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Letter to the Editor

Moderately large amplitude vibrations of a constant tension string: a numerical experiment

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1. Introduction

The differential equation governing the large amplitude vibrations of a constant tension string [1,2] is

$$\frac{T}{m}w'' = [1 + (w')^2]^2\,\ddot{w}, \tag{1}$$

where T is the tension in the string, m is the mass density per unit length, w is the lateral displacement of the string, $(\,)'$ denotes differentiation with respect to the axial co-ordinate x and $(\cdot)$ denotes differentiation with respect to the time.

The physical significance of such a constant tension string is the case of a string of finite vibrating length between end points which are at a fixed distance apart as shown in Fig. 1.

Eq. (1) is solved in a comprehensive way, with rigorous mathematical treatment, by Gottlieb [3]. An interesting discussion on this work can be seen in the study of Pillai and Nageswara Rao [4].

The purpose of the present note is to present a simple numerical experiment, based on intuition, to obtain the moderately large amplitude vibration behaviour of a constant tension string.

2. Numerical experiment

The lateral displacement distribution w of a constant tension string vibrating between two fixed points, separated by a constant distance L, can be taken as

$$w = a\sin\frac{\pi x}{L}, \tag{2}$$

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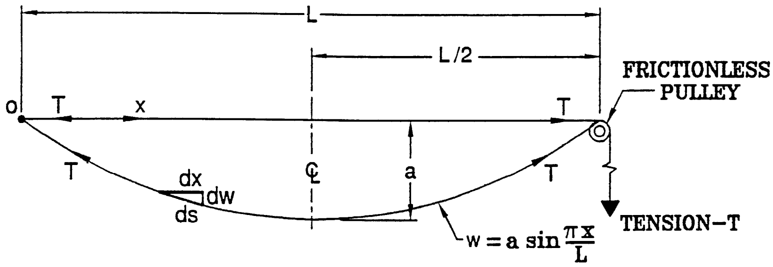

Fig. 1. A constant tension vibrating string with large amplitudes.

where a is the central amplitude. The displacement distribution chosen for w satisfies the boundary conditions

$$w(o) = w(L) = 0. \tag{3}$$

The non-linear, non-dimensional period λ_p is defined as [4]

$$\lambda_p = \frac{2\pi\omega_L}{\omega_{NL}}, \tag{4}$$

where ω_{NL} is the first mode non-linear radian frequency for a given amplitude parameter, defined later, and ω_L is the linear radian frequency, given by [5] (neglecting the non-linear terms in Eq. (1))

$$\omega_L = \frac{\pi}{L}\sqrt{\frac{T}{m}} \tag{5}$$

for the first mode.

At this stage, ω_{NL} is assumed as

$$\omega_{NL} = \frac{\pi}{L'}\sqrt{\frac{T}{m}}, \tag{6}$$

where L' is the length of the deflected string between the two points and ω_{NL} is reinterpreted as the first linear radian frequency of a string with length L'. Then Eq. (4) can be written as

$$\lambda_p = 2\pi\frac{L'}{L}. \tag{7}$$

Knowing L', the non-linear, non-dimensional period can be easily computed.

3. Evaluation of L'

Referring to Fig. 1, the infintesimal curved length ds of the string is expressed in terms of the infinitesimal values dw and dx as

$$ds^2 = dw^2 + dx^2 \tag{8}$$

or

$$ds = \left[1 + \left(\frac{dw}{dx}\right)^2\right]^{1/2} dx. \tag{9}$$

Integrating Eq. (9) between the limits 0 to L, we get

$$\int_0^L ds = L' = \int_0^L \left[1 + \left(\frac{dw}{dx}\right)^2\right]^{1/2} dx. \tag{10}$$

Substituting Eq. (2) into Eq. (10), L' can be written as

$$L' = \int_0^L \left[1 + \alpha^2 \cos^2 \frac{\pi x}{L}\right]^{1/2} dx, \tag{11}$$

where α is the non-dimensional lateral central amplitude defined as

$$\alpha = \frac{a\pi}{L}. \tag{12}$$

Non-dimensionalizing the axial co-ordinate x as

$$X = \frac{x}{L}, \tag{13}$$

Eq. (11) becomes

$$\frac{L'}{L} = \int_0^1 \left[1 + \alpha^2 \cos^2 \pi X\right] dX. \tag{14}$$

The right-hand side of Eq. (14) can be numerically integrated using any standard integration rule and L'/L can be obtained to any desired degree of accuracy. From Eq. (7), knowing the value of L'/L for any α, the non-linear, non-dimensional period λ_p can be evaluated.

4. Error estimate of the intuitive method

It is necessary to obtain a closed form expression for the ratio of non-linear to linear time periods T_{NL}/T_L obtained from the solution of the non-linear differential equation and the intuitive method, as a function of the amplitude parameter $a(\pi/L)$. This complex task is accomplished here with the assumption that w' is smaller compared to unity.

4.1. Differential equation

The governing non-linear differential equation with the assumption mentioned above can be written in a simplified form as

$$\ddot{w} = \frac{T}{m}[1 - 2w'^2]\, w''. \tag{15}$$

Assuming variable separable solution

$$w = w_t w(x), \tag{16}$$

where w_t is a function of time only and the functional form of $w(x)$ is given in Eq. (2).

Substituting Eq. (16) into Eq. (15), eliminating the space variable by using the standard Galerkin method and after simplification, we get

$$\ddot{w}_t + \alpha_1 w_t - \alpha_2 w_t^3 = 0, \tag{17}$$

where the subscript I denotes the solution obtained from the intuitive method:

$$\alpha_1 = \frac{T}{m}\left(\frac{\pi}{L}\right)^2 \tag{18}$$

and

$$\alpha_2 = \frac{T}{2m}\left(\frac{\pi}{L}\right)^4. \tag{19}$$

Eq. (17) is the famous Duffing's with softening non-linearity, the solution of which can be obtained [6] as

$$\left(\frac{T_{NL}}{T_L}\right)_{\dot{D}} = \left[1 + \tfrac{3}{8}\alpha^2\right], \tag{20}$$

where the subscript D denotes the solution obtained from the differential equation.

5. Intuitive method

From Eq. (14), the expression for L'/L, using the aforementioned assumption, can be written as

$$\frac{L'}{L} = \int_0^1 \left[1 + \frac{\alpha^2}{2}\cos^2 \pi X\right] \mathrm{d}X. \tag{21}$$

From Eqs. (17) and (21), the ratio of the non-linear to linear periods of the system for the intuitive method can be written as

$$\left(\frac{T_{NL}}{T_L}\right)_I = \left[1 + \tfrac{1}{4}\alpha^2\right], \tag{22}$$

where the subscript I denotes the solution obtained from the intuitive method.

Now the error ε involved in the intuitive method is

$$\varepsilon = \frac{(T_{NL}/T_L)_D - (T_{NL}/T_L)_I}{(T_{NL}/T_L)_D}, \tag{23}$$

which is a function of the amplitude parameter. It is to be noted here that this error estimate is valid for smaller w' compared to unity, which is the assumption made.

6. Numerical results

The non-linear, non-dimensional period λ_p is evaluated, by numerically integrating Eq. (13) and using Eq. (7), for a given non-dimensional central amplitude α. The values of λ_p obtained from the present numerical experiment are given in Table 1, along with those obtained by Gottlieb [3] and Pillai and Nageswara Rao [4] upto $\alpha = 1.0$, which is a moderately large central amplitude (the central amplitude of vibration is approximately one-third of the distance of the support points). It can be seen that the present results are in good agreement with those of Refs. [3,4]; the error being around 1.3% for $\alpha = 1.0$. It may be stated here that beyond $\alpha = 1.0$, the present results are not matching with those of Refs. [3,4], thus indicating that the numerical experiment, based on intuition, is not valid beyond $\alpha = 1.0$. However, the non-dimensional central amplitude $\alpha = 1.0$ is large enough for all practical purposes.

To show the validity of the intuitive method, theoretical estimates of the non-linear to linear period ratios T_{NL}/T_L are obtained by solving for those, both from the non-linear differential equation and from the present intuitive method. To achieve this complex task, an assumption that w' is smaller than unity is made. The values of λ_p obtained from both the solutions of the non-linear differential equation and the present intuitive method along with the percentage error ε defined in Eq. (23) are given in Table 2. It can be seen that the percentage error is within tolerable

Table 1
Non-linear, non-dimensional period λ_p of a constant tension string

α	λ_p		
	Present solution	Gottlieb [3]	Pillai and Nageswara Rao [4]
0.0	6.2832 (2π)	—	—
0.1	6.2989	6.2950	6.2950
0.2	6.3455	6.3305	6.3305
0.3	6.4223	—	—
0.4	6.5274	—	—
0.5	6.6592	6.5838	6.5838
0.6	6.8153	—	—
0.7	6.9936	—	—
0.8	7.1919	—	—
0.9	7.4082	—	—
1.0	7.6404	7.5415	7.5415

Table 2
Numerical results from error estimates

α	λ_{PD}	λ_{PI}	% Error ε (Eq. (23))
0.0	6.2832 (2π)	6.2832 (2π)	0.0
0.1	6.3067	6.2989	0.1245
0.2	6.3774	6.3460	0.4926
0.3	6.4952	6.4246	1.0883
0.4	6.6602	6.5345	1.8868
0.5	6.8722	6.6759	2.8571

limits upto the value of the amplitude parameter $\alpha = 0.5$. Further, it may be noted that the assumption on w' gives more error in the solution of the non-linear differential equation than the present intuitive method for α greater than 0.5. However, as mentioned earlier, the present intuitive method gives reliable values for λ_p upto $\alpha = 1.0$.

7. Concluding remarks

A numerical experiment, based on intuition, is presented in this note to obtain the non-linear periods of a string, with constant tension, undergoing moderately large amplitude vibrations. The numerical experiment gives results which are valid upto a central amplitude equals to approximately one-third of the distance of the support points, and starts failing when the amplitude exceeds this limit. Theoretical estimates of error presented show the usefulness of the intuitive method. However, further theoretical and numerical investigations are necessary to know why the numerical experiment fails beyond a certain amplitude, even though this amplitude is large enough for all practical purposes.

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