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Letter to the Editor

Relaxation oscillations in a system with a piecewise smooth drag coefficient

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The original Van der Pol equation [1] is

$$\ddot{x} + \varepsilon V(x)\dot{x} + x = 0, \tag{1}$$

where

$$V(x) = x^2 - 1. \tag{2}$$

Piecewise smooth systems occur in a wide variety of applications, including those involving friction, backlash and saturation [2], as well as in electrical systems [3]. As part of a much wider study of these systems (see for example Ref. [4]), $V(x)$ is replaced with a term of the form

$$\begin{aligned} V(x) &= |x| - 1 \\ &= \operatorname{sgn}(x)x - 1. \end{aligned} \tag{3}$$

The form of $V(x)$ in Eq. (3) is chosen such that its lower bound is -1 and it vanishes at $x = \pm 1$, just as in Eq. (2). The behaviour of the system will be investigated for small ε and for large ε. Note that this equation is mentioned in both Refs. [5, p. 150] and [6, p. 134], but an analysis of its solution does not seem to have appeared in print to date.

It will be first demonstrated that a limit cycle exists for Eq. (1), with $V(x)$ given by Eq. (3). Theorem 11.4 of Ref. [5] states that the equation $\ddot{x} + \varepsilon f(x)\dot{x} + g(x) = 0$ has a unique periodic solution if f and g are continuous, if $F(x) \equiv \int_0^x f(u)\,\mathrm{d}u$ is an odd function, if $F(x)$ is zero only at $x = 0$, $x = c$, $x = -c$, for some $c > 0$ and if $F(x) \to \infty$ as $x \to \infty$ monotonically for $x > c$. In addition, $g(x)$ must be an odd function, and $g(x) > 0$ for $x > 0$. Here

$$F(x) = \begin{cases} \left(\dfrac{x^2}{2} - x\right), & x > 0, \\ \left(-\dfrac{x^2}{2} - x\right), & x < 0 \end{cases} \tag{4}$$

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and $g(x) = x$. Hence, the conditions of the theorem are satisfied, with $c = 2$, and so a limit cycle exists.

When ε is small, it is assumed that the period of oscillation is 2π and take the solution to be of the form $x(t) = a\cos t$. There will be no change in the energy of the system over the course of a limit cycle, and so

$$\int_0^{2\pi} h(x,\dot{x})\dot{x}\,\mathrm{d}t = 0, \tag{5}$$

where

$$h(x,\dot{x}) = \begin{cases} (x-1)\dot{x}, & x > 0, \\ (-x-1)\dot{x}, & x < 0. \end{cases} \tag{6}$$

Hence,

$$-a\left\{\int_0^{\pi/2}(x-1)\dot{x}\sin t\,\mathrm{d}t + \int_{\pi/2}^{3\pi/2}(-x-1)\dot{x}\sin t\,\mathrm{d}t + \int_{3\pi/2}^{2\pi}(x-1)\dot{x}\sin t\,\mathrm{d}t\right\} = 0. \tag{7}$$

Using the substitution $\rho = \sin t$ one finds that

$$\int_0^1 a\rho^2\,\mathrm{d}\rho + \int_1^{-1} -a\rho^2\,\mathrm{d}\rho + \int_{-1}^0 a\rho^2\,\mathrm{d}\rho - \int_0^{2\pi}\sin^2 t\,\mathrm{d}t = 0 \tag{8}$$

and so

$$a = \frac{3\pi}{4}. \tag{9}$$

It is straightfoward to show that this limit cycle is stable for $\varepsilon > 0$ and unstable for $\varepsilon < 0$ and that its frequency ω is given by $\omega = 1 + O(\varepsilon^2)$. (Note that when $V(x) = x^2 - 1$, one has $a = 2$ [1] and that when $V(x) = |\dot{x}| - 1$, $a = 3\pi/8$ [5, p. 144].)

One can show how the system evolves to this limit cycle, using the method of multiple scales. Write

$$\mathrm{sgn}(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-)^n}{(2n+1)}\cos(2n+1)t, \tag{10}$$

and set $\tau = t$ to represent the fast time scale of oscillations, and $T = \varepsilon t$ to represent the slow amplitude drift. Hence,

$$\dot{x} = \frac{\partial x}{\partial \tau} + \varepsilon\frac{\partial x}{\partial T} \tag{11}$$

and

$$\ddot{x} = \frac{\partial^2 x}{\partial \tau^2} + 2\varepsilon\frac{\partial^2 x}{\partial \tau\,\partial T} + \varepsilon^2\frac{\partial^2 x}{\partial T^2}. \tag{12}$$

Writing

$$\frac{\partial x}{\partial \tau} = x_\tau, \quad \frac{\partial x}{\partial T} = x_T, \quad \frac{\partial^2 x}{\partial \tau^2} = x_{\tau\tau}, \quad \frac{\partial^2 x}{\partial T^2} = x_{TT}, \quad \frac{\partial^2 x}{\partial \tau\,\partial T} = x_{\tau T},$$

setting

$$x(t) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + O(\varepsilon^2), \tag{13}$$

substituting Eqs. (11)–(13) into Eq. (1) and using Eq. (3), to leading order,

$$x_{0\tau\tau} + x_0 = 0 \tag{14}$$

and at first order,

$$x_{1\tau\tau} + x_1 = x_{0\tau}(1 - \mathrm{sgn}(x)x_0) - 2x_{0\tau T}. \tag{15}$$

The solution to Eq. (14) is

$$x_0 = A(T)\,\mathrm{e}^{i\tau} + A^*(T)\,\mathrm{e}^{-i\tau} \tag{16}$$

and so, using Eqs. (10) and (16), Eq. (15) becomes

$$\begin{aligned} &x_{1\tau\tau} + x_1 \\ &= (\mathrm{i}A\mathrm{e}^{\mathrm{i}\tau} - \mathrm{i}A^*\mathrm{e}^{-\mathrm{i}\tau}) \\ &\quad\times\left(1 - (A\mathrm{e}^{\mathrm{i}\tau} + A^*\mathrm{e}^{-\mathrm{i}\tau})\frac{2}{\pi}\left(\sum_{j=0}^{\infty}\frac{(-)^j}{(2j+1)}A\left[\mathrm{e}^{\mathrm{i}(2j+1)\tau} + \mathrm{e}^{-\mathrm{i}(2j+1)\tau}\right]\right)\right) \\ &\quad - 2\mathrm{i}A_T\mathrm{e}^{\mathrm{i}\tau} + 2\mathrm{i}A_T^*\mathrm{e}^{-\mathrm{i}\tau} \end{aligned} \tag{17}$$

which can be rewritten as

$$\begin{aligned} &x_{1\tau\tau} + x_1 \\ &= -2\mathrm{i}A_T\mathrm{e}^{\mathrm{i}\tau} + 2\mathrm{i}A_T^*\mathrm{e}^{-\mathrm{i}\tau} + \mathrm{i}A\mathrm{e}^{\mathrm{i}\tau} - \mathrm{i}A^*\mathrm{e}^{-\mathrm{i}\tau} \\ &\quad -\frac{2}{\pi}\left\{\left(\sum_{j=0}^{\infty}\frac{(-)^j}{(2j+1)}\mathrm{i}A^2\left[\mathrm{e}^{\mathrm{i}(2j+3)\tau} + \mathrm{e}^{\mathrm{i}(1-2j)\tau}\right]\right) + \left(\sum_{j=0}^{\infty}\frac{(-)^j}{(2j+1)}\mathrm{i}|A|^2\left[\mathrm{e}^{\mathrm{i}(2j+1)\tau} + \mathrm{e}^{-\mathrm{i}(2j+1)\tau}\right]\right)\right. \\ &\quad\left.-\left(\sum_{j=0}^{\infty}\frac{(-)^j}{(2j+1)}\mathrm{i}|A|^2\left[\mathrm{e}^{\mathrm{i}(2j+1)\tau} + \mathrm{e}^{-\mathrm{i}(2j+1)\tau}\right]\right) - \left(\sum_{j=0}^{\infty}\frac{(-)^j}{(2j+1)}\mathrm{i}A^{*2}\left[\mathrm{e}^{\mathrm{i}(2j-1)\tau} + \mathrm{e}^{-\mathrm{i}(2j+3)\tau}\right]\right)\right\}. \end{aligned} \tag{18}$$

The required secularity condition is

$$2A_T = A - \frac{2}{\pi}\left(A^2 + \frac{A^{*2}}{3}\right). \tag{19}$$

Note that the substitution $\mathrm{sgn}(x) = \frac{4}{\pi}\cos t$ leads to the omission of the final term in Eq. (19). In fact it is necessary to keep terms up to and including $j = 1$ at this order.

Setting

$$A = \tfrac{1}{2}a(T)\mathrm{e}^{\mathrm{i}\theta(T)} \tag{20}$$

and substituting this into Eq. (19), it is straightforward to show that $\theta(T)$ is identically zero and that

$$2\frac{\mathrm{d}a}{\mathrm{d}T} = a - \frac{4a^2}{3\pi}. \tag{21}$$

The solution to Eq. (21) is

$$a(T) = \frac{3\pi a_0}{\left[4a_0 - (4a_0 - 3\pi)e^{-(1/2)T}\right]}, \tag{22}$$

where a_0 is the initial value of a. Hence, $a(t) \to 3\pi/4$ as $t \to \infty$, in agreement with Eq. (9). It has been shown that the limit cycle of a piecewise smooth version of the Van der Pol equation can be solved for small ε. The Fourier series expansion of $\mathrm{sgn}(x)$ is required, but only two terms are needed to successfully analyze the system to first order.

For large ε, take $t = \varepsilon t'$, set $\delta = 1/\varepsilon^2$ and drop the primes. Then Eq, (1) becomes

$$\delta\ddot{x} + V(x)\dot{x} + x = 0. \tag{23}$$

Using the Lienard transformation [5] gives

$$\begin{aligned} \dot{y} &= -x, \\ \delta\dot{x} &= y - F(x), \end{aligned} \tag{24}$$

where $F(x)$ is given by Eq. (4). Hence, for large ε (that is, small δ), one can see that $y \to F(x)$. The time taken to complete a limit cycle in this limit is given by

$$S = \oint \mathrm{d}t = \int \frac{\mathrm{d}y}{\dot{y}}. \tag{25}$$

As with the Van der Pol equation, the response in this limit is made up of a fast phase (which is taken to be negligible) and a slow phase. The function $F(x)$ has extreme values of $\pm\frac{1}{2}$ at $x = \mp 1$, respectively. The slow phase begins at $(x, F(x)) = (1 + \sqrt{2}, \frac{1}{2})$ and ends at the minimum of $F(x)$ given by $(x, F(x)) = (1, -\frac{1}{2})$. The slow phase starts again at $(x, F(x)) = (-1 - \sqrt{2}, -\frac{1}{2})$ and ends again at the maximum of $F(x)$ given by $(x, F(x)) = (-1, \frac{1}{2})$. Hence, Eq. (25) becomes

$$\begin{aligned} S &= 2\int \frac{F(x)}{-x}\,\mathrm{d}x \\ &= 2\int_{1+\sqrt{2}}^{1} \left(-1 + \frac{1}{x}\right)\mathrm{d}x \\ &= 2[-x + \ln x]_{1+\sqrt{2}}^{1} \\ &= 2\left[\sqrt{2} - \ln\left(1 + \sqrt{2}\right)\right]. \end{aligned} \tag{26}$$

The period of oscillation of the limit cycle of Eq. (1) and (3) for large ε is therefore given by $\left(2\left[\sqrt{2} - \ln(1 + \sqrt{2})\right]\right)\varepsilon \simeq 1.07\varepsilon$, with the discontinuity in the gradient of $V(x)$ having no effect on the result at this order because it is contained in the fast phase of the response. (Note that for the Van der Pol equation [5], the period of oscillation for large ε is $(3 - \ln 4)\varepsilon \simeq 1.61\varepsilon$.)

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