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Letter to the Editor

Rotating solutions of the parametrically excited pendulum

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1. Introduction

In this letter rotating periodic solutions or orbits of the parametrically excited pendulum are considered. These rotating solutions, sometimes referred to as running solutions by some authors [1,2] are solutions which make revolutions about the pivot point. This work builds on the earlier work of Ref. [3] which considered rotating orbits of the parametrically excited pendulum in the first resonance zone. In this letter the steady state stable rotating solutions of the parametrically excited pendulum are classified into four broad categories. The zones in the parameter space in which the different types of rotating solutions occur are indicated. With appropriate scalings [4] the equation governing the motion of the parametrically excited pendulum in terms of the angle θ which the pendulum makes with the vertical can be shown to be

$$\ddot{\theta} + \beta\dot{\theta} + (1 + p\cos(\omega t))\sin(\theta) = 0. \tag{1}$$

The damping coefficient is denoted by β while ω and p are the frequency of parametric forcing and amplitude of parametric forcing, respectively.

Several attractors exist for this simple system depending on parameter settings and initial conditions. These include: the equilibrium state $(\theta, \dot{\theta}) = (0, 0)$ (in which the pendulum simply goes up and down with the point of suspension), oscillating solutions, chaotic solutions and rotating solutions which are the subject of this letter. The equilibrium state $(\theta, \dot{\theta}) = (0, 0)$, is generally stable except for ω, p within the tongue shaped regions in Fig. 1. These tongue-shaped regions (herein referred to as resonance zones) occur around the values $(\omega, p) = (2/n, 0)$, $n = 1, 2, 3, \ldots$ in the parameter space. With increasing p each resonance zone is twisted to the right and lies above the other so that the resonance zones may possibly merge for arbitrarily large values of p. The dotted lines in Fig. 1 show the position of the resonance zones in the (ω, p) space when $\beta = 0$. Therefore, the damping acts to shift the position of the resonance zones upwards. For simplicity the damping is fixed at a representative level of $\beta = 0.1$. The response prior to rotation corresponds to the

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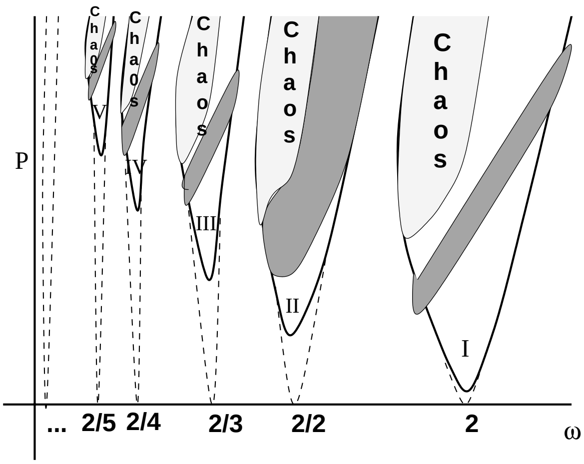

Fig. 1. Schematic stability diagram showing the resonance zones marked I, II, III, IV, V, etc. Inside these resonance zones the equilibrium state $(\theta, \dot{\theta}) = (0, 0)$ is unstable. If the response for a variation of p is investigated, typically stable solutions within these zones are initially oscillatory. But if p is increased further stable rotating orbits and chaotic solutions can also occur depending upon the initial conditions. The dashed lines indicate the position of the resonance zones in the absence of damping.

well-known phenomenon of escape from the potential well [5] which occurs in a wide variety of non-linear oscillators when a system's response crosses a maximum value of the potential energy.

Inside the resonance zones different types of oscillating, rotating and chaotic solutions are realized.

Rotating solutions occur within narrow strips shown in black inside the resonance zones as illustrated in Fig. 1. The resonance zones shown in Fig. 1 are located around $(\omega, p) = (2/n, 0)$, $n = 1, 2, 3, \ldots$ in the parameter space. For n odd these zones are bound by lines of period doubling bifurcation which are subcritical to the left and supercritical to the right. But for n even the resonance zones are bound by lines of symmetry breaking which are subcritical to the left and supercritical to the right. The narrow strips in black in which rotating solutions occur can also extend outside the parametric resonance zones. The steady state rotating orbits of the parametrically excited pendulum are classified into four categories (see Fig. 2 for visualization of these solutions) as follows:

Purely rotating orbits. These rotating solutions are such that the velocity is single signed for all time, that is, either $\dot{\theta}(t) > 0\ \forall t$ or $\dot{\theta}(t) < 0\ \forall t$. They are the dominant type of stable rotating motion in the first resonance zone, that is, they occur around $(\omega, p) = (2/n, 0)$, $n = 1$. A typical purely rotating orbit is shown in Fig. 2(a). These rotating orbits exist in conjugate pairs: one rotating clockwise and another rotating anti-clockwise.

Oscillating rotating orbits. These rotating solutions are such that either average $\dot{\theta}(t) > 0$ or average $\dot{\theta}(t) < 0$. For these rotating solutions a trajectory completes a fixed number of oscillations within a potential energy well before transversing a potential maximum into the next potential energy well through rotating motion. They are the dominant stable rotating attractors within the

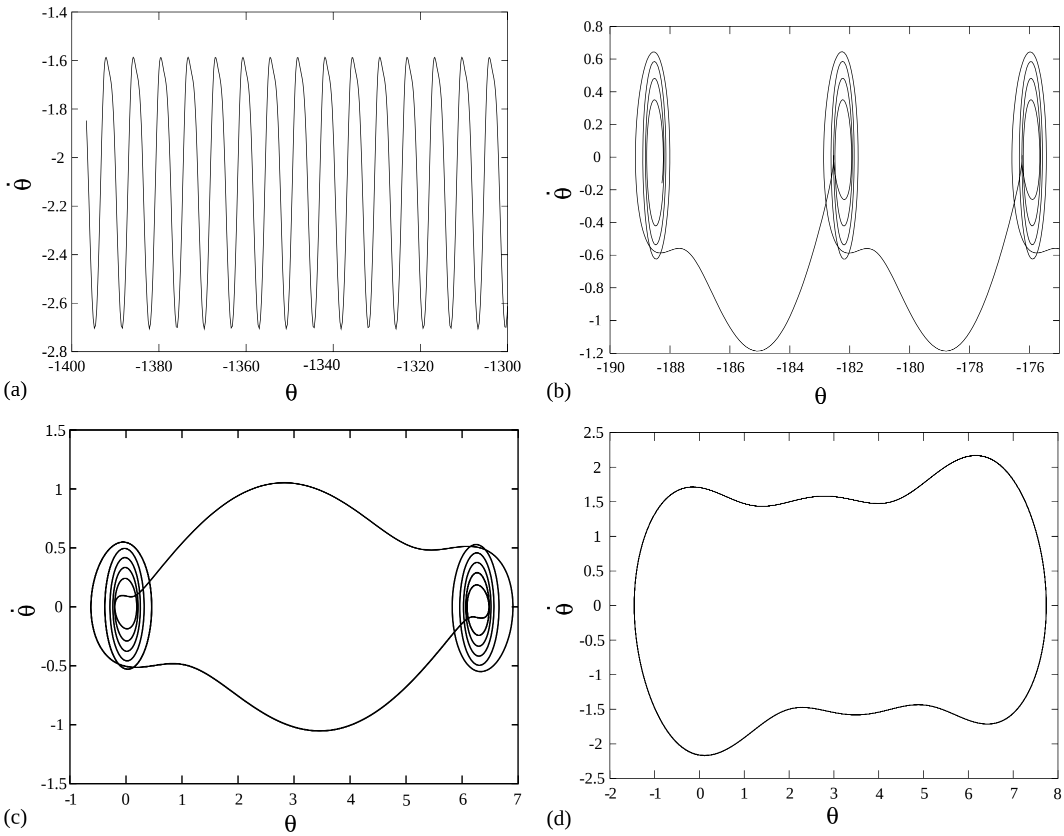

Fig. 2. Phase portraits of stable rotating solutions of the parametrically excited pendulum: (a) purely rotating orbit of period T, for $\omega = 2.0$, $p = 1.7$ in zone I; (b) oscillating rotating orbit of period T, for $\omega = 0.19$, $p = 1.5$ in zone IX; (c) straddling rotating orbit of period $2T$, for $\omega = 0.15$, $p = 1.4$ in zone X; (d) large amplitude rotating orbit of period $2T$, for $\omega = 1.0$, $p = 1.7$ in zone II.

resonance zones located around $(\omega, p) = (2/n, 0)$, $n > 2$, n odd. A typical oscillating rotating orbit is shown in Fig. 2(b) which makes 5 oscillations before transversing into the next well through rotating motion.

Straddling rotating orbits. These have a similar property as oscillating rotating solutions of experiencing oscillations and rotations but differ in that their response only straddles one or two or three wells. The straddling rotating solutions are the dominant stable rotating attractors within the resonance zones located around $(\omega, p) = (2/n, 0)$, $n > 2$, n even. Fig. 2(c) shows a typical straddling rotating solution which straddles two wells.

Large amplitude rotating orbits. These also have similar properties as the straddling rotating solutions in that they straddle one, two or three wells but differ in that they do not make oscillations within the potential wells. They are the dominant stable rotating motions within the second resonance zone, that is the zone located at $(\omega, p) = (2/n, 0)$, $n = 2$. A typical large amplitude rotating orbit is shown in Fig. 2(d).

2. The different types of rotating solutions

In the (ω, p) parameter space the resonance zones in which the pendulum is in parametric resonance occur around $\omega = 2/n$, $n = 1, 2, 3, \ldots$. Now the rotating motions that occur in the narrow black strips within the resonance zones are considered in detail. A characteristic feature of the purely rotating and the oscillating rotating solutions is that they occur in conjugate pairs which are mirror images of each other under the transformation

$$[\theta, \dot{\theta}, t] \rightarrow [-\theta, -\dot{\theta}, t + T/2]. \tag{2}$$

These conjugate pairs are such that one rotates clockwise and another rotates anticlockwise. In the case of the parametrically excited pendulum, one would expect the symmetry with respect to the transformation $[\theta, \dot{\theta}, t] \rightarrow [-\theta, -\dot{\theta}, t + T/2]$ to be generally broken for arbitrary ω and p because

$$\cos(\omega t) \neq \cos[\omega(t + T/2)]. \tag{3}$$

But contrary to this, numerical results show that for the parametrically excited pendulum the symmetry defined by the transformation $[\theta, \dot{\theta}, t] \rightarrow [-\theta, -\dot{\theta}, t + T/2]$ survives in some of the resonance zones located around $(\omega, p) = (2/n, 0)$. This accounts for the observation that the large amplitude rotating solutions and the straddling rotating solutions are symmetric with respect to this transformation and the symmetry is only broken for arbitrarily large values of p.

The purely rotating and the large amplitude rotating solutions share a common feature in that they do not involve oscillations. These solutions are the dominant stable rotating solutions in the first and second resonance zones, respectively. The properties of purely rotating solutions and their subharmonics were studied in detail in Ref. [3].

To illustrate the special features of the oscillating rotating (unsymmetric) and straddling rotating (symmetric) solutions consider the dynamics in the resonance zones located at values around $(\omega, p) = (2/n, 0)$, $n > 2$, in the (ω, p) space. It can be conjectured that in general, symmetric rotating solutions occur within the resonance zones located at $(\omega, p) = (2/n, 0)$, n even, while the unsymmetric rotating solutions occur within the resonance zones located at $(\omega, p) = (2/n, 0)$, n odd, in the (ω, p) space. For the symmetric rotating solutions, their symmetry is only broken for p arbitrarily large.

First the straddling rotating orbits are considered. These occur within the zones located around $(\omega, p) = (2/n, 0)$, $n > 2$, n even. Typical examples of these symmetric orbits for n even are shown in Fig. 3. These straddling rotating orbits were numerically calculated for values of (ω, p) in the IVth (Fig. 3(a)), VIth (Fig. 3(b)) and VIIIth (Fig. 3(c)) resonance zones (i.e., inside the zones located around $(\omega, p) = (2/n, 0)$, $n = 4, 6, 8$.

It is evident from Fig. 3 that as ω decreases, the number of oscillations within each well increases for the straddling rotating attractors (2 oscillations in the local potential energy well for $\omega = 0.45$, three oscillations for $\omega = 0.30$ and 4 oscillations for $\omega = 0.22$). These rotating solutions are symmetric with respect to the transformation formation $[\theta, \dot{\theta}, t] \rightarrow [-\theta, -\dot{\theta}, t + T/2]$, and the symmetry is only broken for large values of p.

Consider rotating solutions which occur in the resonance zones located at $(\omega, p) = (2/n, 0)$, $n > 2$, n odd. Numerical simulations also show that when n is odd we have unsymmetric rotating solutions (oscillating rotating type) of period T. Examples of these unsymmetric rotating orbits

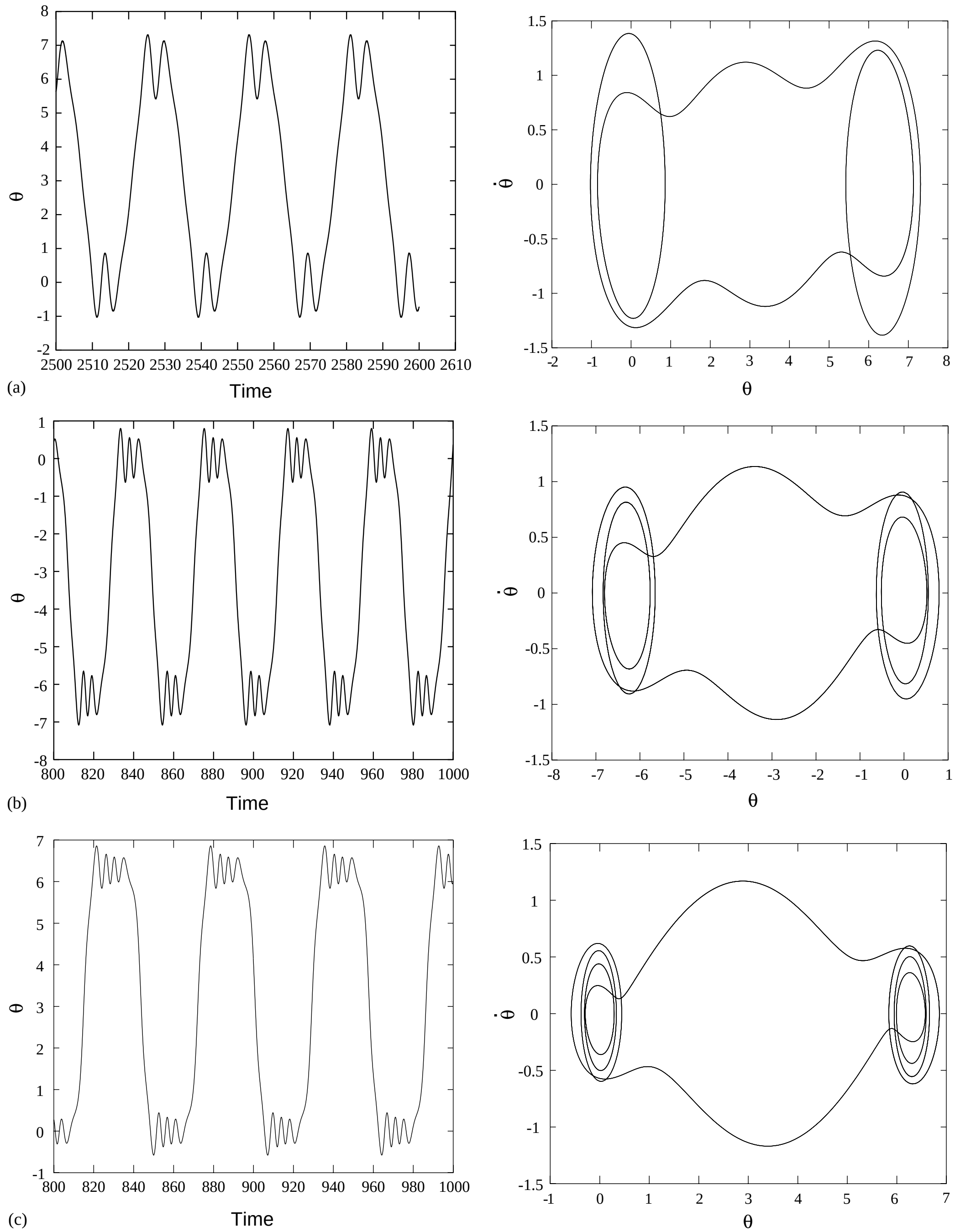

Fig. 3. Straddling rotating orbits of the parametrically excited pendulum for $p = 1.5$ with frequency of forcing (a) $\omega = 0.45$, (b) $\omega = 0.30$, (c) $\omega = 0.22$ for initial conditions $(\theta, \dot{\theta}) = (0.57, 0)$. Left: time series of the trajectories; right: phase space plot of the trajectory. All the orbits are period $2T$. Here the two potential-wells are located at points $(\theta, \dot{\theta}) = (\pm 2m\pi, 0)$, $m = 0, 1, 2, \ldots$ in the phase plane.

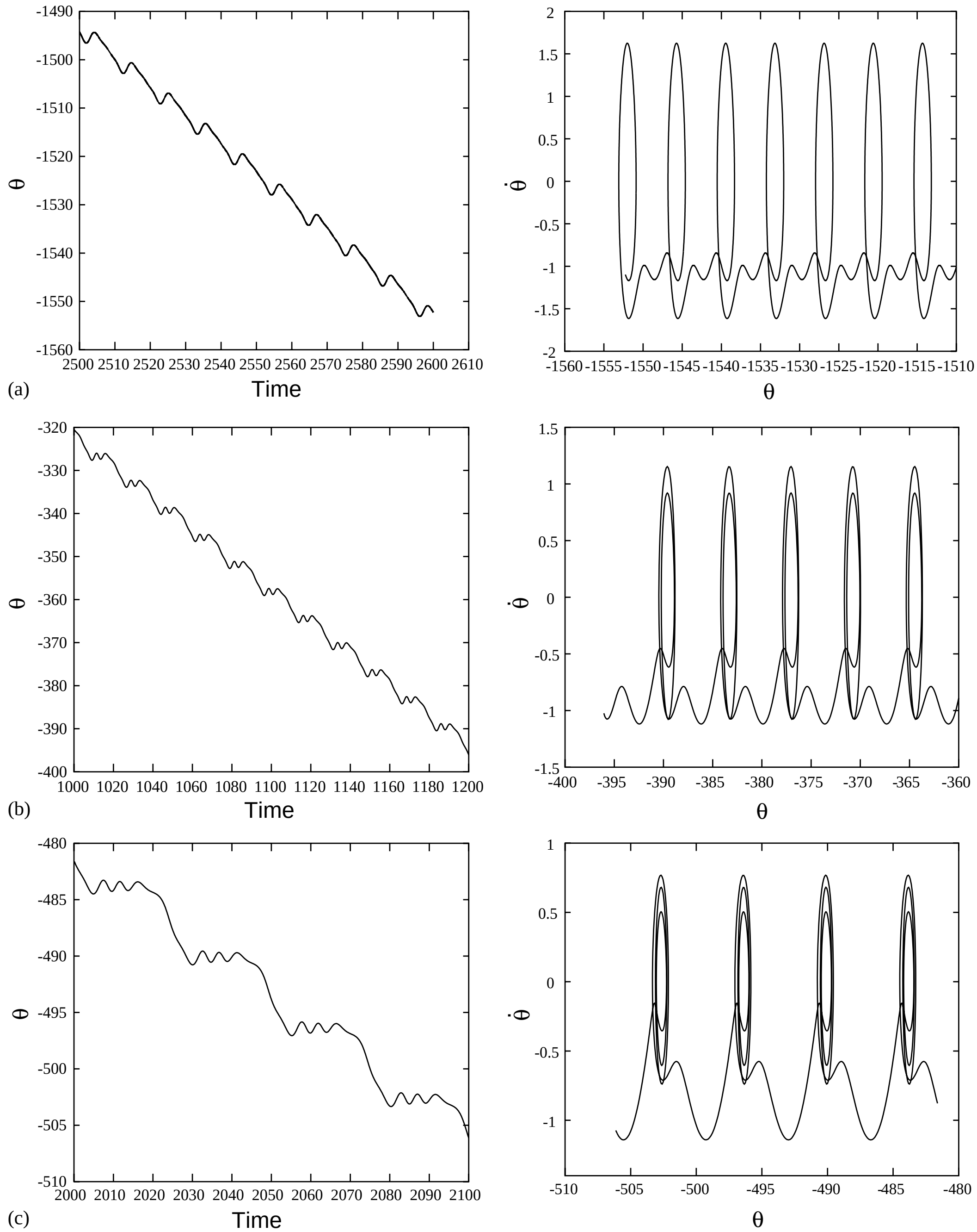

Fig. 4. Oscillating rotating solutions for $p = 1.5$ with frequency of forcing (a) $\omega = 0.60$, (b) $\omega = 0.36$, (c) $\omega = 0.25$ for initial conditions $(\theta, \dot{\theta}) = (1.57, 0)$. Left: time series of the trajectories; Right: phase space plot of the trajectory. Here all the rotating orbits are period T.

are shown in Fig. 4. These oscillating rotating orbits were numerically calculated for values of (ω, p) in the IIIth (Fig. 4(a)), Vth (Fig. 4(b)) and VIIth (Fig. 4(c)) resonance zones (i.e., inside the zones located at $(\omega, p) = (2/n, 0)$, $n = 3, 5, 7$. It can seen from Fig. 4 that the number of oscillations in each well before rotation takes place increases as ω decreases (one oscillation in the well for $\omega = 0.6$, two oscillations for $\omega = 0.36$ and three oscillations for $\omega = 0.25$). These rotating solutions occur in conjugate pairs: one for which the average $\dot{\theta}(t) > 0$ and another for which average $\dot{\theta}(t) < 0$.

3. Final remarks

The properties of rotating periodic motions of the parametrically excited pendulum in the parameter space have been considered. It has been established that within the first two resonance zones the dominant stable rotating solutions do not involve oscillations in the local potential energy wells while in all other resonance zones the rotating solutions involve oscillations within local potential energy wells. Numerical evidence has been presented which shows that the lower resonance zones are alternating regions in which we realize symmetric and non-symmetric rotating solutions.

Acknowledgements

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