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A numerical model for the 3-D non-linear vibrations of an N-string

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Abstract

We present a non-linear numerical model describing the 3-D vibrations of a planar network of N sections of string which are connected together at one common mobile extremity. We call such a network N-string. For small-amplitude vibrations perpendicular to the N-string equilibrium plane, the numerical results coincide with the already known analytical solutions of the linear model. This non-linear model makes it possible to describe small- or large-amplitude 3-D vibrations of any kind of N-string subjected to an initial plucking. The equations of motion are also presented in a dimensionless form and a vast dimensionless physical parameter space is identified. The numerical model can be extended to more complex networks of strings.

1. Introduction

The understanding of the properties of vibrating systems is important because of their commonness in mechanics and mechanical engineering [1, p. 155]. Such systems are also found in molecular physics where the intramolecular bonds are often modelled as springs between masses [2, p. 351]. In the case of systems of strings, the vibrations properties are of great interest in acoustics [3]. Here, we are interested in the non-linear vibrations of certain types of networks of strings. The vibrations of networks of strings have been studied by Schmidt [4], who obtained a non-linear model using Hamilton's principle and examined the corresponding linearized model in the perspective of the controllability of the networks. The exponential stability of a long chain of coupled vibrating strings has also been analyzed [5].

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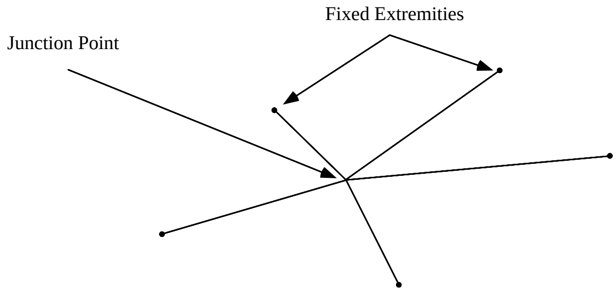

Fig. 1. Schematic of an 5-string. All the extremities are fixed while the junction point is free to move.

We call N-string a network of N tightly stretched flexible strings connected together at one common extremity. This common extremity, or junction point, is mobile while all other extremities are in the same plane and constrained to remain stationary (see Fig. 1). In this paper, we present a numerical model of the 3-D non-linear vibrations of an N-string about its equilibrium position resulting from some initial plucking. We have previously analyzed the linear vibrations which are perpendicular to the N-string equilibrium plane [3]. Analytical solutions were obtained for that case. One of the main results from that study concerns the energy of the vibration modes of a plucked symmetric N-string for $N > 2$. We have shown that higher modes of even orders can be excited to energy levels above that of the fundamental mode by plucking at an appropriate location along one of the strings. This phenomenon is impossible for an ordinary ($N = 2$) string, where the energy of the fundamental mode always dominates.

The methods usually applied to study linear vibrations are of no help for the study of non-linear vibrations since they rely on the superposition principle which is not valid for non-linear problems. Furthermore, it is usually very difficult, if not impossible, to find analytical solutions to non-linear problems. Although non-linear vibration problems can be treated using perturbation methods as in Ref. [6], where the ordinary elastic string is considered (see also Ref. [7]), in this paper we shall obtain quantitative results through numerical approximations of solutions to the equations describing the non-linear vibrations of a discretized model of a continuous N-string. Some general properties of the continuous N-string and its non-linear vibrations will also be deduced.

The paper is organized as follows. In Section 2, we set up the general geometric framework and present a discretized model of the N-string, where each string is replaced by a set of finite masses joined by massless springs. We establish the equations of motion of the discretized N-string in Section 3. In Section 4, we obtain two equations which determine the horizontal position of the junction point. In Section 5, we find the equations describing the position of the points on a plucked continuous N-string in static equilibrium. In Section 6, we present some numerical results which include a validation of the model and various simulations of a symmetric 3-string. Finally, Section 7 contains a discussion of our results.

2. Discretized N-string

Let an N-string whose N strings in equilibrium position are in the same plane, form at their junction point angels θ_i and have length l_i, $i = 1, 2, \ldots, N$. We define θ_i as the angle between the

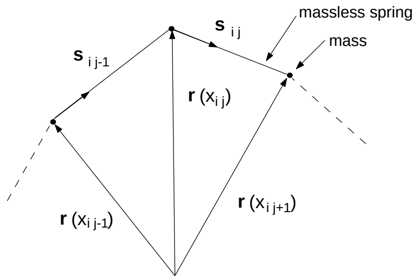

Fig. 2. Continuous strings of the N-string are discretized using finite masses linked by massless springs. The vector $\mathbf{r}$ gives the position of each finite mass, while $\mathbf{s}$ is a unit vector parallel to the displacement vector between adjacent masses.

ith and the $(i+1)$th string for $i = 1, 2, \ldots, N-1$, and between the Nth and the first string if $i = N$. We assume that each string of the N-string is flexible and elastic, and is subjected to no internal nor external friction. The position along the ith string of the N-string in equilibrium is described by the co-ordinates x_i, $0 \leqslant x_i \leqslant l_i$, $i = 1, 2, \ldots, N$, where $x_i = 0$ coincides with the junction point of the N strings. Let $\rho_i(x_i) > 0$ be the function describing the linear mass density of the ith string. From the x_i, we construct N orthogonal right-handed systems of axes $x_i y_i z$ sharing the same z-axis. Following the same methodology as in Ref. [8, p. 437], for instance, we obtain a discretized model of an N-string by replacing each of its N strings with a set of finite masses placed at regular intervals along each of the strings. The spacing between the masses on the ith string is denoted by h_i and the position by x_{ij}, where $i = 1, 2, \ldots, N$, $j = 0, 1, \ldots, J_i$, $J_i = l_i/h_i$ and $x_{i0} = 0$. We assume that each of the discrete masses is linked to its direct neighbours by massless springs (see Fig. 2).

At the point x_{ij}, the finite mass μ_{ij} of the discretized model is defined in terms of the function ρ_i. For $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, J_i - 1$, we set

$$\mu_{ij} = \int_{x_{ij}-h_i/2}^{x_{ij}+h_i/2} \rho_i(x)\,\mathrm{d}x.$$

Similarly, the finite mass μ_0 at the junction point $x_{i0} = 0$ will normally be defined by

$$\mu_0 = \sum_{i=1}^{N} \int_0^{x_{i1}/2} \rho_i(x)\,\mathrm{d}x.$$

Note that it is not necessary to define finite masses at the stationary points x_{iJ_i}.

3. Equations of motion

We shall now determine the equations describing the motion of the masses of the discretized N-string. Let us first observe that in the $x_i y_i z$ system, the position of any point on the ith string at time $t \geqslant 0$ can be given by the vector

$$\mathbf{r}_i(x_i, t) = (x_i + u_i(x_i, t))\mathbf{e}_{i1} + v_i(x_i, t)\mathbf{e}_{i2} + w_i(x_i, t)\mathbf{e}_3, \tag{1}$$

where $\mathbf{e}_{i1}, \mathbf{e}_{i2}, \mathbf{e}_3$ form an orthonormal basis for the system $x_i y_i z$, and u_i, v_i, w_i are twice differentiable functions representing the displacements in the x_i, y_i and z directions, respectively. The N-string is plucked at $t = 0$.

We shall begin with the masses which are not at the junction point, i.e., those located at x_{ij} for $j > 0$. The motion of each of these masses results from the tensions exerted on it by its direct neighbours. If we designate these tensions by $\mathbf{T}_{ij-1} = \mathbf{T}_i(x_{ij-1}, t)$, $\mathbf{T}_{ij} = \mathbf{T}_i(x_{ij}, t)$, and assume a linear configuration of the massless springs between neighbouring masses (see Fig. 2), then from Newton's second law it follows that

$$\mathbf{T}_{ij-1} + \mathbf{T}_{ij} = \mu_{ij}\ddot{\mathbf{r}}_{ij}, \tag{2}$$

where $\ddot{\mathbf{r}}_{ij} = \partial^2 \mathbf{r}_i(x_{ij}, t)/\partial t^2$. Note that it is not assumed that the tensions in the N strings are equal to the same constant, as opposed to the case of the linear model of an ordinary string (see e.g., Ref. [8, p. 437]), and the case of the linear model of an N-string [3].

Let us define the scalar functions $T_{ij-1} = T_i(x_{ij-1}, t)$ and $T_{ij} = T_i(x_{ij}, t)$ such that

$$\mathbf{T}_{ij-1} = -T_{ij-1}\mathbf{s}_{ij-1}, \qquad \mathbf{T}_{ij} = T_{ij}\mathbf{s}_{ij},$$

where $\mathbf{s}_{il} = \mathbf{s}_i(x_{il}, t)$ is the unit vector parallel to $\mathbf{r}_{il+1} - \mathbf{r}_{il}$ at time t for $l = j-1,\ j$, i.e.,

$$\begin{aligned}\mathbf{s}_{il} &= \frac{\mathbf{r}_{il+1} - \mathbf{r}_{il}}{\|\mathbf{r}_{il+1} - \mathbf{r}_{il}\|} \\ &= \frac{(1 + u'_{il})\mathbf{e}_{i1} + v'_{il}\mathbf{e}_{i2} + w'_{il}\mathbf{e}_3}{[(1 + u'_{il})^2 + (v'_{il})^2 + (w'_{il})^2]^{1/2}},\end{aligned}$$

where

$$\begin{aligned}&u'_{ij} = \frac{1}{h_i}[u_i(x_{ij+1}, t) - u_i(x_{ij}, t)], \qquad v'_{ij} = \frac{1}{h_i}[v_i(x_{ij+1}, t) - v_i(x_{ij}, t)],\\ &w'_{ij} = \frac{1}{h_i}[w_i(x_{ij+1}, t) - w_i(x_{ij}, t)].\end{aligned} \tag{3}$$

Vector equation (2) can thus be expressed as the following system of three scalar differential-difference equations:

$$\frac{T_{ij}(1 + u'_{ij})}{[(1 + u'_{ij})^2 + (v'_{ij})^2 + (w'_{ij})^2]^{1/2}} - \frac{T_{ij-1}(1 + u'_{ij-1})}{[(1 + u'_{ij-1})^2 + (v'_{ij-1})^2 + (w'_{ij-1})^2]^{1/2}} = \mu_{ij}\ddot{u}_{ij}, \tag{4}$$

$$\frac{T_{ij}v'_{ij}}{[(1 + u'_{ij})^2 + (v'_{ij})^2 + (w'_{ij})^2]^{1/2}} - \frac{T_{ij-1}v'_{ij-1}}{[(1 + u'_{ij-1})^2 + (v'_{ij-1})^2 + (w'_{ij-1})^2]^{1/2}} = \mu_{ij}\ddot{v}_{ij}, \tag{5}$$

$$\frac{T_{ij}w'_{ij}}{[(1 + u'_{ij})^2 + (v'_{ij})^2 + (w'_{ij})^2]^{1/2}} - \frac{T_{ij-1}w'_{ij-1}}{[(1 + u'_{ij-1})^2 + (v'_{ij-1})^2 + (w'_{ij-1})^2]^{1/2}} = \mu_{ij}\ddot{w}_{ij}. \tag{6}$$

We shall now determine the equations of motion for the mass located at the junction point. Choosing the system x_1y_1z as the reference frame, we represent the tension exerted by the ith string on the mass at the junction point as $(\mathbf{T}_{i0})_1 = (\mathbf{T}_i(0,t))_1$. From Newton's second law, it then follows that

$$\sum_{i=1}^{N} (\mathbf{T}_{i0})_1 = \mu_0 \ddot{\mathbf{r}}_{10}. \tag{7}$$

It is clear from the geometry that

$$(\mathbf{T}_{i0})_1 = (\mathbf{T}_{i0} \cdot (\mathbf{e}_{11})_i)\mathbf{e}_{11} + (\mathbf{T}_{i0} \cdot (\mathbf{e}_{12})_i)\mathbf{e}_{12} + (\mathbf{T}_{i0} \cdot \mathbf{e}_3)\mathbf{e}_3,$$

where $(\mathbf{e}_{11})_i$ and $(\mathbf{e}_{12})_i$ designate the components of $\mathbf{e}_{11}$ and $\mathbf{e}_{12}$ in system x_iy_iz. Also, let

$$\mathbf{T}_{i0} = T_{i0}\mathbf{s}_{i0},$$

where $\mathbf{s}_{i0} = \mathbf{s}_i(0,t)$ is the unit vector parallel to $\mathbf{r}_{i1} - \mathbf{r}_{i0}$ at time t, namely

$$\mathbf{s}_{i0} = \frac{(1+u'_{i0})\mathbf{e}_{i1} + v'_{i0}\mathbf{e}_{i2} + w'_{i0}\mathbf{e}_3}{[(1+u'_{i0})^2 + (v'_{i0})^2 + (w'_{i0})^2]^{1/2}},$$

where u'_{i0}, v'_{i0} and w'_{i0} are given by Eq. (3). We also have

$$\begin{aligned}
(\mathbf{e}_{11})_2 &= \mathbf{e}_{21}\cos\theta_1 - \mathbf{e}_{22}\sin\theta_1,\\
(\mathbf{e}_{11})_3 &= \mathbf{e}_{31}\cos(\theta_1+\theta_2) - \mathbf{e}_{32}\sin(\theta_1+\theta_2),\\
&\vdots\\
(\mathbf{e}_{11})_N &= \mathbf{e}_{N1}\cos\left(\sum_{k=1}^{N-1}\theta_k\right) - \mathbf{e}_{N2}\sin\left(\sum_{k=1}^{N-1}\theta_k\right)
\end{aligned}$$

and

$$\begin{aligned}
(\mathbf{e}_{12})_2 &= \mathbf{e}_{21}\sin\theta_1 + \mathbf{e}_{22}\cos\theta_1,\\
(\mathbf{e}_{12})_3 &= \mathbf{e}_{31}\sin(\theta_1+\theta_2) + \mathbf{e}_{32}\cos(\theta_1+\theta_2),\\
&\vdots\\
(\mathbf{e}_{12})_N &= \mathbf{e}_{N1}\sin\left(\sum_{k=1}^{N-1}\theta_k\right) + \mathbf{e}_{N2}\cos\left(\sum_{k=1}^{N-1}\theta_k\right).
\end{aligned}$$

Vector equation (7) is therefore equivalent to the following system of three scalar differential-difference equations:

$$\sum_{i=1}^{N} \frac{T_{i0}[(1+u'_{i0})\cos(\sum_{k=0}^{i-1}\theta_k) - v'_{i0}\sin(\sum_{k=0}^{i-1}\theta_k)]}{[(1+u'_{i0})^2 + (v'_{i0})^2 + (w'_{i0})^2]^{1/2}} = \mu_0\ddot{u}_{10}, \tag{8}$$

$$\sum_{i=1}^{N} \frac{T_{i0}[(1+u'_{i0})\sin(\sum_{k=0}^{i-1}\theta_k) + v'_{i0}\cos(\sum_{k=0}^{i-1}\theta_k)]}{[(1+u'_{i0})^2 + (v'_{i0})^2 + (w'_{i0})^2]^{1/2}} = \mu_0\ddot{v}_{10}, \tag{9}$$

$$\sum_{i=1}^{N} \frac{T_{i0}w'_{i0}}{[(1+u'_{i0})^2 + (v'_{i0})^2 + (w'_{i0})^2]^{1/2}} = \mu_0\ddot{w}_{10}, \tag{10}$$

where we have set $\theta_0 = 0$. The systems of equations (4)–(6) and (8)–(10) describe the motion of the set of masses that discretizes a continuous N-string. To solve these equations one needs some initial conditions and stress–strain relations giving the tension in each of the strings as a function of its deformations; these points will be discussed later.

Eqs. (8)–(10) are a generalization of the known condition on the slopes of the strings of a continuous N-string at the junction point for vibrations of small amplitude that are perpendicular to its equilibrium plane (see Eq. (8) of Ref. [3]). In fact, in the limit of small-amplitude vibrations, we have $u'_{i0} \ll 1, v'_{i0} \ll 1, w'_{i0} \ll 1$ and the tensions in each of the strings can be considered as being constant. Eqs. (8)–(10) thus become

$$\sum_{i=1}^{N} T_{i0} \cos\left(\sum_{k=0}^{i-1} \theta_k\right) = \mu_0 \ddot{u}_{10}, \qquad \sum_{i=1}^{N} T_{i0} \sin\left(\sum_{k=0}^{i-1} \theta_k\right) = \mu_0 \ddot{v}_{10} \tag{11}$$

and

$$\sum_{i=1}^{N} T_{i0} w'_{i0} = \mu_0 \ddot{w}_{10}. \tag{12}$$

We now assume that the tension is the same in all of the N strings. If μ_0 approaches zero in a manner such that the ratios $\mu_0/h_i,\ i = 1, 2, \ldots, N$, remain constant, as in the case of a continuous N-string, then Eq. (12) becomes

$$\sum_{i=0}^{N} w'_{i0}(0, t) = 0,$$

which is Eq. (8) of Ref. [3]. Under these assumptions, we also have that the two equations of Eq. (11) are identically zero. In the linear limit, it is, moreover, apparent from Eqs. (6) and (10) that the perpendicular component w of the vibrations decouples from the others. This property is pointed out in Ref. [1].

To solve the equations of motion of the discretized N-string, we must specify stress–strain relations for $T_i(x_i, t),\ i = 1, 2, \ldots, N$. As in Ref. [4], we chose them to be of the form

$$T_i(x_i, t) = \tau_{i0} + A_i E_i \{[(1 + u'_i(x_i, t))^2 + (v'_i(x_i, t))^2 + (w'_i(x_i, t))^2]^{1/2} - 1\}, \tag{13}$$

where τ_{i0}, A_i and E_i are, respectively, the tension, the cross-sectional area and the modulus of elasticity of the ith string in its rest position, and u'_i, v'_i, w'_i are calculated using Eq. (3). To solve systems (4)–(6) and (8)–(10) with Eq. (13), we transform each of the second order differential-difference equations into two first order differential-difference equations. This leads to a system of $6(\sum_{i=1}^{N} J_i + 1)$ non-linear first order differential-difference equations for the unknowns u_{ij}, v_{ij}, w_{ij}. The Runge–Kutta methods can be used to solve this system subjected to some initial conditions. But before we find these solutions, we shall point out a property of the junction point which applies to any discretized or continuous geometrically symmetric N-string.

4. The horizontal position of the junction point

Denoting $u_i(x_{i0}, t), v_i(x_{i0}, t), w_i(x_{i0}, t)$ by u_{i0}, v_{i0}, w_{i0}, respectively, it is straightforward to show that

$$\begin{aligned}
u_{20} &= u_{10}\cos\theta_1 + v_{10}\sin\theta_1,\\
u_{30} &= u_{10}\cos(\theta_1+\theta_2) + v_{10}\sin(\theta_1+\theta_2),\\
&\vdots\\
u_{N0} &= u_{10}\cos\left(\sum_{k=1}^{N-1}\theta_k\right) + v_{10}\sin\left(\sum_{k=1}^{N-1}\theta_k\right)
\end{aligned}$$

and

$$\begin{aligned}
v_{20} &= -u_{10}\sin\theta_1 + v_{10}\cos\theta_1,\\
v_{30} &= -u_{10}\sin(\theta_1+\theta_2) + v_{10}\cos(\theta_1+\theta_2),\\
&\vdots\\
v_{N0} &= -u_{10}\sin\left(\sum_{k=1}^{N-1}\theta_k\right) + v_{10}\cos\left(\sum_{k=1}^{N-1}\theta_k\right).
\end{aligned}$$

We thus have

$$\sum_{i=2}^{N} u_{i0} = u_{10}\left[\sum_{k=1}^{N-1}\cos\left(\sum_{j=1}^{k}\theta_j\right)\right] + v_{10}\left[\sum_{k=1}^{N-1}\sin\left(\sum_{j=1}^{k}\theta_j\right)\right]$$

and

$$\sum_{i=2}^{N} v_{i0} = -u_{10}\left[\sum_{k=1}^{N-1}\sin\left(\sum_{j=1}^{k}\theta_j\right)\right] + v_{10}\left[\sum_{k=1}^{N-1}\cos\left(\sum_{j=1}^{k}\theta_j\right)\right].$$

Now, if $\theta_1 = \theta_2 = \cdots = \theta_N$, we have

$$\sum_{i=1}^{N-1}\cos\left(\sum_{j=1}^{i}\theta_j\right) = \frac{\sin(N-1/2)\theta_1}{2\sin\theta_1/2} - \frac{1}{2} = -1 \tag{14}$$

and

$$\sum_{i=1}^{N-1}\sin\left(\sum_{j=1}^{i}\theta_j\right) = \frac{\cos\theta_1/2 - \cos(N-1/2)\theta_1}{2\sin\theta_1/2} = 0, \tag{15}$$

which in turn implies that

$$\sum_{i=1}^{N} u_{i0} = 0 \quad \text{and} \quad \sum_{i=1}^{N} v_{i0} = 0.$$

Therefore, if the N-string is geometrically symmetric, the arithmetic means of the x_i and y_i components of the junction point displacement, i.e., its horizontal position in each of the frames x_iy_iz, are zero for all $t \geqslant 0$.

5. Static equilibrium of a plucked continuous N-string

In order to specify the initial position of a discretized N-string, we shall now determine the position of a plucked continuous N-string in static equilibrium. Let us consider a continuous N-string which at time $t = 0$ is plucked at the point $x_1 = l_1/m,\ m > 1$, in a manner such that the position of this point in the system x_1y_1z is described by

$$\mathbf{r}_1(l_1/m, 0) = \left(\frac{l_1}{m} + U_m\right)\mathbf{e}_{11} + V_m\mathbf{e}_{12} + W_m\mathbf{e}_3,$$

where U_m, V_m and W_m are prescribed constants which correspond to the amplitude of displacement at the plucked point. Assuming a linear static configuration of the plucked N-string, to fully specify the position of all other points on the N-string at $t = 0$, it is sufficient to determine the corresponding position of the junction point. To see that, let

$$\mathbf{r}_1(0, 0) = U_0\mathbf{e}_{11} + V_0\mathbf{e}_{12} + W_0\mathbf{e}_3$$

be the vector giving the junction point position with respect to system x_1y_1z at time $t = 0$. The parameters U_0, V_0, W_0 depend on U_m, V_m, W_m and on the physical characteristics of the N-string, such as the tension in each of the N strings. Assuming that a static equilibrium is achieved in the N-string, it is then easy to show that the positions of the other points of the N-string are given by the linear functions

$$u_1(x_1, 0) = \begin{cases} \frac{mx_1}{l_1}(U_m - U_0) + U_0 & \text{if } 0 \leqslant x_1 \leqslant \frac{l_1}{m}, \\ \frac{mU_m}{m-1}\left(1 - \frac{x_1}{l_1}\right) & \text{if } \frac{l_1}{m} < x_1 \leqslant l_1, \end{cases} \tag{16}$$

$$v_1(x_1, 0) = \begin{cases} \frac{mx_1}{l_1}(V_m - V_0) + V_0 & \text{if } 0 \leqslant x_1 \leqslant \frac{l_1}{m}, \\ \frac{mV_m}{m-1}\left(1 - \frac{x_1}{l_1}\right) & \text{if } \frac{l_1}{m} < x_1 \leqslant l_1, \end{cases} \tag{17}$$

$$w_1(x_1, 0) = \begin{cases} \frac{mx_1}{l_1}(W_m - W_0) + W_0 & \text{if } 0 \leqslant x_1 \leqslant \frac{l_1}{m}, \\ \frac{mW_m}{m-1}\left(1 - \frac{x_1}{l_1}\right) & \text{if } \frac{l_1}{m} < x_1 \leqslant l_1 \end{cases} \tag{18}$$

and for $j = 2, 3, \ldots, N$ and $0 \leqslant x_j \leqslant l_j$,

$$u_j(x_j, 0) = \left(U_0 \cos\sum_{k=1}^{j-1}\theta_k + V_0 \sin\sum_{k=1}^{j-1}\theta_k\right)\left(1 - \frac{x_j}{l_j}\right), \tag{19}$$

$$v_j(x_j, 0) = \left(-U_0 \sin\sum_{k=1}^{j-1}\theta_k + V_0 \cos\sum_{k=1}^{j-1}\theta_k\right)\left(1 - \frac{x_j}{l_j}\right), \tag{20}$$

$$w_j(x_j, 0) = W_0\left(1 - \frac{x_j}{l_j}\right). \tag{21}$$

To determine U_0, V_0, W_0 from U_m, V_m, W_m, system (8)–(10) with Eqs. (13) and (16)–(21), needs to be solved with $\ddot{u}_{10} = \ddot{v}_{10} = \ddot{w}_{10} = 0$. The resulting system of non-linear equations can be expressed as a homogeneous system of the form

$$f_1(U_0, V_0, W_0) = 0, \qquad f_2(U_0, V_0, W_0) = 0, \qquad f_3(U_0, V_0, W_0) = 0, \tag{22}$$

where f_1, f_2 and f_3 result from Eqs. (8)–(10), respectively. System (22) can be solved using a standard non-linear system solver, such as Newton's method.

6. Numerical results

In this section, in order to simplify the presentation we shall assume that the ρ_i are constant on each string. To reduce the number of parameters, we introduce dimensionless physical parameters which will facilitate the analysis of the results. Using the first string as reference, we define $\mathscr{L}_i = l_i/l_1$, $\varrho_i = \rho_i/\rho_1$, $\mathscr{E}_i = E_iA_i/\tau_{10}$ and $\mathscr{T}_{i0} = \tau_{i0}/\tau_{10}$ for $i = 1, 2, \dots, N$. If $\bar{h}_i = \mathscr{L}_i/J_i$, then the dimensionless masses of the corresponding discretized model are given by

$$\bar{\mu}_i = \varrho_i \bar{h}_i = \mu_i/\mathscr{M}_1, \quad i = 1, 2, \dots, N$$

and

$$\bar{\mu}_0 = \frac{1}{2}\sum_{i=1}^{N} \varrho_i \bar{h}_i = \mu_0/\mathscr{M}_1,$$

where $\mathscr{M}_1 = \rho_1 l_1$ is the mass of the first string. In addition to the initial conditions, the behaviour of the N-string may thus depend on at most $4N - 5$ independent physical parameters, where we have taken into account the equilibrium between the N initial tensions. If we use l_1, τ_{10} and $(\mu_1 l_1/\tau_{10})^{1/2} = t_{ref}$ as reference length, initial string tension and time, respectively, and also if we consider N strings whose string lengths are such that we can use the same step size for each string, then Eqs. (4)–(6) can be rendered dimensionless and may be written as

$$\frac{\bar{T}_{ij}(1 + \bar{u}'_{ij})}{[(1 + \bar{u}'_{ij})^2 + (\bar{v}'_{ij})^2 + (\bar{w}'_{ij})^2]^{1/2}} - \frac{\bar{T}_{ij-1}(1 + \bar{u}'_{ij-1})}{[(1 + \bar{u}'_{ij-1})^2 + (\bar{v}'_{ij-1})^2 + (\bar{w}'_{ij-1})^2]^{1/2}} = \varrho_i \ddot{\bar{u}}_{ij}, \tag{23}$$

$$\frac{\bar{T}_{ij}\bar{v}'_{ij}}{[(1 + \bar{u}'_{ij})^2 + (\bar{v}'_{ij})^2 + (\bar{w}'_{ij})^2]^{1/2}} - \frac{\bar{T}_{ij-1}\bar{v}'_{ij-1}}{[(1 + \bar{u}'_{ij-1})^2 + (\bar{v}'_{ij-1})^2 + (\bar{w}'_{ij-1})^2]^{1/2}} = \varrho_i \ddot{\bar{v}}_{ij}, \tag{24}$$

$$\frac{\bar{T}_{ij}\bar{w}'_{ij}}{[(1 + \bar{u}'_{ij})^2 + (\bar{v}'_{ij})^2 + (\bar{w}'_{ij})^2]^{1/2}} - \frac{\bar{T}_{ij-1}\bar{w}'_{ij-1}}{[(1 + \bar{u}'_{ij-1})^2 + (\bar{v}'_{ij-1})^2 + (\bar{w}'_{ij-1})^2]^{1/2}} = \varrho_i \ddot{\bar{w}}_{ij}, \tag{25}$$

where $\bar{T}_{ij} = T_{ij}/\tau_{10}$, $\bar{u}_{ij} = u_{ij}/l_1$, $\bar{v}_{ij} = v_{ij}/l_1, \bar{w}_{ij} = w_{ij}/l_1$, and each dot and the prime represent, respectively, differentiation with respect to $\bar{t} = t/t_{ref}$ and $\bar{x}_i$ for $0 \leqslant \bar{x}_i \leqslant \mathscr{L}_i$. Similarly, the dimensionless equations corresponding to Eqs. (8)–(10) are

$$\sum_{i=1}^{N} \frac{\bar{T}_{i0}[(1 + \bar{u}'_{i0})\cos(\sum_{k=0}^{i-1}\theta_k) - \bar{v}'_{i0}\sin(\sum_{k=0}^{i-1}\theta_k)]}{[(1 + \bar{u}'_{i0})^2 + (\bar{v}'_{i0})^2 + (\bar{w}'_{i0})^2]^{1/2}} = \frac{1}{2}\sum_{i=1}^{N} \varrho_i \ddot{\bar{u}}_{10}, \tag{26}$$

$$\sum_{i=1}^{N} \frac{\bar{T}_{i0}[(1+\bar{u}'_{i0})\sin(\sum_{k=0}^{i-1}\theta_k) + \bar{v}'_{i0}\cos(\sum_{k=0}^{i-1}\theta_k)]}{[(1+\bar{u}'_{i0})^2 + (\bar{v}'_{i0})^2 + (\bar{w}'_{i0})^2]^{1/2}} = \frac{1}{2}\sum_{i=1}^{N} \varrho_i \ddot{\bar{v}}_{10}, \tag{27}$$

$$\sum_{i=1}^{N} \frac{\bar{T}_{i0}\bar{w}'_{i0}}{[(1+\bar{u}'_{i0})^2 + (\bar{v}'_{i0})^2 + (\bar{w}'_{i0})^2]^{1/2}} = \frac{1}{2}\sum_{i=1}^{N} \varrho_i \ddot{\bar{w}}_{10}. \tag{28}$$

The dimensionless expression corresponding to Eq. (13) becomes

$$\bar{T}_i(\bar{x}_i,\bar{t}) = \mathscr{T}_{i0} + \mathscr{E}_i\{[(1+\bar{u}'_i(\bar{x}_i,\bar{t}))^2 + (\bar{v}'_i(\bar{x}_i,\bar{t}))^2 + (\bar{w}'_i(\bar{x}_i,\bar{t}))^2]^{1/2} - 1\}. \tag{29}$$

Finally, Eqs. (16)–(22) are easily rendered dimensionless by dividing by the reference length.

As mentioned in the previous section, system (22) is solved for the initial position of the junction point ($\bar{U}_0$, $\bar{V}_0$, $\bar{W}_0$) using Newton's method, for example. Initial guesses for ($\bar{U}_0$, $\bar{V}_0$, $\bar{W}_0$) are thus required and convergence is extremely rapid. With this ($\bar{U}_0$, $\bar{V}_0$, $\bar{W}_0$), Eqs. (23)–(29) are then solved using a Runge–Kutta method. Validation tests were done to ensure convergence with both spatial ($\bar{h}_i$) and temporal ($\Delta\bar{t}$) step sizes. It was found that the optimal step size $\Delta\bar{t}$ changes as a function of the dimensionless physical parameters in the problem. Some experimentation with $\Delta\bar{t}$ was required in order for the Runge–Kutta method to converge in a reasonable time. Apart from this and long-term accuracy loss, which can be seen in Fig. 3, the numerical method was found to be straightforward and efficient.

In the following numerical examples, we consider a symmetric 3-string whose three strings in equilibrium form equal angles of $2\pi/3$ at their junction point and are such that $\mathscr{L}_i = \varrho_i = \mathscr{T}_{i0} = 1$ for $i = 1, 2, 3$. We first validate our numerical model by comparing, in Fig. 3, the numerical results and the analytical results from Ref. [3] which describe the small-amplitude decoupled vibrations of the corresponding continuous 3-string. The curves shown describe the vibrations of a symmetric 3-string where $\mathscr{E}_i = 1, \; i = 1, 2, 3$, when the first string is initially plucked vertically at $\bar{x}_1 = 0.5$ with an amplitude $\bar{W}_m = 0.01$. The initial position of the 3-string is determined by solving system (22). It is clear from this figure that for small-amplitude vibrations, and hence for linear vibrations, the numerical results agree very well with the analytical solution.

Let us note that the choice of parameters $\mathscr{T}_{i0} = \mathscr{E}_i, \; i = 1, 2, 3$ in Fig. 3 is not accidental. Keller [9] indeed showed that with this special stress–strain law, the three components of *large*-amplitude vibrations of an ordinary string decouple and can be described by the linear wave equation. Keller further showed that certain materials, such as springs and rubber bands, follow this special stress–strain law. In fact, it can easily be seen that when Eq. (29) with $\mathscr{T}_{i0} = \mathscr{E}_i, \; i = 1, 2, 3$, is substituted into Eqs. (23)–(28), then the denominators cancel and components of vibration decouple. Moreover, it is easy to see that if the 3-string is plucked in a manner such that the masses undergo no initial $\bar{u}$ and $\bar{v}$ motion, then the resulting vibrations will be purely perpendicular to the equilibrium plane of the 3-string. This result was shown in Ref. [9] for an ordinary string and generalizes to symmetric N strings. Note that Eqs. (26) and (27) imply that the $\bar{u}$ and $\bar{v}$ vibration components no longer decouple if the N-string is not symmetric in its equilibrium position or if the initial plucking results in a non-symmetric horizontal static configuration of the N-string. We therefore compare in Fig. 3 our numerical results, with this special stress–strain law, to the linear analytical results which correspond to decoupled vibrations governed by the linear wave equation.

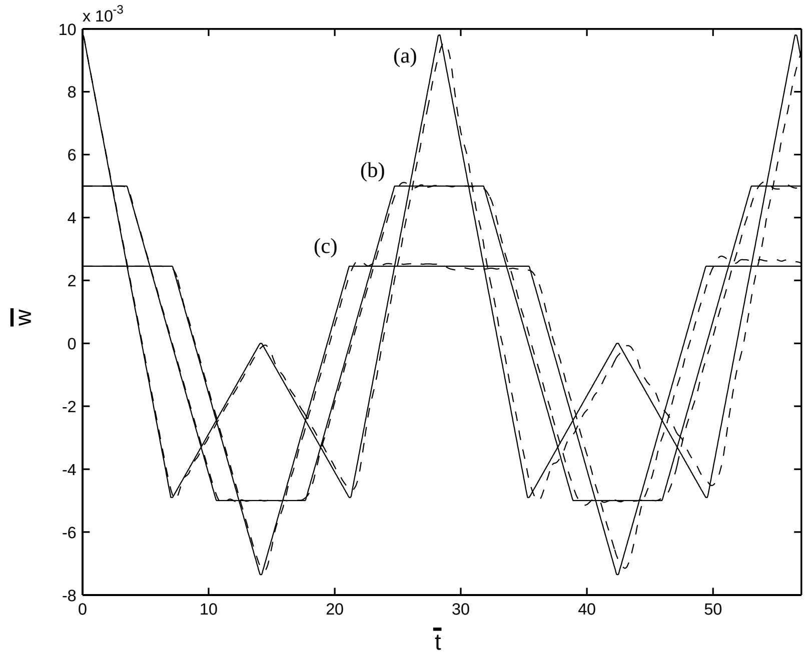

Fig. 3. A comparison of the perpendicular components of vibration, $\bar{w}$, between the numerical model (dashed line) and analytical solutions (solid line) for small-amplitude vibrations. The 3-string is such that $\mathscr{L}_i = \varrho_i = \mathscr{T}_{i0} = \mathscr{E}_i = 1$ and $J_i = 51$ for $i = 1, 2, 3$, and $\bar{U}_m = \bar{V}_m = 0$. The first string has been plucked at $\bar{x}_1 = 0.5$ to an amplitude of $\bar{W}_m = 0.01$. The displacements of $\bar{w}$ shown are (a) at the plucked point $\bar{x}_1 = 0.5$; (b) at the junction point $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0$; and (c) at $\bar{x}_2 = 0.5$ on the second string (and, by symmetry, at $\bar{x}_3 = 0.5$ on the third string).

We shall now present some results which highlight the differences between the linear and the non-linear vibrations of a symmetric 3-string and the capabilities of our numerical model. The results shown in Figs. 4–7 are all for $\mathscr{T}_{i0} = \mathscr{E}_i,\ i = 1, 2, 3$. We start by exploring, in Fig. 4, the effects of a non-small plucking amplitude. The vibrations resulting from a plucked perpendicular amplitude of $\bar{W}_m = 0.5$ and 1 are compared in this figure with those resulting from the numerical model for the small-amplitude case $\bar{W}_m = 0.01$. To compare the different curves, we have normalized the displacement $\bar{w}$ for each curve using the corresponding plucked amplitude, i.e., $\bar{w}/\bar{W}_m = w/W_m$. It appears that a greater plucked amplitude has the predictable effect of increasing the vibration frequency and thus of lowering the relative amplitude of vibration, so that energy is conserved. These effects are due to increased variations of the tension in the strings (see Fig. 5), which results in greater acceleration and velocities. This phenomenon can be seen from Fig. 6. We also note that the vibrations corresponding to greater plucked amplitudes are no longer periodic.

It is important to note that the plucking functions (16)–(21) are general in the sense that one string is plucked at one point and the elements of mass are free to move in any direction in order to establish the static equilibrium. This will generally result in some $\bar{u}$ and $\bar{v}$ motion in the N-string. If the N-string was plucked so that the $\bar{u}$ and $\bar{v}$ initial components of motion of each mass were zero, then our choice of stress–strain law would imply that the curves of each graph superpose in Fig. 4.

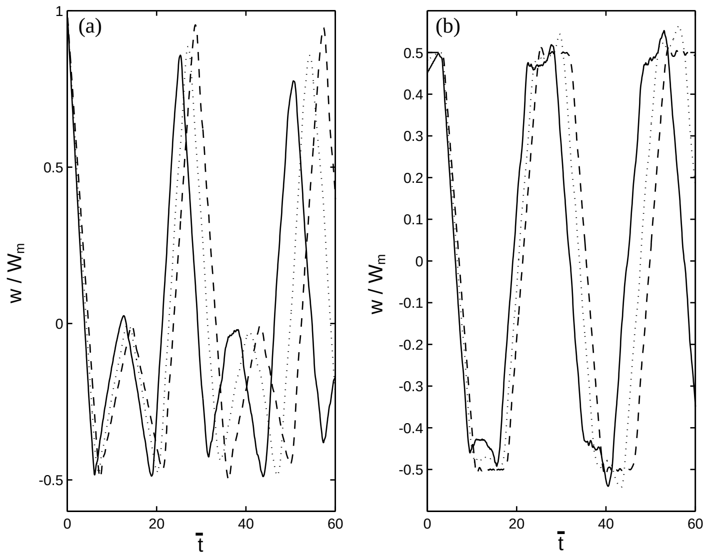

Fig. 4. Perpendicular vibrations normalized with the plucked amplitude $\bar{W}_m$ for the same conditions as in Fig. 3, except for the plucked amplitudes which are $\bar{W}_m = 0.01$ (dashed line), $\bar{W}_m = 0.5$ (dotted line) and $\bar{W}_m = 1$ (solid line). The displacements shown are for (a) the plucked point $\bar{x}_1 = 0.5$ and (b) the junction point $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0$.

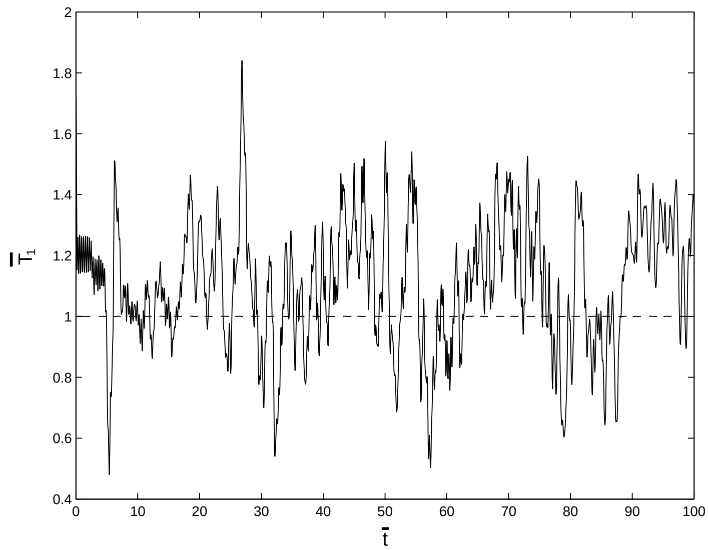

Fig. 5. Dimensionless tension, $\bar{T}_1$, in the first string at $\bar{x}_1 = 0.5$ for a plucked amplitude $\bar{W}_m = 0.01$ (dashed line) and $\bar{W}_m = 0.5$ (solid line). All other parameters are the same as in Fig. 3.

Fig. 5 shows the tension in the symmetric 3-string at the plucked point $\bar{x}_1 = 0.5$. The dashed line represents the tension when the plucked amplitude is $\bar{W}_m = 0.01$. In this case, the vibrations are of small amplitude and the tension in the strings is virtually constant. This result confirms the

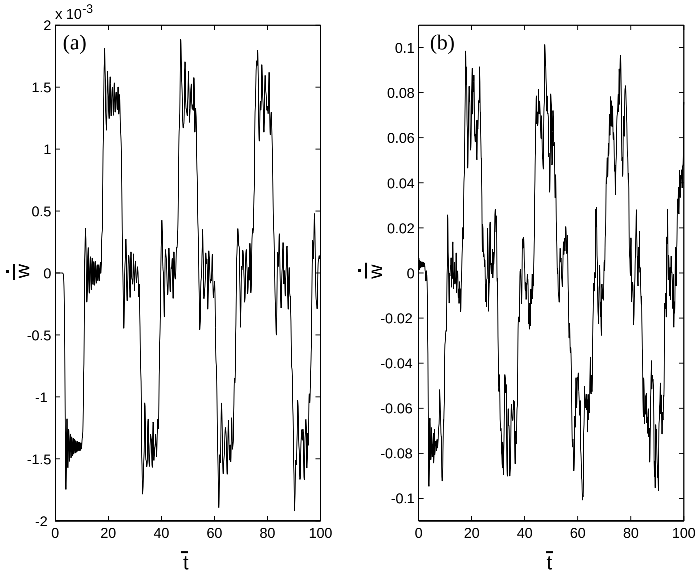

Fig. 6. Speed $d\bar{w}/d\bar{t} = \dot{\bar{w}}$ at the junction point for (a) $\bar{W}_m = 0.01$ and (b) $\bar{W}_m = 0.5$. All the other parameters are the same as in Fig. 3.

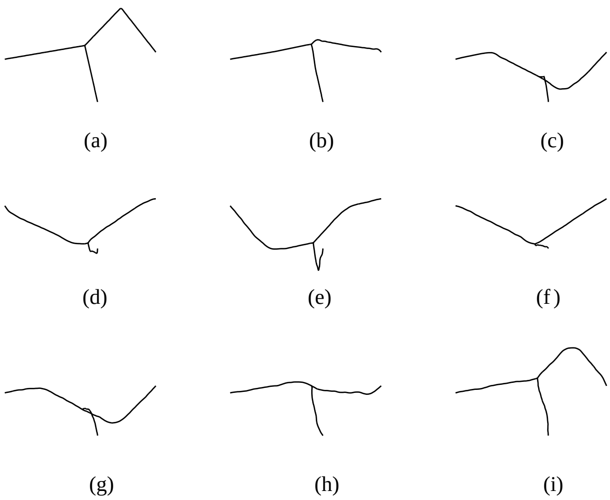

Fig. 7. Movie (a)–(i) of a symmetric 3-string as it vibrates. The parameters are the same as in Fig. 3, with $\bar{W}_m = 0.5$ at $\bar{x}_1 = 0.5$. Frame (a) corresponds to $\bar{t} = 0$.

validity of the assumption of a constant tension in the strings used in the construction of the linear model. The solid curve corresponds to the tension at $\bar{x}_1 = 0.5$ for a plucked height of $\bar{W}_m = 0.5$.

The two graphs of Fig. 6 show the speed, $d\bar{w}/d\bar{t}$, at the junction point. The first graph is for a plucked height $\bar{W}_m = 0.01$, while for the second graph it is $\bar{W}_m = 0.5$. The maximum speed in the second graph is roughly 50 times the maximum speed in the first graph.

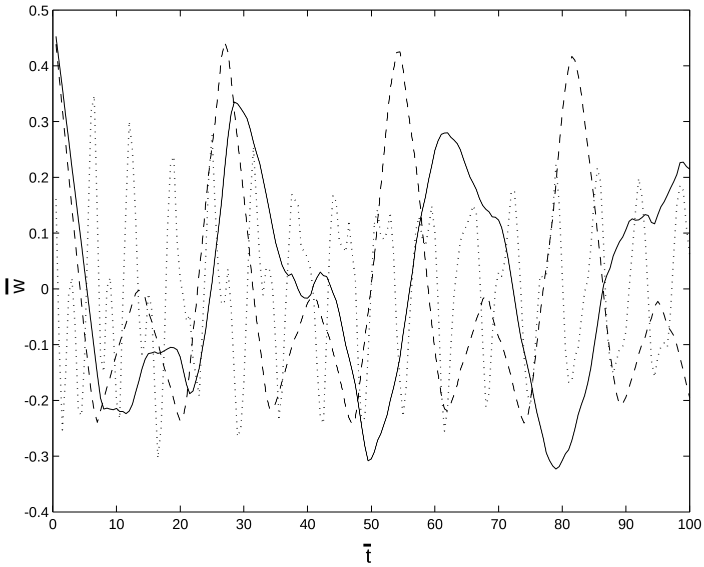

Fig. 8. A comparison of the perpendicular component $\bar{w}$ of vibration for different values of $\mathscr{E}_i$ and $\bar{W}_m = 0.5$ at $\bar{x}_1 = 0.5$. The curves shown represent the displacements at the point $\bar{x}_1 = 0.5$ for the values $\mathscr{E}_i = 200$ (dotted line), $\mathscr{E}_i = 2$ (dashed line) and $\mathscr{E}_i = 0.02$ (solid line), $i = 1, 2, 3$. All the other parameters are the same as in Fig. 3.

Fig. 7 contains nine frames of a movie showing the configurations of a symmetric 3-string as it vibrates. The physical parameters for the simulation are the same here as in Fig. 3, with $\bar{W}_m = 0.5$ at $\bar{x}_1 = 0.5$.

In Fig. 8, the perpendicular component of vibration, $\bar{w}$, is shown for cases where the components of vibration are coupled. The dashed line in this figure corresponds to the case $\mathscr{T}_{i0} = 1$ and $\mathscr{E}_i = 2$, $i = 1, 2, 3$, and still resembles the curves in Fig. 4. However, a dramatic departure from the linear decoupled case is observed for the two other curves which correspond to $\mathscr{E}_i = 200$ and 0.02. These cases could be interpreted as a 3-string where the initial tension has been decreased and increased, respectively, or where the modulus of elasticity has been increased or decreased, respectively. The frequency of vibration does increase with an increased initial tension in the strings, despite appearances in Fig. 8 where the opposite seems true, because of our definition of t_{ref}.

In Fig. 9, a movie is shown of a vibrating symmetric 3-string whose first string has been plucked in its equilibrium plane perpendicular to the string. More precisely, we have $\bar{U}_m = \bar{W}_m = 0$ and $\bar{V}_m = 0.1$. The viewpoint here is perpendicular to the 3-string equilibrium plane.

7. Discussion

In this paper, we have obtained a mathematical model which makes it possible to numerically describe the non-linear vibrations of any kind of N-string initially plucked in any direction at one

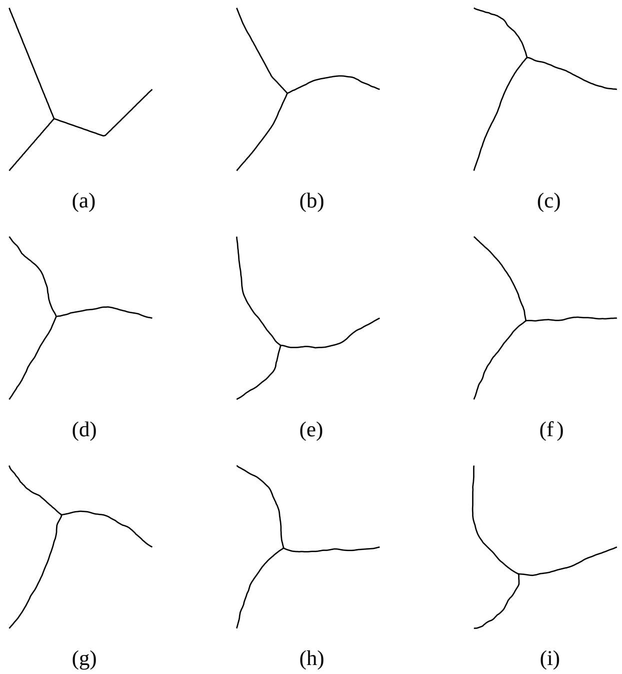

Fig. 9. Movie (a)–(i) of a symmetric 3-string as it vibrates. The 3-string has been plucked in its equilibrium plane, on the first string and perpendicular to the string with $\bar{V}_m = 0.1, \bar{U}_m = \bar{W}_m = 0$, $\mathscr{L}_i = \mathscr{T}_{i0} = 1$, $\mathscr{E}_i = 2$ and $J_i = 51$ for $i = 1, 2, 3$. Frame (a) corresponds to $\bar{t} = 0$.

point. For small-amplitude vibrations perpendicular to the equilibrium plane of the N-string, the numerical results coincide with the analytical solutions found in Ref. [3]. Our numerical model allows one to describe any 3-D vibrations, be they of small or large amplitude, of any N-string whose N strings can be of different or variable densities, of different lengths, have different initial tensions or have different moduli of elasticity.

Let us add a few words about the numerical results of Section 6. We know that the $\bar{u}$ and $\bar{v}$ vibration components are coupled if the projection, in its equilibrium plane, of the static plucked configuration of the 3-string is not geometrically symmetric about the junction point. Considering a symmetric 3-string whose strings are such that $\mathscr{T}_{i0} = \mathscr{E}_i$, $i = 1, 2, 3$, and vibrations that are initiated by a general plucking (which implies that the 3-string does not start from a horizontally symmetric configuration), then the numerical results show only a slight departure from the linear case for plucking amplitudes as large as about half the length of each string. However, the non-linear effects become dominant when $\mathscr{T}_{i0} \neq \mathscr{E}_i$, and this results in non-periodic vibrations. Although not presented in this paper, we have also generated simulations of a 3-string subjected to initial 3-D velocity conditions instead of plucking. This type of initial conditions is actually easier to treat then the one examined in this paper.

No attempt has been made to explore the vast parameter space related to the problem of a vibrating N-string. However, we have laid the groundwork for such a study by recasting the equations in a dimensionless form, and identifying all the independent dimensionless physical parameters, for the case where the ρ_i are constant on each string.

The model presented in this paper can be generalized to study the non-linear vibrations of more complex networks of strings, including the controllability of such networks and their acoustic characteristics. As we noted in Ref. [3], N strings can be used to produce sounds with unique tone colours. A more realistic mathematical model than those of Ref. [3] and the present paper should incorporate friction, such as air resistance. Work on this and on the construction of a physical model of an N-string are in progress and will be reported in a subsequent paper.

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