Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Sound and Vibration 263 (2003) 219–226
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

The use of Mathematica in the dynamic analysis of a beam with a concentrated mass and dashpot

M.A. De Rosa[a,*], N.M. Auciello[a], M.J. Maurizi[b]

[a] *Department of Structural Engineering, University of Basilicata, Potenza 85100, Italy*
[b] *Departamento de Ingenieria, Bahía Blanca, Argentina*

Received 24 May 2002; accepted 8 October 2002

1. Introduction

Since the introduction of symbolic software the whole approach to important engineering fields has changed, and *Mathematica* [1] is universally acknowledged as one the most powerful programming languages, due both to its flexibility and to its integrated environment. For example, the solutions of various dynamic and stability problems have been offered [2,3]. In this paper *Mathematica* is employed in order to find the free frequencies and the vibration mode of an Euler–Bernoulli beam carrying a concentrated mass at an arbitrary position. The beam is supposed to be constrained at both the ends with elastically flexible constraints, and finally a concentrated dashpot is supposed to be placed at an arbitrary position along the span.

The free vibration frequencies of slender beams carrying a concentrated mass at an arbitrary position have already been found [4,5], whereas the presence of flexible ends has been introduced in Refs. [6,7]. All the classical cases can be regarded as particular or limiting cases which can be deduced from this general approach. Finally, quite recently a cantilever beam with a concentrated mass at its free end has been examined [8], in the presence of a concentrated dashpot at an arbitrary position, and the complex free vibration frequencies have been deduced. The aim of the paper is to give the frequency equation for the above-mentioned structural system, in a manner which fully exploits the symbolic properties of *Mathematica*. Various numerical examples end the paper, in which the role of the control parameters is enlightened.

2. Theory

Consider the slender Euler–Bernoulli beam in Fig. 1, with span L, uniform cross-section, Young's modulus E, cross-sectional area A and second moment of area I, constrained at the end by means of elastically flexible constraints defined by the axial stiffnesses k_{TL} and k_{TR} and by the

*Corresponding author. Tel.: +39-971-2051102; fax: +39-971-205070.

doi:10.1016/S0022-460X(02)01434-7

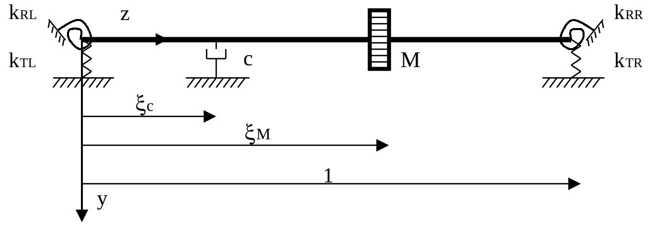

Fig. 1. Structural system.

rotational stiffnesses k_{RL} and k_{RR}. Finally, let $z_c = \xi_c L$ and $z_M = \xi_M L$ be the arbitrary abscissae where the concentrated dashpot and the concentrated mass are located, respectively.

In order to deduce the equation of motion, it is convenient to define the following system of three partial differential equations:

$$EIv_1^{IV}(z,t) + \rho A\ddot{v}_1(z,t) = 0, \quad 0<z<z_c, \tag{1}$$

$$EIv_2^{IV}(z,t) + \rho A\ddot{v}_2(z,t) = 0, \quad z_c<z<z_M, \tag{2}$$

$$EIv_3^{IV}(z,t) + \rho A\ddot{v}_3(z,t) = 0, \quad z_M<z<L, \tag{3}$$

where ρ is the mass density, z the generic cross-section abscissa, v_i the transverse displacement, t the time, IV denotes the spatial derivative with respect to the z variable, and $\dot{v}$ is the temporal derivative with respect to time t.

The solution can be put in the following form:

$$v_h(z,t) = V_h(z)e^{\lambda t}, \tag{4}$$

where λ is the (unknown) complex frequency of the system. This solution can be used in the partial differential equations, leading to a system of three ordinary differential equations in $V_h(z)$:

$$EIV_h^{IV}(z) + \lambda^2 \rho A V_h(z) = 0, \quad h = 1,2,3. \tag{5}$$

Now introduce the non-dimensional abscissa $\xi = z/L$ and the parameter:

$$\beta^4 = -\frac{\lambda^2 \rho A L^4}{EI}, \tag{6}$$

so that Eq. (5) becomes

$$V_h^{IV}(\xi) - \beta^4 V_h(\xi) = 0. \tag{7}$$

The general solutions of these differential equations can be written as

$$V_h(\xi) = C_{1h}e^{\beta\xi} + C_{2h}e^{-\beta\xi} + C_{3h}e^{i\beta\xi} + C_{4h}e^{-i\beta\xi}, \tag{8}$$

where i is the imaginary unit ($i = \sqrt{-1}$), and C_{jh} with $j = 1,\dots,4$ and $h = 1,2,3$ are 12 integration constants, which must be calculated by imposing the boundary and the continuity conditions. More particularly, two boundary conditions will be written at the non-dimensional abscissa $\xi = 0$, four continuity conditions must be imposed at the dashpot abscissa $\xi_c = z_c/L$, and at the mass abscissa $\xi_M = z_M/L$, and finally two other boundary conditions will define the right constraint, at the non-dimensional abscissa $\xi = 1$.

The stiffnesses of the elastic constraints can be conveniently defined by means of the following non-dimensional parameters:

$$R_L = \frac{k_{RL}L}{EI}, \quad T_L = \frac{k_{TL}L^3}{EI}, \tag{9}$$

$$R_R = \frac{k_{RR}L}{EI}, \quad T_R = \frac{k_{TR}L^3}{EI} \tag{10}$$

at the left end and at the right end, respectively.

The 12 boundary and continuity conditions can be written in *Mathematica* as given in Fig. 2, where λ has been deduced from Eq. (6):

$$\lambda = \mathrm{i}\frac{\beta^2}{L^2}\sqrt{\frac{EI}{\rho A}}. \tag{11}$$

All the equations must be equated to zero, and the resulting homogeneous system will have non-trivial solutions only if its determinant is equal to zero. By imposing the nullity of the determinant the frequency equation can be written without any inconvenience, but its complete expression, in terms of the control parameters $(R_L, T_L, R_R, T_R, c, M, \xi_c, \xi_M)$ is too long to be given here. However, a lot of particular cases can be recovered by limiting processes, as for example the frequency equations deduced by Gurgoze et al. [8]. In order to obtain their result, it is necessary to impose the following particular values of the control parameters: $R_L \to \infty$, $T_L \to \infty$, $R_R = 0$,

```
eqn1 = Expand[((T_L*v1 + D[v1, {ξ, 3}]) /. ξ -> 0)];
eqn2 = Expand[(-R_L*D[v1, ξ] + D[v1, {ξ, 2}]) /. ξ -> 0)];
eqn3 = Expand[((v1 - v2) /. ξ -> ξc)];
eqn4 = Expand[((D[v1, ξ] - D[v2, ξ]) /. ξ -> ξc)];
eqn5 = Expand[((D[v1, {ξ, 2}] - D[v2, {ξ, 2}]) /. ξ -> ξc)];
eqn6 = Expand[(D[v1, {ξ, 3}] - D[v2, {ξ, 3}] - c/EI λ v1) /. ξ -> ξc];
eqn7 = Expand[((v2 - v3) /. ξ -> ξM)];
eqn8 = Expand[((D[v2, ξ] - D[v3, ξ]) /. ξ -> ξM)];
eqn9 = Expand[((D[v2, {ξ, 2}] - D[v3, {ξ, 2}]) /. ξ -> ξM)];
eqn10 = Expand[(D[v2, {ξ, 3}] - D[v3, {ξ, 3}] - M/EI λ^2 v2) /. ξ -> ξM];
eqn11 = Expand[((-T_R*v3 + D[v3, {ξ, 3}]) /. ξ -> 1)];
eqn12 = Expand[((R_R*D[v3, ξ] + D[v3, {ξ, 2}]) /. ξ -> 1)];
```

Fig. 2. The boundary conditions as written by the use of *Mathematica*.

$T_R = 0$, and $\xi = 1$, or, in *Mathematica*:

$$Limit[Limit[equation, R_L \to \infty], T_L \to \infty / .\{R_R \to 0, T_R \to 0\}]. \tag{12}$$

In this way, the frequency equation of a cantilever beam with concentrated mass at the free end is obtained, in the presence of a concentrated dashpot at the variable abscissa ξ_c. (see the appendix)

3. Numerical examples

As a first example (Table 1), some numerical comparisons with known results will be performed. Consider a simply supported beam with a concentrated mass at the abscissa $\xi_M = 0.25$, and two rotationally flexible constraints defined by the non-dimensional flexibilities $R_L = R_R = 0.02$. The non-dimensional frequency β is given as a function of the non-dimensional mass ratio $m = M/\rho AL$, and, obviously, the results coincide with the results given in Refs. [6,7], where the same exact approach has been used.

In the following two numerical comparisons and in the following numerical examples the following data will be always assumed: Young's modulus $E = 7 \times 10^{10}$ N/m^2, second moment of area $I = 5.20833 \times 10^{-10}$ m^4, mass per unit length $\rho A = 0.657$ kg/m and concentrated mass $M = 2.025$ kg. The frequencies are given in terms of λ_i, as given in Eq. (11).

A numerical comparison with the results given in Ref. [8] is given in Table 2. The cantilever beam carries a concentrated mass at its right free end, and a concentrated dashpot is placed at the non-dimensional abscissa $\xi_c = 0.2$. The constraints and the frequency equation have already been given in Eq. (12) and in the appendix, and the discrepancies between the results seems to be due to numerical errors.

In Table 3 the exact frequencies are compared with the approximate values given by Wu and Chen [9, Table 1]. The beam is clamped at the left end and free at the right, with a concentrated dashpot at the non-dimensional abscissa $\xi_c = 0.2$. The first five non-dimensional frequencies are reported as functions of the damping coefficient c, as obtained by three different approaches. The agreement between the calculated frequencies is quite good, the FEM gives approximate real and imaginary parts which are always greater than the exact ones, whereas the Analytical and Numerical Method (ANCM) and the other approach give lower values.

The aim of the numerical examples in Tables 4 and 5 is to examine the influence of the right axial stiffness and of the right rotational stiffness for a beam with a clamped end on the left, a

Table 1
Simply supported beam with elastically rotationally flexible ends: first free natural frequency as a function of the non-dimensional mass ratio

m	$\xi_M = 0.25$
0	0.8314
0.2	0.8072
0.4	0.7919
0.6	0.7813
0.8	0.7736
1	0.7676

Table 2
Numerical comparisons with Ref. [8]

λ	This note	Ref. [8]
λ_1	$-0.003660439 \pm i7.076016$	$-0.003660439 \pm i7.076019$
λ_2	$-0.785780 \pm i115.536553$	$-0.785780 \pm i115.536606$
λ_3	$-4.297368 \pm i369.632627$	$-4.297368 \pm i369.632796$
λ_4	$-8.057146 \pm i768.481177$	$-8.057146 \pm i768.481528$
λ_5	$-7.385812 \pm i1312.397325$	$-7.385813 \pm i1312.397925$

Table 3
Numerical comparisons with Ref. [9]

c	Methods	λ_1	λ_2	λ_3	λ_4	λ_5
5.0	FEM	-0.060437	-1.34301	-5.42172	-8.45298	-6.51107
		$\pm i25.8405$	$\pm i161.956$	$\pm i453.570$	$\pm i889.27$	$\pm i1472.36$
	ANCM	-0.060437	-1.34284	-5.41613	-8.41973	-6.44152
		$\pm i25.8405$	$\pm i161.951$	$\pm i453.447$	$\pm i888.41$	$\pm i1468.63$
	[10]	-0.060437	-1.34284	-5.41614	-8.41976	-6.44152
		$\pm i25.8405$	$\pm i161.951$	$\pm i453.448$	$\pm i888.41$	$\pm i1468.63$
	Exact	-0.060437	-1.34283	-5.416174	-8.42056	-6.44336
		$\pm i25.8405$	$\pm i161.951$	$\pm i453.455$	$\pm i888.43$	$\pm i1468.66$
6.0	FEM	-0.0725248	-1.61167	-6.50734	-10.1450	-7.81185
		$\pm i25.8406$	$\pm i161.963$	$\pm i453.579$	$\pm i889.22$	$\pm i1472.29$
	ANCM	-0.0725244	-1.6145	-6.50062	-10.1047	-7.72750
		$\pm i25.8406$	$\pm i161.957$	$\pm i453.454$	$\pm i888.35$	$\pm i1468.55$
	[10]	-0.0725243	-1.61146	-6.50064	-10.1047	-7.72747
		$\pm i25.8406$	$\pm i161.956$	$\pm i453.456$	$\pm i888.35$	$\pm i1468.55$
	Exact	-0.0725243	-1.61145	-6.50069	-10.1060	-7.73064
		$\pm i25.8406$	$\pm i161.957$	$\pm i453.464$	$\pm i888.38$	$\pm i1468.59$
8.0	FEM	-0.0966995	-2.1490	-8.68081	-13.5313	-10.4109
		$\pm i25.8408$	$\pm i161.977$	$\pm i453.604$	$\pm i889.09$	$\pm i1472.10$
	ANCM	-0.0966992	-2.1488	-8.67181	-13.4764	-10.2953
		$\pm i25.8409$	$\pm i161.971$	$\pm i453.473$	$\pm i888.20$	$\pm i1468.33$
	[10]	-0.0966991	-2.1488	-8.67184	-13.4765	-10.2953
		$\pm i25.8409$	$\pm i161.971$	$\pm i453.474$	$\pm i888.20$	$\pm i1468.33$
	Exact	-0.0966990	-2.1488	-8.67196	-13.4796	-10.3028
		$\pm i25.8408$	$\pm i161.972$	$\pm i453.489$	$\pm i888.24$	$\pm i1468.40$
10.0	FEM	-0.12874	-2.68662	-10.8581	-16.9223	-13.0058
		$\pm i25.8412$	$\pm i161.996$	$\pm i453.637$	$\pm i888.92$	$\pm i1471.85$
	ANCM	-0.12874	-2.68631	-10.8469	-16.8512	-12.8562
		$\pm i25.8412$	$\pm i161.989$	$\pm i453.497$	$\pm i888.02$	$\pm i1468.05$
	[10]	-0.12874	-2.68631	-10.8468	-16.8512	-12.8562
		$\pm i25.8412$	$\pm i161.989$	$\pm i453.498$	$\pm i888.00$	$\pm i1468.05$
	Exact	-0.12874	-2.68627	-10.8470	-16.8574	-12.8709
		$\pm i25.8412$	$\pm i161.991$	$\pm i453.521$	$\pm i888.08$	$\pm i1468.16$

Table 4
First and second free complex vibration frequencies as a function of the axial stiffness T_R

T_R	λ_1	λ_2
0	$-0.0079272 \pm i9.545853$	$-1.40587 \pm i161.273281$
0.005	$-0.0079272 \pm i9.553565$	$-1.40590 \pm i161.275960$
0.05	$-0.0079272 \pm i9.622558$	$-1.40621 \pm i161.300145$
0.5	$-0.0081274 \pm i10.283837$	$-1.40925 \pm i161.541319$
5	$-0.0099529 \pm i15.194649$	$-1.43933 \pm i163.894145$
50	$-0.0272368 \pm i32.366245$	$-1.70316 \pm i182.437805$
500	$-0.0750320 \pm i48.228965$	$-2.52378 \pm i226.317733$
5000	$-0.0916338 \pm i51.274219$	$-2.80688 \pm i240.190116$
50 000	$-0.0936571 \pm i51.893732$	$-2.83862 \pm i241.798300$
500 000	$-0.0938637 \pm i51.930767$	$-2.84182 \pm i241.961388$
5 000 000	$-0.0938844 \pm i51.934474$	$-2.84214 \pm i241.977720$
50 000 000	$-0.0938865 \pm i51.934845$	$-2.84218 \pm i241.979353$
500 000 000	$-0.0938867 \pm i51.934882$	$-2.84218 \pm i241.979517$
5 000 000 000	$-0.0938867 \pm i51.934885$	$-2.84218 \pm i241.979533$

Table 5
First and second free complex vibration frequencies as a function of the rotational stiffness R_R

R_R	λ_1	λ_2
0	$-0.0079272 \pm i9.545853$	$-1.40587 \pm i161.273281$
0.005	$-0.0079404 \pm i9.560179$	$-1.40689 \pm i161.342310$
0.05	$-0.0080577 \pm i9.685675$	$-1.41601 \pm i161.954632$
0.5	$-0.0090711 \pm i10.676558$	$-1.49874 \pm i167.301886$
5	$-0.0129776 \pm i13.514227$	$-1.88871 \pm i188.889035$
50	$-0.0155246 \pm i14.893546$	$-2.19661 \pm i203.536652$
500	$-0.0159400 \pm i15.088381$	$-2.24687 \pm i205.831743$
5000	$-0.0159662 \pm i15.108679$	$-2.25219 \pm i206.073792$
50 000	$-0.0159704 \pm i15.110717$	$-2.25273 \pm i206.098128$
500 000	$-0.0159709 \pm i15.110921$	$-2.25278 \pm i206.100563$
5 000 000	$-0.0159709 \pm i15.110941$	$-2.25279 \pm i206.100807$
50 000 000	$-0.0159709 \pm i15.110943$	$-2.25279 \pm i206.100831$
500 000 000	$-0.0159709 \pm i15.110943$	$-2.25279 \pm i206.100831$

dashpot at $\xi_c = 0.2$ with damping coefficient $c = 5N(\mathrm{m/s})$, and a concentrated mass at $\xi_M = 0.8$. In Table 4 the limiting values $T_R \to 0$ and $T_R \to \infty$ correspond to cantilever beams and propped cantilever beams, respectively. In Table 5 the limiting values $R_R \to 0$ and $R_R \to \infty$ correspond to cantilever beams and to clamped-guided beams, respectively.

In Table 6 a simple supported beam is examined, with a concentrated mass at $\xi_M = 0.8$ and a concentrated dashpot at $\xi_c = 0.2$. The frequencies are given for different values of the damping constant c, and, obviously, the purely imaginary frequency corresponds to the undamped case.

Finally, in Table 7 a clamped–clamped beam with concentrated mass at the midspan is examined, in the presence of a dashpot placed at various non-dimensional abscissa ξ_c from 0.1 to 0.9. From the table it is possible to observe that for $\xi_c = 0.5$ the second eigenvalue becomes purely imaginary, because, due to symmetry, its corresponding vibration mode has a node at the midspan.

Table 6
First free complex vibration frequency as a function of damping coefficient

c	λ_1
0	$\pm$i39.490818
1	$-0.1120064\pm$i39.4910775
2	$-0.2240184\pm$i39.4918572
3	$-0.3360414\pm$i39.4931570
4	$-0.4480810\pm$i39.4949774
5	$-0.5601427\pm$i39.4973193
6	$-0.6722321\pm$i39.5001836
7	$-0.7843547\pm$i39.5035715
8	$-0.8965159\pm$i39.5074844
9	$-1.0087214\pm$i39.5119239
10	$-1.1209766\pm$i39.5168920

Table 7
First and second free complex vibration frequencies as a function of the dashpot abscissa

ξ_c	λ_1	λ_2
0.1	$-0.0122425\pm$i55.4505568	$-0.7692006\pm$i453.260046
0.2	$-0.1391719\pm$i55.4511206	$-5.3950390\pm$i453.396131
0.3	$-0.4674316\pm$i55.4517021	$-8.4043499\pm$i453.401784
0.4	$-0.8856946\pm$i55.4462437	$-3.9649440\pm$i453.258505
0.5	$-1.0977028\pm$i55.4396917	$-0\pm$i453.252437
0.6	$-0.8304810\pm$i55.4485871	$-3.6940527\pm$i453.351186
0.7	$-1.1980535\pm$i55.4535043	$-7.4744209\pm$i456.128251
0.8	$-0.1871189\pm$i55.4515064	$-0.6650665\pm$i455.042417
0.9	$-3.1830668\pm$i55.0446628	$-0.0037026\pm$i453.305692

4. Conclusions

In this paper the emphasis is placed on the use of *Mathematica* programming language, and its powerful approach to symbolic analysis of free vibration frequencies for a beam on flexible constraints, a concentrated dashpot at an arbitrary abscissa, and concentrated mass at another concentrated abscissa. The exact results have been compared with some particular cases already studied in the references, and some other numerical examples have been illustrated. The frequency equation is reproduced in the appendix for a particular case, as obtained from the general result by means of limiting processes.

Appendix

Frequency equation for a cantilever beam carrying a concentrated mass at its right free end, and a dashpot at the variable abscissa ξ_c.

$$\frac{1}{\mathrm{EI}\rho\mathrm{A}}\big(128\mathrm{e}^{(-1-\mathrm{i})(2\xi_c\beta+\beta)}\beta^{15}$$
$$\times\,\big(4\mathrm{e}^{(2+2\mathrm{i})\beta\xi_c}\mathrm{EI}((1+\mathrm{i})(\mathrm{i}+\mathrm{e}^{2\mathrm{i}\beta}-\mathrm{e}^{2\beta}-\mathrm{i}\mathrm{e}^{(2+2\mathrm{i})\beta})M\beta$$

$$
\begin{aligned}
&-\ \mathrm{i}(1+\mathrm{e}^{2\mathrm{i}\beta}+4\mathrm{e}^{(1+\mathrm{i})\beta}+\mathrm{e}^{2\beta}+\mathrm{e}^{(2+2\mathrm{i})\beta})\rho A)\beta^3\\
&+\ c\sqrt{\frac{\mathrm{EI}\beta^4}{\rho A}}((\mathrm{e}^{2\beta(\mathrm{i}\xi_c+1)}-\mathrm{i}\mathrm{e}^{2\beta((1+\mathrm{i})+\mathrm{i}\xi_c)}-(1+\mathrm{i})\mathrm{e}^{(2+2\mathrm{i})\beta\xi_c}-\mathrm{e}^{(2+4\mathrm{i})\beta\xi_c}\\
&+\ (2+2\mathrm{i})\mathrm{e}^{(3+3\mathrm{i})\beta\xi_c}-\mathrm{i}\mathrm{e}^{(4+2\mathrm{i})\beta\xi_c}+\mathrm{i}\mathrm{e}^{2\beta(\mathrm{i}+\xi_c)}-(1+\mathrm{i})\mathrm{e}^{(2+2\mathrm{i})\beta(\xi_c+1)}\\
&-\ \mathrm{e}^{2\beta((1+\mathrm{i})+\xi_c)}+(2+2\mathrm{i})\mathrm{e}^{(1+\mathrm{i})\beta(\xi_c+2)}+(1+\mathrm{i})\mathrm{e}^{2\beta(\mathrm{i}+(1+\mathrm{i})\xi_c)}\\
&+\ \mathrm{e}^{2\beta(\mathrm{i}+(2+\mathrm{i})\xi_c)}+(1+\mathrm{i})\mathrm{e}^{2((1+\mathrm{i})\xi_c\beta+\beta)}+\mathrm{i}\mathrm{e}^{2((1+2\mathrm{i})\xi_c\beta+\beta)}\\
&-\ (2+2\mathrm{i})\mathrm{e}^{(1+3\mathrm{i})\xi_c\beta+2\beta}-(2+2\mathrm{i})\mathrm{e}^{2\mathrm{i}\beta+(3+\mathrm{i})\xi_c\beta})(1+\mathrm{i})M\beta\\
&+\ (\mathrm{e}^{2\mathrm{i}\beta+(2+2\mathrm{i})\xi_c\beta}(1-\mathrm{i})+\mathrm{e}^{(2+4\mathrm{i})\beta\xi_c}+\mathrm{e}^{2\beta(\mathrm{i}+\xi_c)}-\mathrm{e}^{2\beta((1+\mathrm{i})+\xi_c)}-\mathrm{e}^{2((1+2\mathrm{i})\xi_c\beta+\beta)}\\
&-\ (2+2\mathrm{i})\mathrm{e}^{(3+3\mathrm{i})\beta\xi_c}-(1+\mathrm{i})\mathrm{e}^{(2+2\mathrm{i})\beta(\xi_c+1)}-(2+2\mathrm{i})\mathrm{e}^{(1+\mathrm{i})(3\xi_c\beta+\beta)}\\
&-\ (1-\mathrm{i})\mathrm{e}^{2((1+\mathrm{i})\xi_c\beta+\beta)}-(2-2\mathrm{i})\mathrm{e}^{2\mathrm{i}\beta+(3+\mathrm{i})\xi_c\beta}-(2-2\mathrm{i})\mathrm{e}^{(1+\mathrm{i})\beta+(3+\mathrm{i})\xi_c\beta}\\
&-\ \mathrm{i}\mathrm{e}^{2\mathrm{i}\beta+(4+2\mathrm{i})\xi_c\beta}+\mathrm{e}^{2\beta((1+\mathrm{i})+\mathrm{i}\xi_c)}(-\mathrm{i})+\mathrm{e}^{(4+2\mathrm{i})\beta\xi_c}\mathrm{i}+\mathrm{e}^{2\mathrm{i}\xi_c\beta+2\beta}\mathrm{i}\\
&+\ \mathrm{e}^{(2+2\mathrm{i})\beta\xi_c}(1+\mathrm{i})+\mathrm{e}^{(1+\mathrm{i})\beta(\xi_c+1)}(2+2\mathrm{i})+\mathrm{e}^{(1+\mathrm{i})\beta(\xi_c+2)}(2+2\mathrm{i})\\
&+\mathrm{e}^{(1+\mathrm{i})((2+\mathrm{i})\xi_c\beta+\beta)}(2-2\mathrm{i})+\mathrm{e}^{(1+3\mathrm{i})\xi_c\beta+2\beta}(2-2\mathrm{i}))\rho A)\big)=0.
\end{aligned}
$$

References

[1] S. Wolfram, The Mathematica Book, 3rd Edition, Wolfram Media/Cambridge University Press, Cambridge, 1996.

[2] M.A. De Rosa, C. Franciosi, M.J. Maurizi, On the dynamic behaviour of slender beams with elastic ends carrying a concentrated mass, Computer and Structures 58 (6) (1996) 1145–1159.

[3] M.A. De Rosa, C. Franciosi, Higher order Timoshenko quotient in the stability and dynamic analysis of smoothly tapered beams, Journal of Sound and Vibration 196 (1996) 253–262.

[4] P.A.A. Laura, J.L. Pombo, E.L. Susemihl, A note on the vibrations of a clamped-free beam with a mass at the free end, Journal of Sound and Vibration 37 (1974) 161–168.

[5] P.A.A. Laura, M.J. Maurizi, J.L. Pombo, A note on the dynamic analysis of an elastically restrained-free beam with a mass at the free end, Journal of Sound and Vibration 41 (1975) 397–405.

[6] M.A. De Rosa, S. Ascoli, S. Nicastro, Exact dynamic analysis of beam-mass systems, Journal of Sound and Vibration 196 (1996) 529–533.

[7] M.A. De Rosa, S. Ascoli, S. Nicastro, Comprehensive exact analysis of beam-mass systems, Internal Report DiSGG 196 (1995) 529–533.

[8] M. Gürgöze, V. Mermertas, On the eigenvalues of a viscously damped cantilever carrying a tip mass, Journal of Sound and Vibration 216 (1998) 309–314.

[9] J.S. Wu, D.W. Chen, Dynamic analysis of a uniform cantilever beam carrying a number of elastically mounted point masses with dampers, Journal of Sound and Vibration 229 (2000) 549–578.

[10] M. Gürgöze, On the eigenvalues of viscously damped carrying heavy masses and restrained by linear and torsional springs, Journal of Sound and Vibration 208 (1997) 153–158.