (12) in the complex plane, obtained by the method of increment of impedances, as the specific acoustic impedances of two parallel room walls vary according to $\varepsilon\zeta$ (b) for the specific terminal impedances $\zeta_1 = 5 - \mathrm{i}2$, $\zeta_2 = 82.8$, and for $\eta = 20.7$.

found. This is particularly critical for the higher order roots, and as a rule, the step size should be reduced as the order of the root increases. For roots of orders corresponding to room natural frequencies of a thousand Hertz, the step size should be as low as 10^{-6}. The need of having to work with such small step sizes slows down considerably the numerical iterative scheme based on small increments of impedances.

Since, on the one hand, no solution should be attributed more than once to a root, on the other hand, no important solution should be missed by the numerical procedure. The latter difficulty was also experienced by this numerical procedure. As discussed earlier, there is the possibility of more than one root in some strips of unity width running parallel to the imaginary axis. When

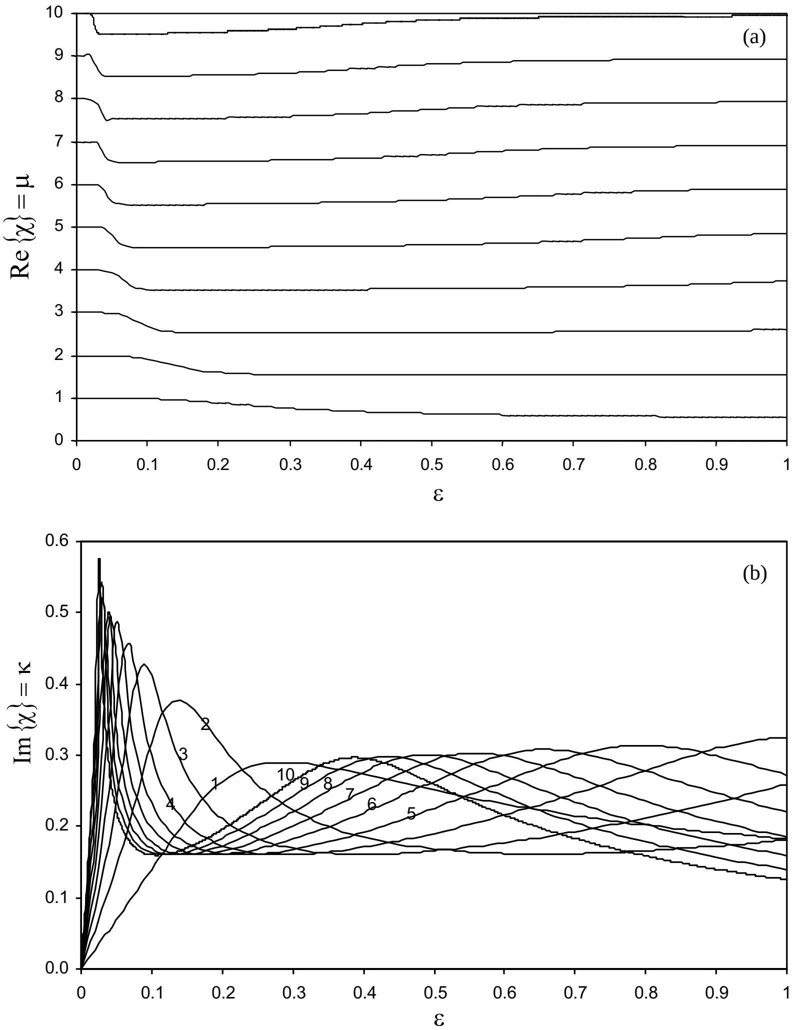

Fig. 2. (a) Real parts and (b) imaginary parts as functions of ε of the first 10 non-trivial roots of Eq. (12), obtained by the method of increment of impedances, for the specific terminal impedances of two parallel room walls equal to $\zeta_1 = 5 - i2$ and $\zeta_2 = 82.8$, and for $\eta = 20.7$.

that is the case, it was found that the method of increment of impedances usually finds only the root with smaller imaginary part. As shown in Table 1, in the fourth strip there is the root $3.7386 + i0.3247$. It is known that in this same strip there is also the root $3.5688 + i1.4271$, which is missed by this method.

3.2. Homotopic continuation procedure

This procedure poses the eigenvalue problem as one of homotopic continuation from a non-physical reference configuration in which all eigenvalues are known and obvious. The continuation is performed by the numerical integration of two differential equations. The

Table 1
Real and imaginary parts of the first 10 non-trivial roots of Eq. (12), obtained by the method of increment of impedances, for a pair of room walls having the specific acoustic impedances $\zeta_1 = 5 - i2$ and $\zeta_2 = 82.8$ and for $\eta = 20.7$

Root	$\mathrm{Re}\{\chi\} = \mu$	$\mathrm{Im}\{\chi\} = \kappa$
1	0.5537	0.1828
2	1.5455	0.1807
3	2.6004	0.2593
4	3.7386	0.3247
5	4.8525	0.2717
6	5.8969	0.2202
7	6.9195	0.1848
8	7.9334	0.1593
9	8.9429	0.1402
10	9.9498	0.1253

homotopic transformation preserves the multiplicity and the number of roots, the domains and their relation to each other.

The nth root χ_n can be viewed as a pure function of ζ_1/η and ζ_2/η, defined by the constraint of Eq. (12). It is then possible to define the differential equations

$$\frac{\partial \chi_n}{\partial(\zeta_1/\eta)} = \frac{\partial f/\partial(\zeta_1/\eta)}{\partial f/\partial \chi} \tag{13a}$$

and

$$\frac{\partial \chi_n}{\partial(\zeta_2/\eta)} = \frac{\partial f/\partial(\zeta_2/\eta)}{\partial f/\partial \chi}, \tag{13b}$$

where

$$f(\chi, \zeta_1/\eta, \zeta_2/\eta) = \mathrm{e}^{\mathrm{i}\pi\chi}\left(1 + \frac{\zeta_1}{\eta}\chi\right)\left(1 + \frac{\zeta_2}{\eta}\chi\right) - \mathrm{e}^{-\mathrm{i}\pi\chi}\left(1 - \frac{\zeta_1}{\eta}\chi\right)\left(1 - \frac{\zeta_2}{\eta}\chi\right), \tag{14}$$

$$\frac{\partial f}{\partial(\zeta_1/\eta)} = \frac{\partial f}{\partial(\zeta_2/\eta)} = -\mathrm{e}^{-\mathrm{i}\pi\chi}\left(1 - \frac{\zeta_1}{\eta}\chi\right)\left(1 - \frac{\zeta_2}{\eta}\chi\right) \tag{15}$$

and

$$\frac{\partial f}{\partial \chi} = -\mathrm{e}^{-\mathrm{i}\pi\chi}\left(1 - \frac{\zeta_1}{\eta}\chi\right)\left(1 - \frac{\zeta_2}{\eta}\chi\right) + \mathrm{i}\pi\mathrm{e}^{\mathrm{i}\pi\chi}\left(1 + \frac{\zeta_1}{\eta}\chi\right)\left(1 + \frac{\zeta_2}{\eta}\chi\right). \tag{16}$$

Since the solution set is the positive integers for $\zeta_1'/\eta = \zeta_2'/\eta = 0$, it is then possible to find the nth root by integrating Eqs. (13). Starting from the solution $\chi_n = n$, $\zeta_1'/\eta = \zeta_2'/\eta = 0$, integrate Eq. (13a) to ζ_1/η. From this new solution, integrate Eq. (13b) to ζ_2/η.

For a pair of room walls with the specific acoustic impedance $\zeta = 31 - i87$ and for $\eta = 20.7$, Table 2 shows the first 10 non-trivial roots of Eq. (12) obtained by the method of increment of impedances and by the homotopic continuation procedure. It can be seen that, in this case, the method of increment of impedances fails to find two roots in the first strip of unity width running parallel to the imaginary axis.

Table 2
Real and imaginary parts of the first 10 non-trivial roots of Eq. (12), obtained by two numerical procedures, for a pair of room walls having the specific acoustic impedances $\zeta = 31 - i87$ and for $\eta = 20.7$

Root	Numerical procedure			
	Increment of impedances		Homotopic continuation	
	$\mathrm{Re}\{\chi\} = \mu$	$\mathrm{Im}\{\chi\} = \kappa$	$\mathrm{Re}\{\chi\} = \mu$	$\mathrm{Im}\{\chi\} = \kappa$
1	—	—	0.0762	0.3924
2	—	—	0.8479	0.0645
3	1.9308	0.0254	1.9308	0.0254
4	2.9546	0.0163	2.9546	0.0163
5	3.9661	0.0121	3.9661	0.0121
6	4.9729	0.0096	4.9729	0.0096
7	5.9775	0.0080	5.9775	0.0080
8	6.9807	0.0068	6.9807	0.0068
9	7.9831	0.0060	7.9831	0.0060
10	8.9850	0.0053	8.9850	0.0053

4. Modal natural frequencies and damping constants

The solution of Eq. (12) for the three pairs of walls makes it possible to compute the eigenfunctions and eigenvalues of the boundary value problem. The eigenfunctions given by Eq. (1) satisfy the wave equation. Therefore, for each room mode N, a direct substitution of Eq. (1) into the wave equation results in

$$\nabla^2\psi_N(\omega) + (1/c^2)[\omega_N(\omega) + \mathrm{i}k_N(\omega)]^2\psi_N(\omega) = 0, \tag{17}$$

where N stands for the trio of numbers n_x, n_y, n_z. The corresponding eigenvalues $[\omega_N(\omega) + ik_N(\omega)]$ are obtained from

$$[\omega_N(\omega) + \mathrm{i}k_N(\omega)]^2 = (\pi c)^2[(\chi_{x,n_x}(\omega)/L_x)^2 + (\chi_{y,n_y}(\omega)/L_y)^2 + (\chi_{z,n_z}(\omega)/L_z)^2], \tag{18}$$

where $\chi_{x,n_x}(\omega)$ is the n_x root of Eq. (12) for the driving frequency ω and for the x-walls, and similarly for $\chi_{y,n_y}(\omega)$ and $\chi_{z,n_z}(\omega)$.

The eigenvalue $[\omega_N(\omega) + \mathrm{i}k_N(\omega)]$ is the *complex angular frequency*, whose real part $\omega_N(\omega)$ is the natural angular frequency of the damped standing wave $\psi_N(\omega)$ and whose imaginary part $k_N(\omega)$ is the temporal absorption coefficient, also called the damping constant. Since k_N is usually much smaller than ω_N, the following equations, in terms of the wave number parameters and attenuation parameters for the three pair of walls, are good approximations for the natural angular frequency and damping constant of the standing wave ψ_N:

$$\omega_N(\omega) \cong \pi c[(\mu_x^2 - \kappa_x^2)/L_x^2 + (\mu_y^2 - \kappa_y^2)/L_y^2 + (\mu_z^2 - \kappa_z^2)/L_z^2]^{1/2}, \tag{19}$$

$$k_N(\omega) \cong \pi c[(\mu_x\kappa_x/\eta_x L_x) + (\mu_y\kappa_y/\eta_y L_y) + (\mu_z\kappa_z/\eta_z L_z)]. \tag{20}$$

A room supports many of these standing waves. The number of standing waves (modes) in a given frequency band of width Δf increases as the center frequency of the band, or the size of the room, is increased.

4.1. Collective-modal-decay

An application of these results is the prediction of sound decays in rooms. Sound decays are usually measured in frequency bands. Each individual standing wave (mode) in the room, with natural angular frequency given by Eq. (19), decays with a damping constant given by Eq. (20). Therefore, in each frequency band there are a certain number of room modes that determine the sound decay process. The decay in each band of width Δf is characterized by the collective-modal-decay curve given by [6]

$$\mathit{Collective\ decay}(\Delta f, t) = 10 \log \left[\frac{\sum_{\Delta f} (1/k_N) \exp(-2k_N t)}{\sum_{\Delta f} 1/k_N} \right] \text{in dB}, \tag{21}$$

where the summation is taken over all the modes whose natural frequencies fall within the band Δf.

The methods developed here were applied to obtain the impedance of the hard walls of a rectangular room. The room dimensions are 9.20 m long $\times$ 4.67 m wide and 3.56 m high. The walls are hard, consisting of painted, non-porous masonry. The roots of Eq. (12) were generated for each pair of room walls for various specific acoustic impedances ζ, under the assumption that ζ is real, and having the same value for all room walls. The natural angular frequency and damping constant of each standing wave in the room could then be found with the aid of Eqs. (19) and (20). These were then used to obtain the collective-modal-decay curves in octave frequency bands with the aid of Eq. (21). The collective-modal-decay curves thus obtained were then compared with decay curves experimentally obtained in the rectangular room described above. The ζ values were chosen among those giving the best decay resemblance.

The sound decays experimentally obtained in octave frequency bands are compared with computational results in Fig. 3. The experimental decays are the averages of six microphones positions. The ζ values for each octave frequency band generated by the procedure described above are shown across the top of Fig. 3 with the corresponding decay curves. The ζ values thus obtained are typical of hard surfaces were the impedance is known to be real and large. It can be seen that the decay is linear for the higher frequency bands, whereas for the 500, 250, and 125 Hz frequency bands, the decay curves reveal a monotonic curvature. This behavior has been independently observed in reverberation chambers at low frequencies by various investigators [6]. Therefore, as far as the prediction of sound decays in hard-walled rectangular rooms, these results seem to support the adequacy of the normal mode solution.

4.2. Modal reverberation time

Another application of the methods presented here is in the prediction of modal decays in rooms. In room acoustics, it is more convenient to talk in terms of the reverberation time of an individual mode T_N, rather than in terms of its damping constant. The reverberation time of the mode T_N is equal to $3 \ln 10/k_N$. In general, there are as many damping constants (reverberation times), as there are modes. By plotting the number of modes having a reverberation time in a specified time interval gives information about the type of modes that should dominate the sound decay process.

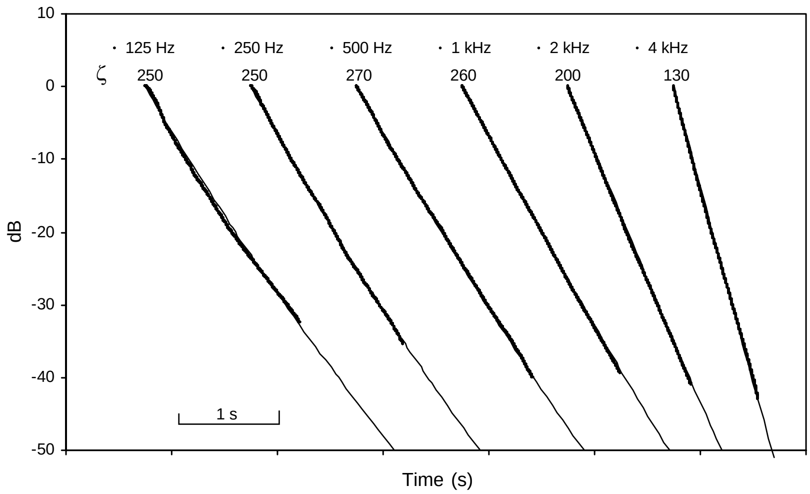

Fig. 3. Collective-modal-decay curves in octave frequency bands for a $9.20 \times 4.67 \times 3.56\,\mathrm{m}^3$ rectangular room with hard walls. Measured average decay curves of six microphone positions (······). Collective-modal-decay curves (——). The specific acoustic impedances ζ estimated for the room walls are shown across the top of the decay curves in octave frequency bands.

As usual, the modes can be separated into three groups. The modal numbering of Morse and Bolt [2] will be adopted; that is, the mode with $n = 0$ is given to the value of χ with the smallest value of μ. It should be mentioned that the smallest value of μ does not take into account the trivial root for which $\mu = \kappa = 0$. One group is composed of *axial* modes, for which two of the n's are zero. A second group is composed of *tangential* modes, for which only one n is zero. Finally, the third group is composed of *oblique* modes, for which no n is zero.

The oblique modes, which are the majority of modes in a given frequency band, have nearly the same reverberation time; however, the reverberation times of the tangential and axial modes are quite different, and the resulting collective-decay curve is not a straight line, even under very light absorption and at the lower frequency bands as shown in Fig. 3.

To find the modal distribution of reverberation times, the modes, according to their natural frequencies, were grouped into five octave frequency bands from 125 Hz to 2 kHz and, for each band, they were re-grouped into oblique, tangential and axial-type modes. The reverberation times of the modes in the hard-walled rectangular room are known from the damping constants used to generate the results of Fig. 3. In the same room, the modal distribution of reverberation times were also found for the case where the room floor is lined with sound absorbing material of known specific acoustic impedance. The specific acoustic impedances of the lining material in the five octave frequency bands from 125 Hz to 2 kHz are: 0.80–i12.61; 0.92–i6.44; 0.80–i3.38; 0.83–i1.45; 0.95–i0.53. The other five walls were assumed to have specific-acoustic-impedance values shown across the top of Fig. 3. The roots of Eq. (12) were then generated for each pair of room walls, from which the natural angular frequency and damping constant of each standing wave in the room could be calculated with the aid of Eqs. (19) and (20). From the knowledge of the damping constants, the reverberation time of each individual mode could be found.

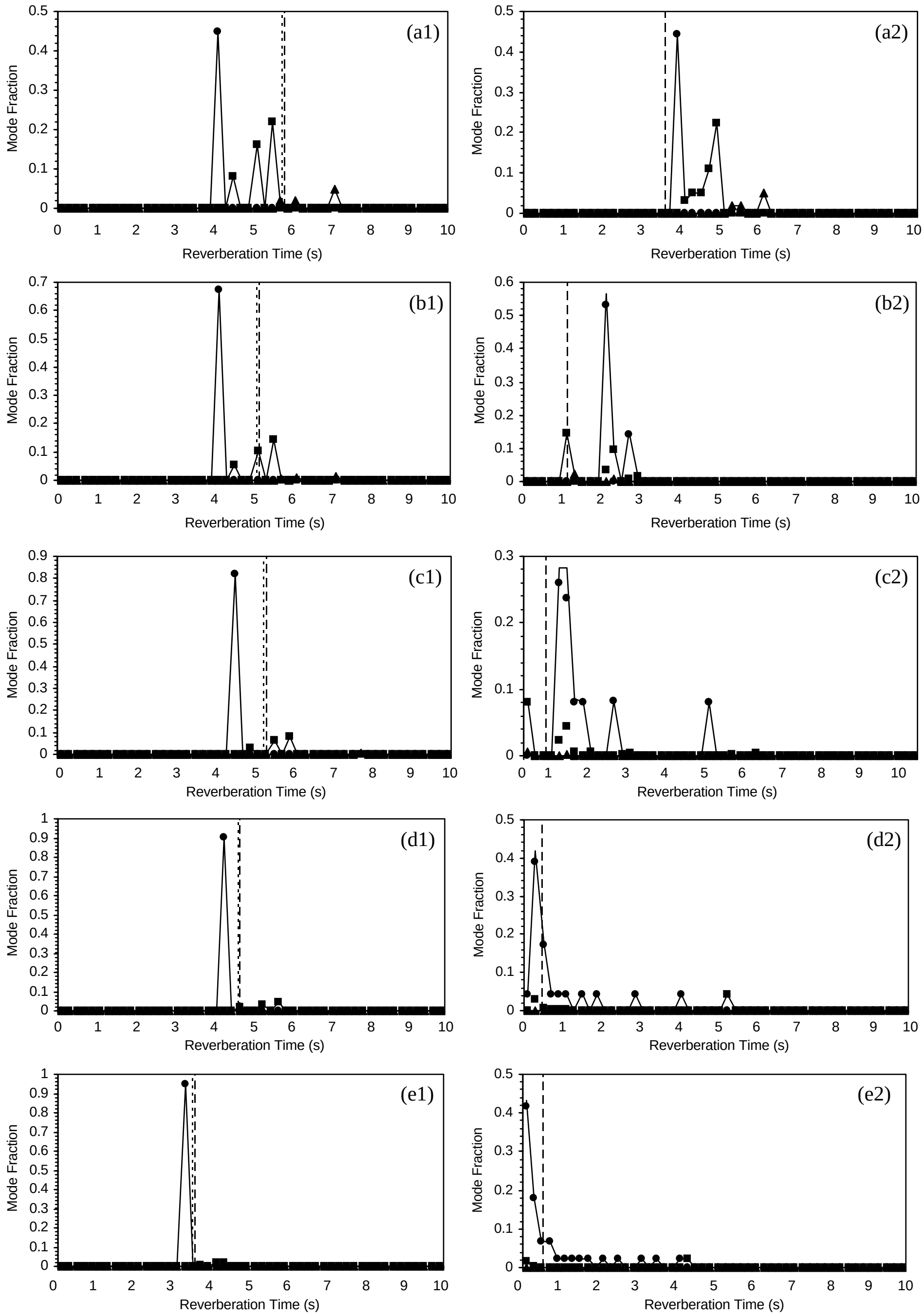

Fig. 4. Mode fraction distribution as a function of the reverberation time for modes that have natural frequencies lying in the octave frequency bands of (a1, a2) 125 Hz, (b1, b2) 250 Hz, (c1, c2) 500 Hz, (d1, d2) 1 kHz and (e1, e2) 2 kHz, for a rectangular room of dimensions $9.20\,\mathrm{m} \times 4.67\,\mathrm{m} \times 3.56\,\mathrm{m}^3$ with hard walls (subfigures on the left with index 1), and with the floor lined with sound absorbing material (subfigures on the right with index 2). Axial modes (▲); tangential modes (■); oblique modes (●); all modes (——). Reverberation time according to Sabine's prediction (– – – –). Measured reverberation time in the hard-walled room (- - - - -).

Easwaran and Craggs [7] used the finite element model to obtain the reverberation times of the modes in rooms with different shapes and sound absorbing characteristics. The same type of plots of Easwaran and Craggs are shown in Fig. 4. This figure shows the mode fraction distribution of reverberation times in octave frequency bands, which was obtained using the procedure described above. The values on the abscissa of these plots are the mid-interval values of the reverberation time interval. Also shown on these plots are the measured reverberation times in the hard-walled room, and the reverberation times according to Sabine's prediction for the hard-walled room and for the room with the floor lined with sound absorbing material. The continuous lines on these plots correspond to the sum of mode fractions that have reverberation times in the same interval. For the room with hard walls, this sum is always equal to the mode fraction of one particular type of mode. For the room with the floor lined with sound absorbing material, there may be more than one group of modes with reverberation times in the same interval.

Sabine's theory indicates that a single reverberation time would exist. From the results of the normal mode theory, Fig. 4 shows that this is only true for the room with hard walls and for the higher frequency bands (1 kHz and 2 kHz). This is consistent with the assumption for the validity of Sabine's formula; that is, a diffuse sound field, a situation in general encountered in rooms with low absorption, and at the higher frequencies. In the room with hard walls and at the lower frequency bands, and in the room with added absorption in all frequency bands, a single reverberation time cannot be said to exist. This is consistent with the results using finite element models [7]. Modes with rather distinct reverberation times can be excited. Even within the same group of modes, there may be different reverberation times. For instance, axial and tangential modes with three different reverberation times can be supported in the room with hard walls in the 125 Hz octave frequency band. In the same frequency band, in the room with the floor lined, the tangential modes group has five different reverberation times. In Fig. 4 the reverberation time corresponding to the dominant peaks coincides with the reverberation time of the oblique modes.

5. Summary and concluding remarks

In the present work, two numerical procedures were developed for finding the solutions of the acoustic eigenvalue equation in the rectangular with arbitrary (uniform) wall impedances. These numerical procedures were made possible by a transformation of the original eigenvalue equation of Morse and Bolt [2] into an entire function. One numerical procedure finds the eigenvalues using Newton's method and the other the homotopic continuation procedure. The latter procedure is faster and tracks multiple eigenvalues in strips of unit width running parallel to the imaginary axis.

Once the eigenvalues were found, the natural frequencies and damping constants of the room modes could be obtained. The specific acoustic impedances in octave frequency bands of a hard-walled rectangular room were estimated by comparing the collective-modal-decay curve with measured decays. The damping constants were used to obtain the reverberation times of the modes. It was found that a single reverberation time, for all modes, is only supported in the rectangular room with hard walls and at the higher frequency bands, consistent with Sabine's theory, which assumes a diffuse sound field. In the rectangular room with hard walls and at the lower frequency bands, and in the room with the floor lined with sound absorbing material and for all frequency bands, modes with rather distinctive reverberation times may produce sound decays not always consistent with Sabine's prediction.

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