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Letter to the Editor

In-plane rotational and thickness-twist vibrations of polygonal plates and spherical shallow shells

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1. Introduction

Vibration of a simply supported polygonal plate can be linked to a membrane vibration problem. The relevant literature has been reviewed in Refs. [1,2] (see references cited) for a single-layer plate and a symmetrically sandwich plate. More recently, the linking relations are extended to an inhomogeneous plate and a spherical shallow shell [3–7]. Among the numerous works, Irschik [8] is the only one who has studied the thickness-twist mode of a vibrating shear deformable single-layer plate. The others only considered flexural and thickness-shear modes for a mid-plane-symmetric plate as well as an in-plane dilatational mode for a materially asymmetric plate about its mid-plane and a spherical shallow shell. Therefore, the missing in-plane rotational and thickness-twist modes need an additional treatment.

There are five unknowns to be solved in Reddy's third order plate theory [9–11] for an inhomogeneous plate. For the vibration problem of a simply supported polygonal plate, the predominantly in-plane dilatational, flexural and thickness-shear modes have been addressed in Ref. [5]. It is governed by Dirichlet's eigenvalue problem, mathematically similar to a vibrating membrane with fixed edges. As will be seen in the present paper, the vibration associated with in-plane rotational and thickness-twist modes, which is decoupled from the other three modes, is governed by Neumann's eigenvalue problem. The frequencies associated with the in-plane rotational and thickness-twist modes are linked to the frequency of a vibrating membrane with sliding edges. The frequency correspondences are established between the third order, first order and classical plate theories via the membrane analogy. These correspondence relations also apply to a spherical shallow shell. It is found that the new frequency correspondences are independent of the elastic foundation parameters, hydrostatic in-plane pressure and radius of the spherical shallow shell.

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2. Reddy's third order theory for inhomogeneous isotropic plates

The present work starts from a plate of uniform thickness h, resting on a Winkler–Pasternak elastic foundation, and subjected to in-plane initial hydrostatic pressure N per unit edge length. The plate is made of an inhomogeneous material, and the material properties vary only in its thickness direction. Let $\{x_i\}$ be the Cartesian co-ordinate system and $x_3 = 0$ is the mid-plane of the undeformed plate. Hereafter, a comma followed by a subscript i denotes the partial derivative with respect to x_i, and a repeated index implies summation over the range of the index with Latin indices ranging from 1 to 3 and Greek indices from 1 to 2.

Using the Reddy third order plate theory [9–11], the steady state linear governing equations are expressed as [5]

$$N_{\alpha\beta,\beta} + I_0\omega^2 u_\alpha - I_4\omega^2 u_{3,\alpha} + I_5\omega^2\varphi_\alpha = 0, \tag{1}$$

$$M_{\alpha\beta,\alpha\beta} - Nu_{3,\alpha\alpha} - ku_3 + Gu_{3,\alpha\alpha} + I_4\omega^2 u_{\alpha,\alpha} + I_0\omega^2 u_3 - I_1\omega^2 u_{3,\alpha\alpha} + I_2\omega^2\varphi_{\alpha,\alpha} = 0, \tag{2}$$

$$P_{\alpha\beta,\beta} - R_\alpha + I_5\omega^2 u_\alpha - I_2\omega^2 u_{3,\alpha} + I_3\omega^2\varphi_\alpha = 0, \tag{3}$$

where u_α, u_3 and φ_α are five basic unknowns on the plate mid-plane, ω denotes an angular frequency, k and G denote the Winkler–Pasternak foundation parameters [12], and

$$\begin{bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\alpha\beta} \end{bmatrix} = (\mathbf{a} - \mathbf{b}) \begin{bmatrix} u_{\omega,\omega} \\ -u_{3,\omega\omega} \\ \varphi_{\omega,\omega} \end{bmatrix} \delta_{\alpha\beta} + \mathbf{b} \begin{bmatrix} \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \\ -u_{3,\alpha\beta} \\ \frac{1}{2}(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}) \end{bmatrix}, \quad R_\alpha = c\varphi_\alpha,$$

$$\mathbf{a} \equiv \begin{bmatrix} a_0 & a_4 & a_5 \\ a_4 & a_1 & a_2 \\ a_5 & a_2 & a_3 \end{bmatrix} = \int_{-h/2}^{h/2} \mathbf{F}\frac{E}{1-v^2}\,\mathrm{d}x_3, \quad \mathbf{b} \equiv \begin{bmatrix} b_0 & b_4 & b_5 \\ b_4 & b_1 & b_2 \\ b_5 & b_2 & b_3 \end{bmatrix} = \int_{-h/2}^{h/2} \mathbf{F}\frac{E}{1+v}\,\mathrm{d}x_3, \tag{4}$$

$$\mathbf{I} \equiv \begin{bmatrix} I_0 & I_4 & I_5 \\ I_4 & I_1 & I_2 \\ I_5 & I_2 & I_3 \end{bmatrix} = \int_{-h/2}^{h/2} \mathbf{F}\rho\,\mathrm{d}x_3, \quad c = \int_{-h/2}^{h/2} (g_{,3})^2\mu\,\mathrm{d}x_3, \tag{5}$$

$$\mathbf{F} = \begin{bmatrix} 1 & x_3 & g \\ x_3 & x_3^2 & x_3 g \\ g & x_3 g & g^2 \end{bmatrix}, \quad g(x_3) = x_3\left(1 - \frac{4x_3^2}{3h^2}\right). \tag{6}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. $E \equiv E(x_3)$, $v \equiv v(x_3)$, $\mu \equiv \mu(x_3)$ and $\rho \equiv \rho(x_3)$ are Young's modulus, the Poisson ratio, the shear modulus and the mass density, respectively. For an isotropic material, $\mu = E/2(1+v)$. For a transversely isotropic material, μ is taken as an extra independent material parameter, i.e., the shear modulus normal to the isotropy plane $x_3 = \text{constant}$. Substituting Eq. (4) into Eqs. (1)–(3) yields

$$\begin{aligned} &\tfrac{1}{2}b_0 u_{\alpha,\beta\beta} + (a_0 - \tfrac{1}{2}b_0)u_{\beta,\beta\alpha} - a_4 u_{3,\alpha\beta\beta} + \tfrac{1}{2}b_5\varphi_{\alpha,\beta\beta} + (a_5 - \tfrac{1}{2}b_5)\varphi_{\beta,\beta\alpha} \\ &\quad + I_0\omega^2 u_\alpha - I_4\omega^2 u_{3,\alpha} + I_5\omega^2\varphi_\alpha = 0, \end{aligned} \tag{7}$$

$$a_4 u_{\alpha,\alpha\beta\beta} - a_1 u_{3,\alpha\alpha\beta\beta} + a_2 \varphi_{\alpha,\alpha\beta\beta} - N u_{3,\alpha\alpha} - k u_3 + G u_{3,\alpha\alpha} + I_4\omega^2 u_{\alpha,\alpha} + I_0\omega^2 u_3 - I_1\omega^2 u_{3,\alpha\alpha} + I_2\omega^2\varphi_{\alpha,\alpha} = 0, \tag{8}$$

$$\tfrac{1}{2}b_5 u_{\alpha,\beta\beta} + (a_5 - \tfrac{1}{2}b_5)u_{\beta,\beta\alpha} - a_2 u_{3,\alpha\beta\beta} + \tfrac{1}{2}b_3\varphi_{\alpha,\beta\beta} + (a_3 - \tfrac{1}{2}b_3)\varphi_{\beta,\beta\alpha} - c\varphi_\alpha + I_5\omega^2 u_\alpha - I_2\omega^2 u_{3,\alpha} + I_3\omega^2\varphi_\alpha = 0. \tag{9}$$

We decompose u_α and φ_α into their potential and solenoidal parts as

$$u_\alpha = U_{,\alpha} + \varepsilon_{\alpha\omega} V_{,\omega}, \quad \varphi_\alpha = \Phi_{,\alpha} + \varepsilon_{\alpha\omega}\Psi_{,\omega}, \tag{10}$$

where $\varepsilon_{\alpha\omega}$ is the two-dimensional permutation tensor. Then, Eqs. (7)–(9) become

$$\left(a_0\nabla^2 U - a_4\nabla^2 u_3 + a_5\nabla^2\Phi + I_0\omega^2 U - I_4\omega^2 u_3 + I_5\omega^2\Phi\right)_{,\alpha} + \varepsilon_{\alpha\omega}\left(\tfrac{1}{2}b_0\nabla^2 V + \tfrac{1}{2}b_5\nabla^2\Psi + I_0\omega^2 V + I_5\omega^2\Psi\right)_{,\omega} = 0, \tag{11}$$

$$a_4\nabla^4 U - a_1\nabla^4 u_3 + a_2\nabla^4\Phi - N\nabla^2 u_3 - k u_3 + G\nabla^2 u_3 + I_4\omega^2\nabla^2 U + I_0\omega^2 u_3 - I_1\omega^2\nabla^2 u_3 + I_2\omega^2\nabla^2\Phi = 0, \tag{12}$$

$$\left(a_5\nabla^2 U - a_2\nabla^2 u_3 + a_3\nabla^2\Phi - c\Phi + I_5\omega^2 U - I_2\omega^2 u_3 + I_3\omega^2\Phi\right)_{,\alpha} + \varepsilon_{\alpha\omega}\left(\tfrac{1}{2}b_5\nabla^2 V + \tfrac{1}{2}b_3\nabla^2\Psi - c\Psi + I_5\omega^2 V + I_3\omega^2\Psi\right)_{,\omega} = 0, \tag{13}$$

in which $\nabla^2 = \partial^2/\partial x_\alpha \partial x_\alpha$ is the two-dimensional Laplace operator. Note that the following are invariants under the rotation about the x_3-axis:

$$u_{\alpha,\alpha} = \nabla^2 U, \quad \varepsilon_{\alpha\beta}u_{\alpha,\beta} = \nabla^2 V, \quad \varphi_{\alpha,\alpha} = \nabla^2\Phi, \quad \varepsilon_{\alpha\beta}\varphi_{\alpha,\beta} = \nabla^2\Psi. \tag{14}$$

The vibration modes associated with each term in Eq. (14) are called in-plane dilatational, in-plane rotational, thickness-shear and thickness-twist modes, respectively.

We can recognize that Eq. (11) is the Cauchy–Riemann equation in a tensor form. Both parts enclosed by a pair of parentheses in Eq. (11) are harmonic functions and thus satisfy a harmonic equation. This is the same case for Eq. (13). We may regroup Eqs. (11)–(13) into the following two matrix equations

$$\mathbf{K}\begin{bmatrix}\nabla^2 U \\ u_3 \\ \nabla^2\Phi\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \tag{15}$$

$$\begin{bmatrix}\tfrac{1}{2}b_0\nabla^2 + I_0\omega^2 & \tfrac{1}{2}b_5\nabla^2 + I_5\omega^2 \\ \tfrac{1}{2}b_5\nabla^2 + I_5\omega^2 & \tfrac{1}{2}b_3\nabla^2 - c + I_3\omega^2\end{bmatrix}\begin{bmatrix}\nabla^2 V \\ \nabla^2\Psi\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \tag{16}$$

where there are three equations in Eqs. (15) and two equations in Eq. (16). Eqs. $(15)_1$ and $(16)_1$ are two harmonic equations associated with the first and second harmonic functions in Eq. (11), Eq. $(15)_2$ is Eq. (12), and Eq. $(15)_3$ and Eq. $(16)_2$ are two harmonic equations associated with the first and second harmonic functions in Eq. (13). In Eq. (15), $\mathbf{K} = (K_{IJ})$ is a 3×3 operator matrix defined by

$$\begin{aligned}
&K_{11}(\nabla^2) = a_0\nabla^2 + I_0\omega^2, \quad K_{12}(\nabla^2) = -a_4\nabla^4 - I_4\omega^2\nabla^2,\\
&K_{13}(\nabla^2) = a_5\nabla^2 + I_5\omega^2, \quad K_{21}(\nabla^2) = a_4\nabla^2 + I_4\omega^2,\\
&K_{22}(\nabla^2) = -a_1\nabla^4 - (N - G + I_1\omega^2)\nabla^2 - k + I_0\omega^2,\\
&K_{23}(\nabla^2) = a_2\nabla^2 + I_2\omega^2, \quad K_{31}(\nabla^2) = a_5\nabla^2 + I_5\omega^2,\\
&K_{32}(\nabla^2) = -a_2\nabla^4 - I_2\omega^2\nabla^2, \quad K_{33}(\nabla^2) = a_3\nabla^2 - c + I_3w^2.
\end{aligned} \tag{17}$$

We notice that the functions U, u_3 and Φ have been decoupled from V and Ψ.

3. Simply supported polygonal plates

For a simply supported polygonal plate, the boundary conditions are

$$N_{NN} = 0, \quad M_{NN} = 0, \quad P_{NN} = 0, \tag{18}$$

$$u_3 = 0, \quad u_T = 0, \quad \varphi_T = 0, \quad u_{3,T} = 0, \tag{19}$$

where the upper case subscripts N and T denote, respectively, normal and tangential directions to the boundary, and the implicit summation convention does not apply to them. In terms of equations $(4)_1$ and (19), the boundary conditions (18) reduce to

$$\begin{bmatrix} N_{NN} \\ M_{NN} \\ P_{NN} \end{bmatrix} = \mathbf{a} \begin{bmatrix} u_{N,N} \\ -u_{3,NN} \\ \varphi_{N,N} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{20}$$

Recalling the Gram theorem [13], we have $\det(\mathbf{a}) > 0$ in the third order plate theory. Thus, Eq. (20) gives

$$u_{N,N} = 0, \quad u_{3,NN} = 0, \quad \varphi_{N,N} = 0 \tag{21}$$

or

$$\nabla^2 U = 0, \quad \nabla^2 u_3 = 0, \quad \nabla^2 \Phi = 0. \tag{22}$$

Using Eqs. (15), $(19)_1$ and (22) leads to

$$\mathbf{a} \begin{bmatrix} \nabla^4 U \\ -\nabla^4 u_3 \\ \nabla^4 \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

or

$$\nabla^4 U = 0, \quad \nabla^4 u_3 = 0, \quad \nabla^4 \Phi = 0, \tag{24}$$

and more generally,

$$\nabla^{2J} U = 0, \quad \nabla^{2J-2} u_3 = 0, \quad \nabla^{2J} \Phi = 0 \quad (J = 1, 2, 3, \cdots). \tag{25}$$

Using Eqs. (11), (13), (19) and (22) leads to

$$\begin{bmatrix} b_0 & b_5 \\ b_5 & b_3 \end{bmatrix} \begin{bmatrix} \nabla^2 V \\ \nabla^2 \Psi \end{bmatrix}_{,N} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{26}$$

As $b_0 b_3 - b_5^2 > 0$, we have

$$\nabla^2 V_{,N} = 0, \quad \nabla^2 \Psi_{,N} = 0, \tag{27}$$

and furthermore, using Eq. (16),

$$\nabla^4 V_{,N} = 0, \quad \nabla^4 \Psi_{,N} = 0. \tag{28}$$

4. Membrane analogy

Eq. (15) associated with the boundary conditions (25) has been studied in Ref. [5]. Eq. (16) and the associated boundary conditions (27) and (28), which were missing in that work, constitute a new eigenvalue problem. Because Eqs (7)–(9) underlying the Reddy third order plate theory are of the 12th order, it is therefore expected that there are six characteristic roots λ's. Four characteristic roots are associated with Eq. (15) and have been addressed in Ref. [5]. The remaining two come from Eq. (16).

Following the same procedure adopted in Ref. [5], we may obtain the following eigenvalue problem from Eqs. (16), (27) and (28):

$$\det[\mathbf{L}(\nabla^2)]\Lambda = \tfrac{1}{4}(b_0 b_3 - b_5^2)(\nabla^2 + \lambda_1)(\nabla^2 + \lambda_2)\Lambda = 0, \quad \Lambda_{,N}\big|_{\Gamma} = (\nabla^2 \Lambda_{,N})\big|_{\Gamma} = 0 \tag{29}$$

where

$$\mathbf{L}(\nabla^2) = \begin{bmatrix} \frac{1}{2} b_0 \nabla^2 + I_0 \omega^2 & \frac{1}{2} b_5 \nabla^2 + I_5 \omega^2 \\ \frac{1}{2} b_5 \nabla^2 + I_5 \omega^2 & \frac{1}{2} b_3 \nabla^2 - c + I_3 \omega^2 \end{bmatrix}, \tag{30}$$

Λ is either $\nabla^2 V$ or $\nabla^2 \Psi$, Γ denotes the polygonal edges, and λ_1 and λ_2 are two roots of the quadratic equation

$$\det[\mathbf{L}(-\lambda)] = 0. \tag{31}$$

Let λ_m and H be the eigenvalue and the transverse deflection of a vibrating membrane with sliding edges. The corresponding governing equation is the Helmholtz equation and the corresponding boundary condition is of Neumann's type [14]:

$$(\nabla^2 + \lambda_{\rm m})H = 0, \quad H_{,N} = 0. \tag{32}$$

By analogy, the boundary value problem (29) for the polygonal plate is similar to Eq. (32) providing that the membrane has the same contour as the plate. The plate executes membrane-like vibration of Neumann's type and their vibration frequencies are connected by

$$\det[\mathbf{L}(-\lambda_{\rm m})] = \det\left\{ -\tfrac{1}{2}\lambda_{\rm m} \begin{bmatrix} b_0 & b_5 \\ b_5 & b_3 + 2c/\lambda_{\rm m} \end{bmatrix} + \omega^2 \begin{bmatrix} I_0 & I_5 \\ I_5 & I_3 \end{bmatrix} \right\} = 0. \tag{33}$$

Moreover, a Neumann-type eigenvalue problem contains a denumerably infinite sequence of discrete non-negative eigenvalues corresponding to non-trivial real eigenfuctions. The zero eigenvalue is associated with a constant non-zero deflection of a sliding membrane. Therefore, we have $\lambda_{\mathrm{m}} \geqslant 0$. The vibration frequencies of the polygonal plate at $\lambda_m = 0$ are

$$\omega^2_{minimum} = 0, \quad I_0 c/(I_0 I_3 - I_5^2), \tag{34}$$

where $\sqrt{I_0 c/(I_0 I_3 - I_5^2)}$ is a lower bound of the vibration frequency associated with the predominantly thickness-twist mode.

Clearly, Eq. (15) provides three natural frequencies associated with one predominantly in-plane dilatational ($u_{\alpha,\alpha}$), one predominantly flexural (u_3) and one predominantly thickness-shear ($\varphi_{\alpha,\alpha}$) vibrating mode, while Eq. (16) provides two natural frequencies associated with one predominantly in-plane rotational ($\varepsilon_{\alpha\beta} u_{\alpha,\beta}$) and one predominantly thickness-twist ($\varepsilon_{\alpha\beta}\varphi_{\alpha,\beta}$) vibrating mode. This is consistent with such a shear deformable plate theory that yields five natural frequencies. Eqs (15) and (25) constitute a boundary value problem of Dirichlet's type and have been discussed in Ref. [5]. This boundary value problem is mathematically similar to a uniform membrane whose shape coincides with the same contour as the plate, is fixed at the edges and is executing small transverse vibration. Eqs (16), (27) and (28) constitute a boundary value problem of Neumann's type. This boundary value problem is mathematically similar to a vibrating uniform membrane with sliding edges.

5. The first-order and classical plate theories

The first order shear deformation plate theory and the classical plate theory correspond to $g(x_3) = x_3$ and 0 instead of Eq. $(6)_2$. Without giving details of the derivation, the final result of the frequency correspondence between a polygonal inhomogeneous plate using the first order and classical plate theories and a membrane with sliding edges of same contour is

$$\det\left\{ -\tfrac{1}{2}\lambda_m \begin{bmatrix} b_0 & b_4 \\ b_4 & b_1 + 2c_f/\lambda_m \end{bmatrix} + \omega_f^2 \begin{bmatrix} I_0 & I_4 \\ I_4 & I_1 \end{bmatrix} \right\} = 0, \tag{35}$$

$$-\tfrac{1}{2}\lambda_m b_0 + I_0 \omega_c^2 = 0, \tag{36}$$

where ω_f and ω_c are frequencies associated with the first-order and classical plate theories, and

$$c_f = \kappa \int_{-h/2}^{h/2} \mu \, \mathrm{d}x_3 \tag{37}$$

with the shear correction factor κ. Due to the assumption in the classical plate theory that a normal to the mid-plane of an undeformed plate remains normal to the mid-plane during deformation, the frequency ω_c is only associated with the in-plane rotational vibration mode and the thickness-twist vibration cannot be predicted by the classical plate theory.

6. A spherical shallow shell of polygonal planform

The vibration problem of a simply supported spherical shallow shell of polygonal planform has been studied in Refs. [6,7]. It is found that the shell exhibits membrane-like vibration. Using the third order and the first order shear deformation theories and the classical theory, exact vibration frequencies of the spherical shallow shell associated with the predominantly in-plane dilatational, flexural and thickness-shear vibrations are obtained in terms of the frequency of a vibrating membrane. The corresponding membrane is flat, fixed at its edges, and its contour coincides with that of the shell planform. However, the vibration problem associated with predominantly in-plane rotational and thickness-twist modes were not addressed therein.

The detailed governing equations and boundary conditions for a simply supported spherical shallow shell of polygonal planform have been given in Ref. [7], where the hydrostatic in-surface pressure N was excluded. They are not duplicated herein for brevity. The vibration eigenvalue problem associated with predominantly in-plane rotational and thickness-twist modes can be shown to be exactly the same as Eqs. (16), (27) and (28). Thus, the final frequency correspondences between a vibrating membrane with sliding edges and a simply supported spherical shallow shell of polygonal planform using the Reddy third order plate theory, the first-order plate theory and the classical plate theory are the same as Eqs. (33), (35) and (36). It is interesting to find that these relations are independent of the radius of the spherical shallow shell.

7. Independent check

In order to conduct an independent check on the frequency correspondences between the membrane eigenvalues and the vibration frequencies of a polygonal plate, we select a homogeneous single-layer rectangular plate as an example for which the three-dimensional elastic solution is available. Because the plate is symmetric about its mid-plane, the vibration frequency associated with the in-plane rotational mode is decoupled with the frequency associated with the thickness-twist mode. From Eqs (33) and (35), the frequency associated with the thickness-twist mode is predicted by Reddy's third order plate theory and Mindlin's first order plate theory [15] as

$$\bar{\omega}^2 = \bar{\lambda}_m + 168/17, \quad \bar{\omega}_f^2 = \bar{\lambda}_m + \pi^2, \tag{38}$$

where

$$\bar{\omega} = \omega h\sqrt{\rho/\mu}, \quad \bar{\omega}_f = \omega_f h\sqrt{\rho/\mu}, \quad \bar{\lambda}_m = \lambda_m h^2. \tag{39}$$

It is known that the vibration eigenvalue and mode of a rectangular membrane is given by [14]

$$\lambda_m = \pi^2\left(\frac{k^2}{a^2} + \frac{l^2}{b^2}\right), \quad H = H_0 \cos\frac{k\pi x_1}{a}\cos\frac{l\pi x_2}{b}. \tag{40}$$

Table 1 shows some numerical results calculated from Eq. (38) for $a = b = 10h$. These results are exactly the same as those obtained in Ref. [16] by directly solving the shear deformable plate problem. This confirms the correctness of our frequency correspondences presented in this paper.

Table 1
Independent check on the correspondence relation of the frequencies associated with the thickness-twist mode ($a = b = 10h$)

k	l	Membrane eigenvalue $\bar{\lambda}_m$	First order/3-D exact $\bar{\omega}_f$ ($\kappa = \pi^2/12$)	Reddy's 3rd order $\bar{\omega}$
1	1	0.1974	3.1729	3.1749
1	2	0.4935	3.2192	3.2212
1	3	0.9870	3.2949	3.2969
2	2	0.7896	3.2648	3.2668
2	3	1.2830	3.3396	3.3415
2	4	1.9739	3.4414	3.4433
3	3	1.7765	3.4126	3.4145
4	4	3.1583	3.6094	3.6112

Note that the exact three-dimensional elastic solution matches with that from the Mindlin plate theory. This is because the shear correction factor was purposely taken as $\kappa = \pi^2/12$ in the Mindlin first order plate theory [15] in order to match the three-dimensional solution.

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