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Moment Lyapunov exponents of a two-dimensional system under bounded noise parametric excitation

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Abstract

The moment Lyapunov exponents of a two-dimensional system under bounded noise parametric excitation are studied in this paper. The method of regular perturbation is applied to obtain weak noise expansions of the moment Lyapunov exponent, Lyapunov exponent, and stability index in terms of the small fluctuation parameter.
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1. Introduction

Loadings imposed on structures are quite often random forces, such as those arising from earthquakes, wind and ocean waves, which can be described satisfactorily only in probabilistic terms. Under the action of such loadings, the parameters that describe the motion of the structure will fluctuate in a stochastic manner. The response of the structure is governed by stochastic differential equations, in which the parameters or coefficients are stochastic processes. Investigations of stability under parametric stochastic excitation have become increasingly important.

The dynamic stability behaviour of the following dimensionless, parametrically excited, two-dimensional system is of interest:

$$\frac{d^2 q(\tau)}{d\tau^2} + 2\beta\frac{dq(\tau)}{d\tau} + [\omega_0^2 - \varepsilon_0\eta(\tau)]q(\tau) = 0, \tag{1}$$

where τ is the time variable, $q(\tau)$ the generalized co-ordinate, β the damping constant, ω_0 the circular natural frequency of the system, $\varepsilon_0 > 0$ a small fluctuation parameter, and $\eta(\tau)$ the parametric noise process.

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The sample or almost-sure stability of the trivial solution of system (1) is determined by the Lyapunov exponent, which characterizes the average exponential rate of growth of the solutions of system (1) for τ large, defined as

$$\lambda_{q(\tau)} = \lim_{\tau\to\infty} \frac{1}{\tau}\log\|\mathbf{q}(\tau)\|, \tag{2}$$

where $\mathbf{q}(\tau) = \{q(\tau), q'(\tau)\}^{\mathrm{T}}$ and $\|\cdot\|$ denotes the Euclidean vector norm. Depending on the initial conditions $q(0)$ and $q'(0)$, there are two Lyapunov exponents for system (1). The trivial solution of system (1) is stable with probability one if the top Lyapunov exponent is negative, whereas it is unstable with probability one if the top Lyapunov exponent is positive.

On the other hand, the stability of the pth moment, for any real values of p, of the trivial solution of system (1), $E[\|\mathbf{q}(\tau)\|^p]$, is determined by the moment Lyapunov exponent

$$\Lambda_{q(\tau)}(p) = \lim_{\tau\to\infty} \frac{1}{\tau}\log E[\|\mathbf{q}(\tau)\|^p], \tag{3}$$

where $E[\cdot]$ denotes expected value. If $\Lambda_{q(\tau)}(p)<0$, then $E[\|\mathbf{q}(t)\|^p]\to 0$ as $\tau\to\infty$.

The pth moment Lyapunov exponent $\Lambda_{q(\tau)}(p)$ is a convex analytic function in p with $\Lambda_{q(\tau)}(0) = 0$ and $\Lambda'_{q(\tau)}(0)$ equal to the top Lyapunov exponent $\lambda_{q(\tau)}$. The non-trivial zero $\delta_{q(\tau)}$ of $\Lambda_{q(\tau)}(p)$, i.e. $\Lambda_{q(\tau)}(\delta_{q(\tau)}) = 0$, is called the stability index.

However, suppose the top Lyapunov exponent $\lambda_{q(\tau)}$ is negative, implying that system (1) is sample stable, the pth moment typically grows exponentially for large enough p, implying that the pth moment of system (1) is unstable. This can be explained by large deviation. Although the solution of the system $\|\mathbf{q}(\tau)\|\to 0$ as $\tau\to\infty$ with probability one at an exponential rate $\lambda_{q(\tau)}$, there is a small probability that $\|\mathbf{q}(\tau)\|$ is large, which makes the expected value $E[\|\mathbf{q}(\tau)\|^p]$ of this rare event large for large enough values of p, leading to pth moment instability.

To have a complete picture of the dynamic stability of system (1), it is important to study both the sample stability and the pth moment stability for all real values of p, and to determine both the top Lyapunov exponent and the pth moment Lyapunov exponent.

A systematic study of moment Lyapunov exponents is presented in Ref. [1] for linear Itô systems and in Ref. [2] for linear stochastic systems under real-noise excitations. The connection between moment Lyapunov exponents and the large deviation theory was studied by Baxendale [3], Arnold and Kliemann [4], and Baxendale and Stroock [5]. A systematic presentation of the theory of random dynamical systems and a comprehensive list of references can be found in Ref. [6].

Although the moment Lyapunov exponents are important in the study of dynamic stability of stochastic systems, the actual evaluations of the moment Lyapunov exponents are very difficult. Very few results on the moment Lyapunov exponents have been published. Using the analytic property of the moment Lyapunov exponents, Arnold et al. [7] obtained expansions in terms of $\varepsilon_0 p$ under both white and real-noise excitations. However, for system (1), moment instability usually occurs for large values of p. This makes the results obtained by Arnold et al. [7] inappropriate for determining the stability index. Khasminskii and Moshchuk [8] obtained an asymptotic expansion of the moment Lyapunov exponent of system (1) under white-noise parametric excitation in terms of the small fluctuation parameter ε_0, from which the stability index was obtained.

In a recent study, Xie [9] applied a procedure similar to that employed in Khasminskii and Moshchuk [8] to obtain weak-noise expansions of the moment Lyapunov exponent, the Lyapunov exponent, and the stability index of system (1) under real noise excitation in terms of the small fluctuation parameter ε_0. The real-noise excitation $\eta(\tau)$ is characterized by an Ornstein–Uhlenbeck process given by [11]

$$\mathrm{d}\eta(\tau) = -\alpha_0 \eta(\tau)\,\mathrm{d}\tau + \sigma_0 \circ \mathrm{d}W(\tau),$$

where $W(\tau)$ is a standard Wiener process. It is well known that $\eta(\tau)$ is a normally distributed random variable, which is not bounded and may take arbitrarily large values with small probabilities, and hence may not be a realistic model of noise in many engineering applications.

In this paper, the noise $\eta(\tau)$ in system (1) is considered as a bounded noise given by

$$\eta(\tau) = \cos\left[\nu_0\tau + \sigma_0 W(\tau) + \theta\right], \tag{4}$$

in which θ is a uniformly distributed random number in $(0,\ 2\pi)$. The inclusion of the phase angle θ in Eq. (4) makes $\eta(\tau)$ a stationary process. The mean square stability, i.e., the pth moment stability with $p = 2$, of system (1) under the bounded noise excitation (4) was studied by Dimentberg [10].

Eq. (4) may be written as

$$\begin{aligned} \eta(\tau) &= \cos Z(\tau), \\ \mathrm{d}Z(\tau) &= \nu_0\,\mathrm{d}\tau + \sigma_0 \circ \mathrm{d}W(\tau), \end{aligned} \tag{5}$$

where the initial condition of $Z(\tau)$ is $Z(0) = \theta$. The correlation function of $\eta(\tau)$ is given by

$$E[\eta(\tau_1)\eta(\tau_2)] = R(\tau_1 - \tau_2) = \tfrac{1}{2}\cos \nu_0(\tau_1 - \tau_2)\exp\left(-\frac{\sigma_0^2}{2}|\tau_1 - \tau_2|\right),$$

and the spectral density function of $\eta(\tau)$ is

$$S(\omega) = \int_{-\infty}^{+\infty} R(\tau)\,\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau = \frac{\sigma_0^2(\omega^2 + \nu_0^2 + \frac{1}{4}\sigma_0^4)}{2[(\omega-\nu_0)^2 + \frac{1}{4}\sigma_0^4][(\omega+\nu_0)^2 + \frac{1}{4}\sigma_0^4]}.$$

It may be noted that the mean-square value of the bounded noise process $\eta(\tau)$ is fixed at $E[\eta^2(\tau)] = \frac{1}{2}$. The spectral density function can be made to approximate the well-known Dryden and von Karman spectra of wind turbulence by suitable choice of the parameters ν_0, σ_0, and ε_0. In the limit as σ_0 approaches infinite, the bounded noise becomes a white noise of constant spectral density. However, since the mean-square value is fixed at $\frac{1}{2}$, this constant spectral density level reduces to zero in the limit. On the other hand, in the limit as σ_0 approaches zero, the bounded noise becomes a deterministic sinusoidal function.

The bounded noise process (4) was first employed by Stratonovich [12] and has since been applied in certain engineering applications by Dimentberg [13], Wedig [14], Lin and Cai [15], and Ariaratnam [16].

It is advantageous to remove the damping term in Eq. (1) by applying the transformation $q(\tau) = x(\tau)\,\mathrm{e}^{-\beta\tau}$ to yield

$$\frac{\mathrm{d}^2x(\tau)}{\mathrm{d}\tau^2} + [\omega^2 - \varepsilon_0 \cos Z(\tau)]\, x(\tau) = 0,$$
$$\mathrm{d}Z(\tau) = \nu_0\, \mathrm{d}\tau + \sigma_0 \circ \mathrm{d}W(\tau), \tag{6}$$

where $\omega^2 = \omega_0^2 - \beta^2$.

Eq. (6) can be further simplified by employing the time scaling $t = \omega\tau$ to give

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + [1 - \varepsilon \cos \zeta(t)]\, x(t) = 0,$$
$$\mathrm{d}\zeta(t) = \nu\, \mathrm{d}t + \sigma \circ \mathrm{d}W(t), \tag{7}$$

where $\varepsilon = \varepsilon_0/\omega^2$, $\nu = \nu_0/\omega$, $\sigma = \sigma_0/\sqrt{\omega}$, and $W(t)$ is a standard Wiener process in time t.

From the definitions of the Lyapunov exponent (2) and the moment Lyapunov exponent (3), it can be easily shown that the Lyapunov exponents and the moment Lyapunov exponents of systems (1), (6), and (7) are related as follows:

$$\lambda_{q(\tau)} = -\beta + \lambda_{x(\tau)} = -\beta + \omega\lambda_{x(t)},$$
$$\Lambda_{q(\tau)}(p) = -p\beta + \Lambda_{x(\tau)}(p) = -p\beta + \omega\Lambda_{x(t)}(p). \tag{8}$$

Without loss of generality, the moment Lyapunov exponent of system (7) is studied in the remaining part of this paper.

2. Formulation

Considering the two-dimensional system (7) under bounded noise parametric excitation, the generator of process $\zeta(t)$ is

$$G = \frac{\sigma^2}{2}\frac{\partial^2}{\partial\zeta^2} + \nu\frac{\partial}{\partial\zeta}. \tag{9}$$

Letting $x_1 = x$, $x_2 = \dot{x}$, the two-dimensional system may be written in the form of a state equation:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \mathbf{A}(\zeta)\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{A}(\zeta) = \begin{bmatrix} 0 & 1 \\ -1 + \varepsilon\cos\zeta & 0 \end{bmatrix}. \tag{10}$$

Apply the Khasminskii transformation [17]

$$s_1 = \frac{x_1}{a} = \cos\varphi, \quad s_2 = \frac{x_2}{a} = \sin\varphi, \quad a = \|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}, \tag{11}$$

and denote $\mathbf{s} = \{s_1, s_2\}^{\mathrm{T}} = \{\cos\varphi, \sin\varphi\}^{\mathrm{T}}$. From the general theory of moment Lyapunov exponents [2], it is well known that the moment Lyapunov exponent $\Lambda_{x(t)}(p)$ of system (10) is the principal simple eigenvalue of the infinitesimal operator $L(p)$

$$L(p)T(\zeta, \mathbf{s}) = \Lambda_{x(t)}(p)T(\zeta, \mathbf{s}), \quad L(p) = \mathscr{L} + pQ(\zeta, \mathbf{s}), \tag{12}$$

where

$$\mathscr{L} = G + \mathbf{h}^{\mathrm{T}} \frac{\partial}{\partial \mathbf{s}},$$

$$Q(\zeta, \mathbf{s}) = \mathbf{s}^{\mathrm{T}} \mathbf{A}(\zeta)\, \mathbf{s} = \varepsilon \cos\zeta \cos\varphi \sin\varphi,$$

$$\mathbf{h}(\zeta, \mathbf{s}) = (\mathbf{A}(\zeta) - Q(\zeta, \mathbf{s})\mathbf{I})\, \mathbf{s} = \begin{Bmatrix} -\varepsilon\cos\zeta\cos^2\varphi\sin\varphi + \sin\varphi \\ (-1+\varepsilon\cos\zeta)\cos\varphi - \varepsilon\cos\zeta\cos\varphi\sin^2\varphi \end{Bmatrix}.$$

The generator G of the bounded noise $\zeta(t)$ is strongly elliptic, which is a required condition for the validity of Eq. (12) and the uniqueness of its solution. Since

$$\frac{\partial}{\partial s_1} = -\sin\varphi \frac{\partial}{\partial \varphi}, \qquad \frac{\partial}{\partial s_2} = \cos\varphi \frac{\partial}{\partial \varphi},$$

one has

$$\mathbf{h}^{\mathrm{T}} \frac{\partial}{\partial \mathbf{s}} = h_1 \frac{\partial}{\partial s_1} + h_2 \frac{\partial}{\partial s_2} = (-1 + \varepsilon\cos\zeta\cos^2\varphi)\frac{\partial}{\partial \varphi},$$

and the infinitesimal operator $L(p)$ is obtained as

$$L(p) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial\zeta^2} + v\frac{\partial}{\partial\zeta} + (-1+\varepsilon\cos\zeta\cos^2\varphi)\frac{\partial}{\partial\varphi} + \varepsilon p\cos\zeta\cos\varphi\sin\varphi. \tag{13}$$

The infinitesimal operator $L(p)$ of the eigenvalue problem (13) for the pth moment Lyapunov exponent can also be derived using a more straightforward approach without resorting to the general theory of moment Lyapunov exponents. This approach was first applied by Wedig [18] to derive the eigenvalue problem for the moment Lyapunov exponents of a two-dimensional linear Itô stochastic system.

Eqs. (7) may be considered as a three-dimensional system.

$$\mathrm{d}\begin{Bmatrix} x_1 \\ x_2 \\ \zeta \end{Bmatrix} = \begin{Bmatrix} x_2 \\ (-1+\varepsilon\cos\zeta)x_1 \\ v \end{Bmatrix} \mathrm{d}t + \begin{Bmatrix} 0 \\ 0 \\ \sigma \end{Bmatrix} \mathrm{d}W.$$

Apply the Khasminskii transformation (11) and define a pth norm $P = a^p$. The Itô equations for P and φ can be obtained by Itô's lemma:

$$\mathrm{d}P = \varepsilon p P \cos\zeta\cos\varphi\sin\varphi\,\mathrm{d}t, \quad \mathrm{d}\varphi = (-1+\varepsilon\cos\zeta\cos^2\varphi)\,\mathrm{d}t. \tag{14}$$

Applying a linear stochastic transformation,

$$S = T(\zeta,\varphi)P, \quad P = T^{-1}(\zeta,\varphi)S, \quad -\infty < \zeta < \infty, \quad 0 \leqslant \varphi < \pi,$$

the Itô equation for the new pth norm process S is given by, from Itô's lemma,

$$\mathrm{d}S = [\tfrac{1}{2}\sigma^2 T_{\zeta\zeta} + vT_\zeta + (-1+\varepsilon\cos\zeta\cos^2\varphi)T_\varphi + \varepsilon p\cos\zeta\cos\varphi\sin\varphi T]P\,\mathrm{d}t + \sigma T_\zeta P\,\mathrm{d}W. \tag{15}$$

For bounded and non-singular transformation $T(\zeta,\varphi)$, both processes P and S are expected to have the same stability behaviour. Therefore, $T(\zeta,\varphi)$ is chosen so that the drift term of the Itô differential equation (15) is independent of the noise process $\zeta(t)$ and the phase process φ so that

$$dS = \Lambda S\,dt + \sigma T_\zeta T^{-1} S\,dW. \tag{16}$$

Comparing Eqs. (15) and (16), it is seen that such a transformation $T(\zeta,\varphi)$ is given by the equation

$$\begin{aligned}&\tfrac{1}{2}\sigma^2 T_{\zeta\zeta} + \nu T_\zeta + (-1 + \varepsilon\cos\zeta\cos^2\varphi)T_\varphi + \varepsilon p\cos\zeta\cos\varphi\sin\varphi\, T = \Lambda T,\\ &-\infty<\zeta<\infty,\quad 0\leqslant\varphi<\pi,\end{aligned} \tag{17}$$

in which $T(\zeta,\varphi)$ is a periodic function in φ of period π and is bounded when $\zeta\to\pm\infty$. Eq. (17) defines an eigenvalue problem for a second order differential operator with Λ being the eigenvalue and $T(\zeta,\varphi)$ the associated eigenfunction. From Eq. (16), the eigenvalue Λ is seen to be the Lyapunov exponent of the pth moment of system (7), i.e., $\Lambda = \Lambda_{x(t)}(p)$. It is obvious that the differential operator in the eigenvalue problem (17) is the same as the infinitesimal operator $L(p)$ given by Eq. (13).

In the following section, for the case when σ is not small so that the eigenvalue problem (12) is not singular, the method of regular perturbation is applied to obtain a weak-noise expansion of the moment Lyapunov exponent for system (7).

3. Weak-noise expansion of the moment Lyapunov exponent

For weak-noise excitation, i.e., small ε, the infinitesimal operator $L(p)$ can be written as

$$L(p) = L_0(p) + \varepsilon L_1(p), \tag{18}$$

where

$$L_0(p) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial\zeta^2} + \nu\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\varphi},\quad L_1(p) = \cos\zeta\left(\cos^2\varphi\frac{\partial}{\partial\varphi} + p\cos\varphi\sin\varphi\right).$$

Applying the method of regular perturbation, both the eigenvalue $\Lambda_{x(t)}(p)$ and the eigenfunction $T(\zeta,\varphi)$, a periodic function in φ of period π, are expanded in power series of ε as

$$\Lambda_{x(t)}(p) = \sum_{k=0}^{\infty}\varepsilon^k \Lambda_k(p),\quad T(\zeta,\varphi) = \sum_{k=0}^{\infty}\varepsilon^k T_k(\zeta,\varphi), \tag{19}$$

in which $T_i(\zeta,\varphi)$, $i = 0,1,\ldots$, are periodic functions in φ of period π.

Substituting Eqs. (18) and (19) into the eigenvalue problem (12) and equating terms of equal power of ε yields the equations

$$\begin{aligned}&\text{zeroth order}:\quad L_0T_0 = \Lambda_0T_0,\\ &k\text{th order}:\quad L_0T_k + L_1T_{k-1} = \sum_{m=0}^{k}\Lambda_mT_{k-m},\quad k=1,2,\ldots\,.\end{aligned} \tag{20}$$

3.1. Zeroth order perturbation

The equation for the zeroth order perturbation is

$$L_0 T_0 = \Lambda_0 T_0 \tag{21}$$

or

$$\frac{\sigma^2}{2}\frac{\partial^2 T_0}{\partial \zeta^2} + v\frac{\partial T_0}{\partial \zeta} - \frac{\partial T_0}{\partial \varphi} - \Lambda_0 T_0 = 0.$$

Applying the method of separation of variables and letting $T_0(\zeta, \varphi) = Z_0(\zeta)\Phi_0(\varphi)$ results in

$$\frac{\sigma^2}{2}\frac{\ddot{Z}_0}{Z_0} + v\frac{\dot{Z}_0}{Z_0} - \Lambda_0 = \frac{\Phi_0'}{\Phi_0} = k.$$

Solving the equation for Φ_0 yields $\Phi_0(\varphi) = A\,\mathrm{e}^{k\varphi}$. For $\Phi_0(\varphi)$ to be a periodic function, it is required that $k = 0$ and hence $\Phi_0(\varphi)$ can be chosen as 1.

The equation for $Z_0(\zeta)$ becomes

$$\tfrac{1}{2}\sigma^2 \ddot{Z}_0 + v\dot{Z}_0 - \Lambda_0 Z_0 = 0. \tag{22}$$

From the property of moment Lyapunov exponent, it is known that

$$\Lambda_{x(t)}(0) = \Lambda_0(0) + \varepsilon\Lambda_1(0) + \cdots + \varepsilon^n \Lambda_n(0) + \cdots = 0,$$

which results in $\Lambda_0(0) = 0$. Since the eigenvalue problem (22) does not contain p, the eigenvalue $\Lambda_0(p)$ is independent of p. Hence, $\Lambda_0(0) = 0$ leads to $\Lambda_0(p) = 0$.

Eq. (22) can be easily solved to yield

$$Z_0(\zeta) = C_0 + C_1 \exp\left(-\frac{2v}{\sigma^2}\zeta\right), \quad -\infty < \zeta < \infty.$$

For $Z_0(\zeta)$ to be bounded, it is required that $C_1 = 0$ and hence $Z_0(\zeta)$ can be taken as 1. Therefore,

$$\Lambda_0(p) = 0, \quad T_0(\zeta, \varphi) = Z_0(\zeta)\Phi_0(\varphi) = 1. \tag{23}$$

Since $\Lambda_0(p) = 0$, the associated adjoint differential equation of Eq. (21) is

$$L_0^* T_0^* = \frac{\sigma^2}{2}\frac{\partial^2 T_0^*}{\partial \zeta^2} - v\frac{\partial T_0^*}{\partial \zeta} + \frac{\partial T_0^*}{\partial \varphi} = 0. \tag{24}$$

Applying the method of separation of variables and letting $T_0^*(\zeta, \varphi) = Z_0^*(\zeta)\Phi_0^*(\varphi)$ leads to

$$\frac{\sigma^2}{2}\frac{\ddot{Z}_0^*}{Z_0^*} - v\frac{\dot{Z}_0^*}{Z_0^*} = -\frac{\Phi_0^*}{\Phi_0^*} = \kappa.$$

The equation for Φ_0^* yields $\Phi_0^*(\varphi) = B\,\mathrm{e}^{-\kappa\varphi}$. For $\Phi_0^*(\varphi)$ to be a period function, $\kappa = 0$ and $\Phi_0^*(\varphi)$ can be taken as

$$\Phi_0^*(\varphi) = \frac{1}{\pi}, \quad 0 \leqslant \varphi < \pi, \tag{25}$$

which is the probability density function of a uniformly distributed random variable φ between 0 and π.

The equation for Z_0^* becomes

$$\tfrac{1}{2}\sigma^2 \ddot{Z}_0^* - v\dot{Z}_0^* = 0. \tag{26}$$

Eq. (26) can be easily solved to give

$$Z_0^*(\zeta) = D_0 + D_1 \exp\left(\frac{2v}{\sigma^2}\zeta\right), \quad -\infty < \zeta < \infty.$$

For $Z_0^*(\zeta)$ to be bounded, it is required that $D_1 = 0$ and hence $Z_0^*(\zeta) = D_0$, a constant. Note that $\zeta(t) = vt + \sigma W(t) + \theta$, which is a linear function vt with superimposed noise $\sigma W(t)$, and $\zeta(t)$ appears as an angle of a sinusoidal function $\cos\zeta$, which is a periodic function of period 2π. Hence, after folding, the angle $\zeta(t)$ may be considered as taking values between 0 and 2π. $Z_0^*(\zeta)$ may then be chosen as

$$Z_0^*(\zeta) = \frac{1}{2\pi}, \quad 0 \leqslant \zeta < 2\pi, \tag{27}$$

which is the probability density function of a uniformly distributed random variable between 0 and 2π.

Hence $T_0^*(\zeta, \varphi) = Z_0^*(\zeta)\Phi_0^*(\varphi)$ represents the joint stationary probability density function of the independent random variables ζ and φ, in which ζ is uniformly distributed between 0 and 2π and φ is uniformly distributed between 0 and π.

3.2. First order perturbation

The first order perturbation equation is

$$L_0 T_1 = \Lambda_1 T_0 - L_1 T_0. \tag{28}$$

Since the homogeneous equation $L_0 T_0 = 0$ has a non-trivial solution as given by Eq. (23), for Eq. (28) to have a solution it is required that, from the Fredholm alternative,

$$(\Lambda_1 T_0 - L_1 T_0, T_0^*) = 0, \tag{29}$$

where $T_0^*(\zeta, \varphi)$ is the solution of the adjoint equation (24) as obtained in Section 3.1, and (S_1, S_2) denotes the inner product of functions $S_1(\zeta, \varphi)$ and $S_2(\zeta, \varphi)$ defined by

$$(S_1, S_2) = \int_0^{2\pi} \int_0^{\pi} S_1(\zeta, \varphi) S_2(\zeta, \varphi)\, \mathrm{d}\varphi\, \mathrm{d}\zeta.$$

From Eq. (29), the first order perturbation of the moment Lyapunov exponent is

$$\Lambda_1 = (L_1 T_0, T_0^*), \tag{30}$$

because $(T_0, T_0^*) = 1$.

It is easy to show that

$$L_1 T_0 = \cos\zeta \left(\cos^2\varphi \frac{\partial T_0}{\partial \varphi} + p\cos\varphi \sin\varphi\, T_0 \right) = f_{\cos,1}^{(1)}(\varphi)\cos\zeta,$$

where $f_{\cos,1}^{(1)}(\varphi) = p\cos\varphi\sin\varphi$. Hence, using Eqs. (25), (27), and (30) results in

$$\Lambda_1 = (L_1 T_0, T_0^*) = \overline{f_{\cos,1}^{(1)}(\varphi)} E[\cos\zeta] = 0, \tag{31}$$

where

$$\overline{a(\varphi)} = \frac{1}{\pi}\int_0^{\pi} a(\varphi)\,\mathrm{d}\varphi$$

denotes the expected value of the random variable $a(\varphi)$, in which φ is the uniformly distributed random variable between 0 and π as defined in Eq. (25), and

$$E[b(\zeta)] = \frac{1}{2\pi}\int_0^{2\pi} b(\zeta)\,\mathrm{d}\zeta$$

denotes the expected value of the random variable $b(\zeta)$, in which ζ is the uniformly distributed random variable between 0 and 2π as defined in Eq. (27).

Hence, Eq. (28) becomes

$$L_0 T_1 = g^{(1)}_{\cos,\,1}(\varphi)\cos\zeta, \tag{32}$$

where $g^{(1)}_{\cos,\,1}(\varphi) = -f^{(1)}_{\cos,\,1}(\varphi) = -p\cos\varphi\sin\varphi$. Eq. (32) is of form (A.1) and the solution is given in Appendix A by Eq. (A.16) as

$$T_1(\zeta,\varphi) = G^{(1)}_{\sin,1}(\varphi)\sin\zeta + G^{(1)}_{\cos,\,1}(\varphi)\cos\zeta, \tag{33}$$

where

$$G^{(1)}_{\sin,1}(\varphi) = -\int_0^{s} g^{(1)}_{\cos,\,1}(\psi - r)\,s_1(r-s)\,\mathrm{d}r,$$
$$= -\frac{2pv[4(v^2-4+\frac{1}{4}\sigma^4)\sin 2\varphi - 8v^2\cos 2\varphi]}{16v^4+(8\sigma^4-128)v^2+\sigma^8+32\sigma^4+256},$$

$$G^{(1)}_{\cos,\,1}(\varphi) = \int_0^{s} g^{(1)}_{\cos,\,1}(\psi - r)\,c_1(r-s)\,\mathrm{d}r,$$
$$= \frac{p[4\sigma^2(v^2+4+\frac{1}{4}\sigma^4)\sin 2\varphi + 16(v^2-4-\frac{1}{4}\sigma^4)\cos 2\varphi]}{16v^4+(8\sigma^4-128)v^2+\sigma^8+32\sigma^4+256},$$

in which $s_1(r-s)$ and $c_1(r-s)$ are as defined in Eq. (A.17) of Appendix A, and $\psi - s = \varphi$, $s \to -\infty$ have been employed after integration.

3.3. Second order perturbation

The equation for the second order perturbation is

$$L_0 T_2 = \Lambda_2 T_0 - L_1 T_1. \tag{34}$$

From the Fredholm alternative, for Eq. (34) to have a solution it is required that

$$(\Lambda_2 T_0 - L_1 T_1, T_0^*) = 0, \tag{35}$$

where

$$L_1 T_1 = \cos\zeta\left(\cos^2\varphi\frac{\partial T_1}{\partial\varphi} + p\cos\varphi\sin\varphi\, T_1\right)$$
$$= f_0^{(2)}(\varphi) + f^{(2)}_{\sin,\,2}(\varphi)\sin 2\zeta + f^{(2)}_{\cos,\,2}(\varphi)\cos 2\zeta,$$

$f_0^{(2)}(\varphi) = f_{\cos,2}^{(2)}(\varphi)$, and $f_{\sin,2}^{(2)}(\varphi)$, $f_{\cos,2}^{(2)}(\varphi)$ are given by, with $m = 2$,

$$f^{(m)}_{\{\substack{\sin\\\cos}\},m}(\varphi) = \tfrac{1}{2}\left[\cos^2\varphi\, G^{(m-1)\,\prime}_{\{\substack{\sin\\\cos}\},m-1}(\varphi) + p\cos\varphi\sin\varphi\, G^{(m-1)}_{\{\substack{\sin\\\cos}\},m-1}(\varphi)\right], \tag{36}$$

in which the prime denotes differentiation with respect to φ. Hence, since $E[\sin 2\zeta] = E[\cos 2\zeta] = 0$, from Eq. (35) one obtains

$$\Lambda_2 = (L_1 T_1, T_0^*) = \overline{f_0^{(2)}(\varphi)} = \frac{p(p+2)\,S(2)}{16}, \tag{37}$$

where

$$S(2) = \frac{\sigma^2(4 + \nu^2 + \frac{1}{4}\sigma^4)}{2[(2+\nu)^2 + \frac{1}{4}\sigma^4][(2-\nu)^2 + \frac{1}{4}\sigma^4]}$$

is the spectral density function $S(\omega)$ with $\omega = 2$ of the bounded noise $\zeta(t)$.

Eq. (34) becomes

$$L_0 T_2 = g_0^{(2)}(\varphi) + g_{\sin,2}^{(2)}(\varphi)\sin 2\zeta + g_{\cos,2}^{(2)}(\varphi)\cos 2\zeta, \tag{38}$$

where

$$g_0^{(2)}(\varphi) = \Lambda_2 - f_0^{(2)}(\varphi), \quad g_{\sin,2}^{(2)}(\varphi) = -f_{\sin,2}^{(2)}(\varphi), \quad g_{\cos,2}^{(2)}(\varphi) = -f_{\cos,2}^{(2)}(\varphi).$$

From Appendix A, the solution of Eq. (38) given by Eqs. (A.15) and (A.16) is

$$T_2(\zeta,\varphi) = G_0^{(2)}(\varphi) + G_{\sin,2}^{(2)}(\varphi)\sin 2\zeta + G_{\cos,2}^{(2)}(\varphi)\cos 2\zeta, \tag{39}$$

where $G_0^{(2)}(\varphi)$ is given by, with $m = 2$,

$$G_0^{(m)}(\varphi) = \int_0^s g_0^{(m)}(\psi - r)\,\mathrm{d}r; \tag{40}$$

$G_{\sin,2}^{(2)}(\varphi)$ and $G_{\cos,2}^{(2)}(\varphi)$ are given by, with $m = j = 2$,

$$G^{(m)}_{\{\substack{\sin\\\cos}\},j}(\varphi) = \int_0^s \left[g^{(m)}_{\sin,j}(\psi - r)\begin{Bmatrix} c_j(r-s) \\ s_j(r-s) \end{Bmatrix} + g^{(m)}_{\cos,j}(\psi - r)\begin{Bmatrix} -s_j(r-s) \\ c_j(r-s) \end{Bmatrix}\right]\mathrm{d}r, \tag{41}$$

where $s_j(r-s)$, $c_j(r-s)$ are as defined in Eqs. (A.17) of Appendix A, and $\psi - s = \varphi$ and $s \to -\infty$ are taken after integration.

3.4. Higher order perturbation

Based on the results obtained in Sections 3.1–3.3, the method of mathematical induction can be applied to determine the moment Lyapunov exponents of higher order perturbations.

For the $(2n)$th order perturbation, $n = 1, 2, \ldots$, the perturbation equation is

$$L_0 T_{2n} = \sum_{k=1}^{n} \Lambda_{2k} T_{2n-2k} - L_1 T_{2n-1}, \tag{42}$$

because $\Lambda_0 = \Lambda_1 = \Lambda_3 = \cdots = \Lambda_{2n-1} = 0$. From the Fredholm alternative, for Eq. (43) to have a solution, it is required that

$$\sum_{k=1}^{n} \Lambda_{2k}(T_{2n-2k}, T_0^*) - (L_1 T_{2n-1}, T_0^*) = 0. \tag{43}$$

Since $L_1 T_{2n-1}$ is of the form

$$L_1 T_{2n-1} = f_0^{(2n)}(\varphi) + \sum_{k=1}^{n} [f_{\sin,\,2k}^{(2n)}(\varphi)\sin 2k\zeta + f_{\cos,\,2k}^{(2n)}(\varphi)\cos 2k\zeta],$$

where

$$f_0^{(2n)}(\varphi) = \tfrac{1}{2}[\cos^2\varphi G_{\cos,\,1}^{(2n-1)\,\prime}(\varphi) + p\cos\varphi\sin\varphi G_{\cos,\,1}^{(2n-1)}(\varphi)],$$

$f_{\sin,\,2n}^{(2n)}(\varphi)$ and $f_{\cos,\,2n}^{(2n)}(\varphi)$ are given by Eq. (36) with $m = 2n$, $f_{\sin,\,2k}^{(2n)}(\varphi)$ and $f_{\cos,\,2k}^{(2n)}(\varphi)$, for $k = 1, 2, \ldots, n-1$, are given by, with $m = 2n,\ j = 2k$,

$$\begin{aligned} f_{\{\substack{\sin\\ \cos}\},j}^{(m)}(\varphi) = \tfrac{1}{2}\Bigg\{ & \cos^2\varphi \left[G_{\{\substack{\sin\\ \cos}\},j-1}^{(m-1)\,\prime}(\varphi) + G_{\{\substack{\sin\\ \cos}\},j+1}^{(m-1)\,\prime}(\varphi) \right] \\ & + p\cos\varphi\sin\varphi \left[G_{\{\substack{\sin\\ \cos}\},j-1}^{(m-1)}(\varphi) + G_{\{\substack{\sin\\ \cos}\},j+1}^{(m-1)}(\varphi) \right], \Bigg\} \end{aligned} \tag{44}$$

and

$$T_{2k}(\zeta, \varphi) = G_0^{(2k)}(\varphi) + \sum_{m=1}^{k} [G_{\sin,\,2m}^{(2k)}(\varphi)\sin 2m\zeta + G_{\cos,\,2m}^{(2k)}(\varphi)\cos 2m\zeta].$$

Eq. (43) leads to

$$\begin{aligned} \Lambda_{2n} &= (L_1 T_{2n-1}, T_0^*) - \sum_{k=1}^{n-1} \Lambda_{2k}(T_{2n-2k}, T_0^*) \\ &= \overline{f_0^{(2n)}(\varphi)} - \sum_{k=1}^{n-1} \Lambda_{2k}\ \overline{G_0^{(2n-2k)}(\varphi)}. \end{aligned} \tag{45}$$

Eq. (42) is then of the form

$$L_0 T_{2n} = g_0^{(2n)}(\varphi) + \sum_{k=1}^{n} [g_{\sin,\,2k}^{(2n)}(\varphi)\sin 2k\zeta + g_{\cos,\,2k}^{(2n)}(\varphi)\cos 2k\zeta], \tag{46}$$

where

$$g_0^{(2n)}(\varphi) = \sum_{m=1}^{n} \Lambda_{2m}\, G_0^{(2n-2m)}(\varphi) - f_0^{(2n)}(\varphi), \quad G_0^{(0)}(\varphi) = 1,$$

$$g^{(2n)}_{\{{\sin \atop \cos}\},2k}(\varphi) = \sum_{m=1}^{n-k} \Lambda_{2m} G^{(2n-2m)}_{\{{\sin \atop \cos}\},2k}(\varphi) - f^{(2n)}_{\{{\sin \atop \cos}\},2k}(\varphi), \quad k = 1, 2, \ldots, n-1,$$

$$g^{(2n)}_{\{{\sin \atop \cos}\},2n}(\varphi) = -f^{(2n)}_{\{{\sin \atop \cos}\},2n}(\varphi).$$

From Eqs. (A.15) and (A.16), the solution of Eq. (46) may be obtained as

$$T_{2n}(\zeta, \varphi) = G_0^{(2n)}(\varphi) + \sum_{k=1}^{n} [G^{(2n)}_{\sin,\,2k}(\varphi) \sin 2k\zeta + G^{(2n)}_{\cos,\,2k}(\varphi) \cos 2k\zeta], \tag{47}$$

where $G_0^{(2n)}(\varphi)$ is given by Eq. (40) with $m = 2n$, and $G^{(2n)}_{\sin,\,2k}(\varphi)$, $G^{(2n)}_{\cos,\,2k}(\varphi)$, for $k = 1, 2, \ldots, n$, are given by Eq. (41) with $m = 2n,\ j = 2k$.

For the $(2n+1)$th order perturbation, $n = 0, 1, \ldots$, the perturbation equation is

$$L_0 T_{2n+1} = \sum_{k=1}^{n} \Lambda_{2k} T_{2n-2k+1} + \Lambda_{2n+1} T_0 - L_1 T_{2n}. \tag{48}$$

From the Fredholm alternative, for Eq. (48) to have a solution, it is required that

$$\Lambda_{2n+1} = (L_1 T_{2n}, T_0^*) - \sum_{k=1}^{n} \Lambda_{2k} (T_{2n-2k+1}, T_0^*). \tag{49}$$

Since $L_1 T_{2n}$ is of the form

$$L_1 T_{2n} = \sum_{k=0}^{n} [f^{(2n+1)}_{\sin,\,2k+1}(\varphi) \sin{(2k+1)\zeta} + f^{(2n+1)}_{\cos,\,2k+1}(\varphi) \cos{(2k+1)\zeta}],$$

where

$$f^{(2n+1)}_{\sin,\,1}(\varphi) = \tfrac{1}{2}[\cos^2\varphi G^{(2n)}_{\sin,\,2}(\varphi) + p \cos\varphi \sin\varphi\ \ G^{(2n)}_{\sin,\,2}(\varphi)],$$

$$f^{(2n+1)}_{\cos,\,1}(\varphi) = \cos^2\varphi[G_0^{(2n)}(\varphi) + \tfrac{1}{2}\, G^{(2n)}_{\cos,\,2}(\varphi)] + p \cos\varphi \sin\varphi[G_0^{(2n)}(\varphi) + \tfrac{1}{2}\, G^{(2n)}_{\cos,\,2}(\varphi)],$$

$f^{(2n+1)}_{\sin,\,2n+1}(\varphi)$ and $f^{(2n+1)}_{\cos,\,2n+1}(\varphi)$ are given by Eq. (36) with $m = 2n+1$, $f^{(2n+1)}_{\sin,\,2k+1}(\varphi)$ and $f^{(2n+1)}_{\cos,\,2k+1}(\varphi)$, for $k = 1, 2, \ldots, n-1$, are given by Eq. (44) with $m = 2n+1,\ j = 2k+1$, and

$$T_{2k+1}(\zeta, \varphi) = \sum_{m=1}^{k} [G^{(2k)}_{\sin,\,2m+1}(\varphi) \sin{(2m+1)\zeta} + G^{(2k)}_{\cos,\,2m+1}(\varphi) \cos{(2m+1)\zeta}].$$

Eq. (49) leads to

$$\Lambda_{2n+1} = 0. \tag{50}$$

Eq. (48) is then of the form

$$L_0 T_{2n+1} = \sum_{k=0}^{n} [g^{(2n+1)}_{\sin,\,2k+1}(\varphi) \sin{(2k+1)\zeta} + g^{(2n+1)}_{\cos,\,2k+1}(\varphi) \cos{(2k+1)\zeta}], \tag{51}$$

where

$$g^{(2n+1)}_{\{{\sin \atop \cos}\},2k+1}(\varphi) = \sum_{m=1}^{n-k} \Lambda_{2m} G^{(2n-2m+1)}_{\{{\sin \atop \cos}\},2k+1}(\varphi) - f^{(2n+1)}_{\{{\sin \atop \cos}\},2k+1}(\varphi), \quad k = 0, 1, \ldots, n-1,$$

$$g^{(2n+1)}_{\{{\sin \atop \cos}\},2n+1}(\varphi) = -f^{(2n+1)}_{\{{\sin \atop \cos}\},2n+1}(\varphi).$$

From Eqs. (A.15) and (A.16), the solution of Eq. (51) may be obtained as

$$T_{2n+1}(\zeta, \varphi) = \sum_{k=0}^{n} [G^{(2n+1)}_{\sin,\,2k+1}(\varphi) \sin(2k+1)\zeta + G^{(2n+1)}_{\cos,\,2k+1}(\varphi) \cos(2k+1)\zeta], \tag{52}$$

where, for $k = 0, 1, \ldots, n$, $G^{(2n+1)}_{\sin,\,2k+1}(\varphi)$ and $G^{(2n+1)}_{\cos,\,2k+1}(\varphi)$ are given by Eq. (41) with $m = 2n+1$, $j = 2k+1$.

The algebraic manipulation of higher order perturbations can be performed using a symbolic computation software such as *maple* so that higher order approximations can be easily obtained. Following this procedure, a weak-noise expansion of the moment Lyapunov exponent is obtained as

$$\Lambda_{x(t)}(p) = \varepsilon^2 \Lambda_2 + \varepsilon^4 \Lambda_4 + O(\varepsilon^6), \tag{53}$$

where Λ_2 is given by Eq. (37) and

$$\Lambda_4 = -\frac{p(p+2)\sigma^2[N_0^{(4)} + p(p+2)N_p^{(4)}]}{D^{(4)}}, \tag{54}$$

in which the values of $N_0^{(4)}$, $N_p^{(4)}$, and $D^{(4)}$ are given in Appendix B.

The Lyapunov exponent for system (7) can be obtained from Eq. (53) by using the property of the moment Lyapunov exponent,

$$\lambda_{x(t)} = \left.\frac{\mathrm{d}\Lambda_{x(t)}(p)}{\mathrm{d}\,p}\right|_{p=0} = \varepsilon^2 \lambda_2 + \varepsilon^4 \lambda_4 + O(\varepsilon^6), \tag{55}$$

where

$$\lambda_2 = \frac{S(2)}{8}, \quad \lambda_4 = -\frac{2\sigma^2 N_0^{(4)}}{D^{(4)}}.$$

λ_2 is the same as the well-known result obtained using the standard stochastic averaging method for a more general rapidly fluctuating noise, namely,

$$\lambda_2 = \tfrac{1}{8}\,\omega_0^2\, S(2\omega_0),$$

with $\omega_0 = 1$, where $S(\omega)$ is the spectral density function of the noise process (see, e.g., Eq. (10.44) of Ref. [12, p. 288]).

3.5. Stability index

As mentioned in the introduction, the stability index is the non-trivial zero of the moment Lyapunov exponent. For system (7), the moment Lyapunov exponent is given by Eq. (53). It is seen that $p = 0$ and -2 are the two values that lead to $\Lambda_{x(t)}(p) = 0$, and hence the stability index $\delta_{x(t)} = -2$.

For system (1) with parametric excitation (4), the moment Lyapunov exponent is

$$\Lambda_{q(\tau)}(p) = -p\beta + \omega\Lambda_{x(t)}(p),$$

and the stability index $\delta_{q(\tau)}$ is given by the non-trivial zero of $\Lambda_{q(\tau)}(p)$, i.e.,

$$\Lambda_{x(t)}(\delta_{q(\tau)}) - \varepsilon^2\tilde{\beta}\delta_{q(\tau)} = 0, \tag{56}$$

where $\varepsilon^2\tilde{\beta} = \beta/\omega$.

Expanding the stability index $\delta_{q(\tau)}$ in power series of ε as

$$\delta_{q(\tau)} = \sum_{k=0}^{\infty} \varepsilon^k \delta_k, \tag{57}$$

substituting Eqs. (53) and (57) into Eq. (56), expanding and equating terms of equal power of ε yields the equations

$$\begin{aligned}
&\varepsilon^2: \quad \delta_0\left[-\tilde{\beta} + \frac{(\delta_0+2)\sigma^2 N_0^{(2)}}{D^{(2)}}\right] = 0,\\
&\varepsilon^3: \quad \delta_1\left[-\tilde{\beta} + \frac{2(\delta_0+1)\sigma^2 N_0^{(2)}}{D^{(2)}}\right] = 0,\\
&\varepsilon^4: \quad -\tilde{\beta}\delta_2 + \frac{[2(\delta_0\delta_2+1)+\delta_1^2]\sigma^2 N_0^{(2)}}{D^{(2)}} - \frac{\delta_0(\delta_0+2)\sigma^2[N_0^{(4)} + \delta_0(\delta_0+2)N_p^{(4)}]}{D^{(4)}} = 0,\\
&\vdots \qquad\qquad \vdots
\end{aligned} \tag{58}$$

These equations can be easily solved for $\delta_i,\ i = 0, 1, \ldots,$ to result in

$$\begin{aligned}
\delta_0 &= -2 + \frac{\tilde{\beta}D^{(2)}}{\sigma^2 N_0^{(2)}},\\
\delta_1 &= 0,\\
\delta_2 &= \tilde{\beta}[D^{(2)}]^2\{\sigma^4[N_0^{(2)}]^2 N_0^{(4)} - 2\tilde{\beta}\sigma^2 D^{(2)} N_0^{(2)} N_p^{(4)} + \tilde{\beta}^2[D^{(2)}]^2 N_p^{(4)}\}/\{\sigma^6 D^{(4)}[N_0^{(2)}]^4\},
\end{aligned} \tag{59}$$

where $\tilde{\beta} = \beta/(\varepsilon^2\omega)$.

4. Numerical results and conclusions

In this paper, the moment Lyapunov exponents of a two-dimensional system under bounded noise parametric excitation are studied. The method of regular perturbation is applied to obtain a weak noise expansion of the moment Lyapunov exponent in terms of the small fluctuation parameter. Weak noise expansions of the Lyapunov exponent and stability index are also obtained.

Typical results of the moment Lyapunov exponents $\Lambda_{x(t)}(p)$ for system (7) given by Eq. (53) are shown in Figs. 1 and 2 for $\nu = 1.0, 2.0$, respectively, $\sigma = 1.0$, and various values of ε. The moment Lyapunov exponents $\Lambda_{x(t)}(p)$ are shown in Figs. 3, 4, and 5 for $\sigma = 0.3, 1.0, 2.0$, respectively, $\varepsilon = 0.5$, and various values of ν.

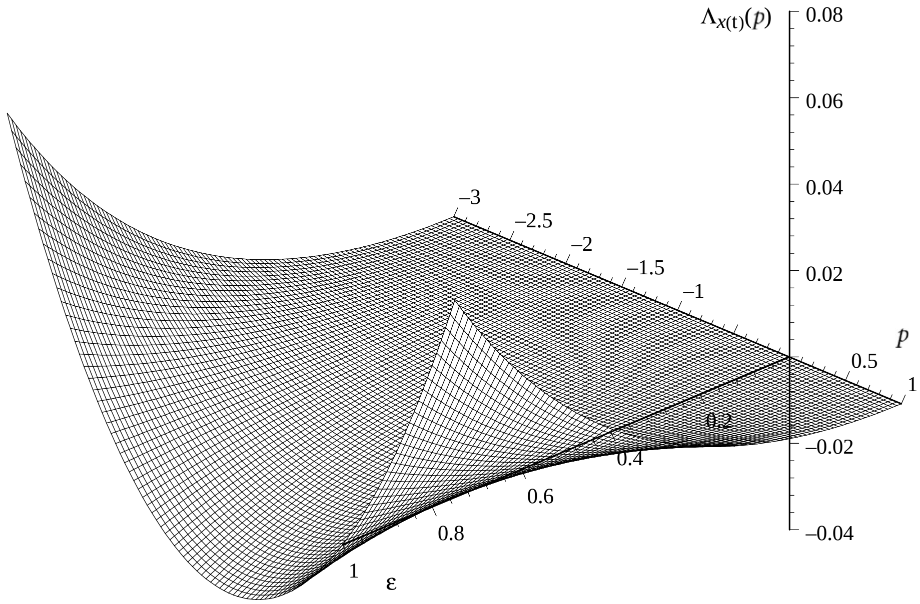

Fig. 1. Moment Lyapunov exponent: $\sigma = 1.0$, $v = 1.0$.

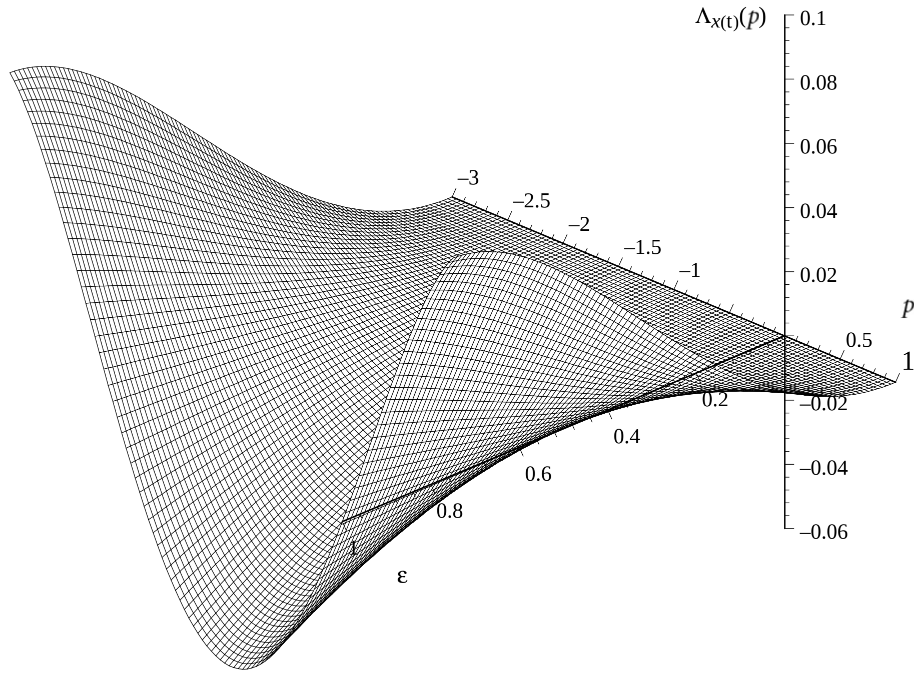

Fig. 2. Moment Lyapunov exponent: $\sigma = 1.0$, $v = 2.0$.

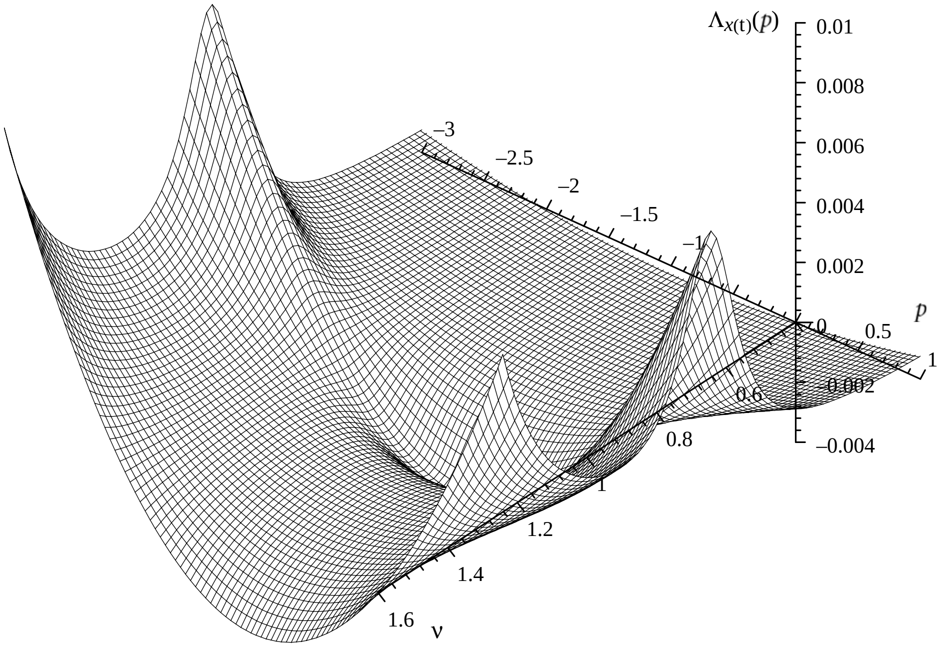

Fig. 3. Moment Lyapunov exponent: $\varepsilon = 0.5$, $\sigma = 0.3$.

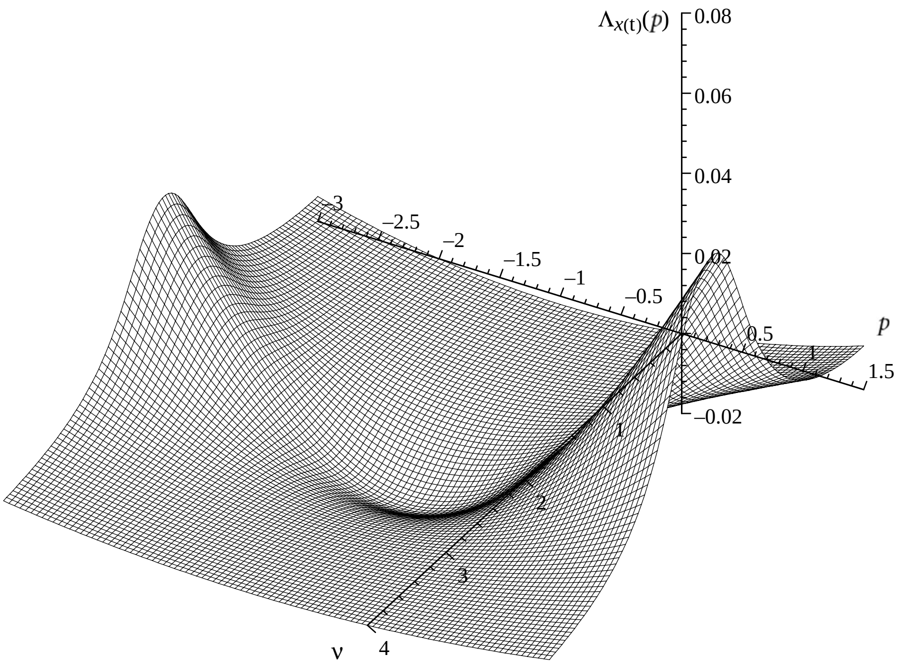

Fig. 4. Moment Lyapunov exponent: $\varepsilon = 0.5$, $\sigma = 1.0$.

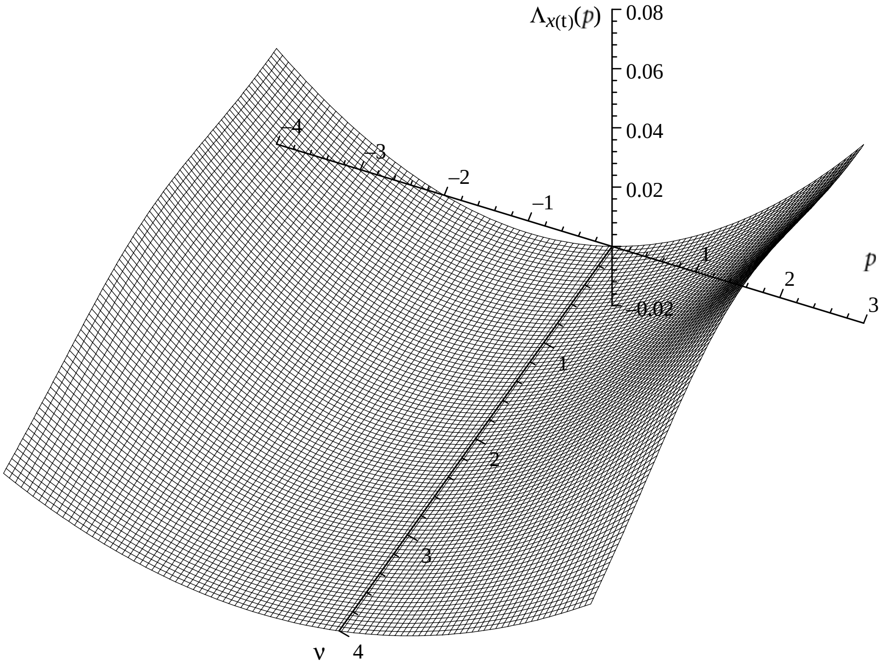

Fig. 5. Moment Lyapunov exponent: $\varepsilon = 0.5$, $\sigma = 2.0$.

In the absence of noise perturbation, i.e., when $\sigma_0 = 0$, the two-dimensional system (1) under bounded noise parametric excitation (4) reduces to the Mathieu's equation. It is well known that parametric resonance occurs when $v_0/(2\omega_0)$ is in the vicinity of $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$. For system (7), if the noise fluctuation parameter $\sigma = 0$, the primary parametric resonance occurs in the vicinity of $v = 2$, while the secondary parametric resonance occurs in the vicinity of $v = 1$. From Eq. (37), it is seen that Λ_2 is singular at $v = 2$ when $\sigma = 0$. However, from Eq. (54), Λ_4 is singular at $v = 2$ and 1 when $\sigma = 0$. Hence the effect of the primary parametric resonance appears in the second order perturbation results; whereas the effect of the secondary parametric resonance is noticeable only in the fourth order perturbation. The effects of higher order parametric resonance can be observed only in the higher order perturbation results.

When the noise fluctuation parameter σ is not zero, the bounded noise is a sinusoidal function with noise superimposed. The larger the value of σ, the noisier the bounded noise $\cos\zeta(t)$, resulting in a smaller effect of the parametric resonance. This is clearly seen by comparing Figs. 3, 4, and 5.

Typical results of the Lyapunov exponent $\lambda_{x(t)}$ for system (7) given by Eq. (52) are shown in Figs. 6 and 7 for $\sigma = 0.3, 1.0$, respectively, and various values of v and ε. The effects of parametric resonance when v is in the vicinity of 2 and 1 can be clearly seen.

Typical results of the stability index $\delta_{q(\tau)}$ given by Eqs. (57) and (59) are shown in Fig. 8 for $\beta = 0.05$, $\omega_0 = 1.0$, $\sigma_0 = 1.0$, and various values of ε_0 and v_0. The stability indices are shown in Figs. 9 and 10 for $\sigma_0 = 1.0, 2.0$, respectively, $\omega_0 = 1.0$, $\varepsilon_0 = 0.5$, and various values of β and v_0. From the definition of the stability index, it is clear that the larger the value of the stability index, the more stable the system is in the sense of moment stability. From Fig. 8, it is seen that the

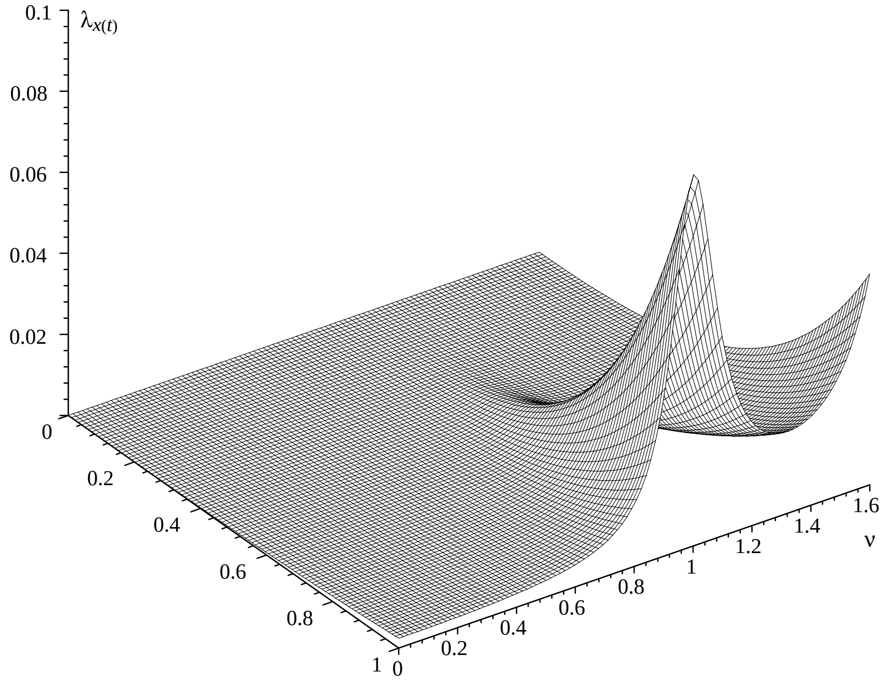

Fig. 6. Lyapunov exponent: $\sigma = 0.3$.

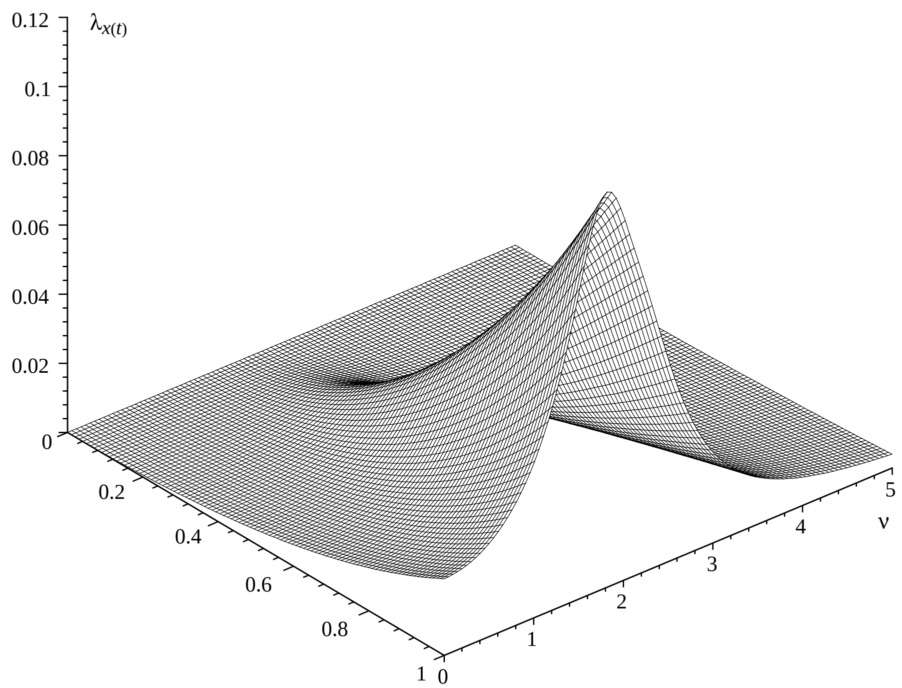

Fig. 7. Lyapunov exponent: $\sigma = 1.0$.

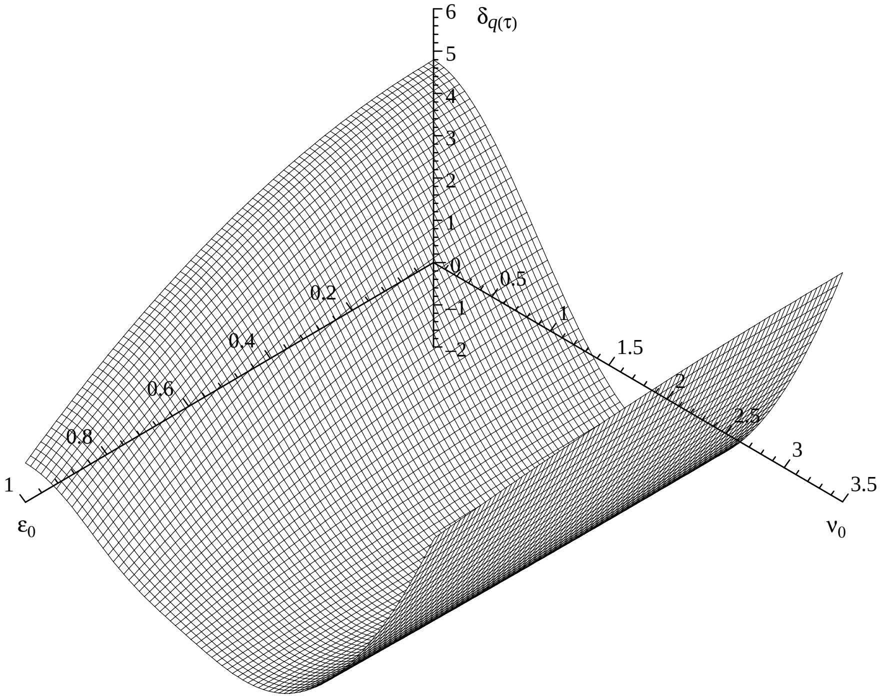

Fig. 8. Stability index: $\beta = 0.05$, $\omega_0 = 1.0$, $\sigma_0 = 1.0$.

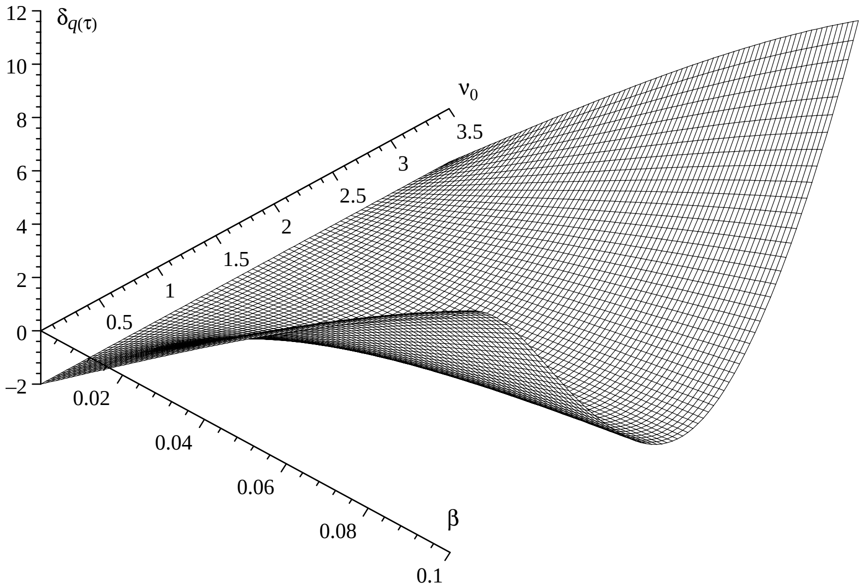

Fig. 9. Stability index: $\varepsilon_0 = 0.5$, $\omega_0 = 1.0$, $\sigma_0 = 1.0$.

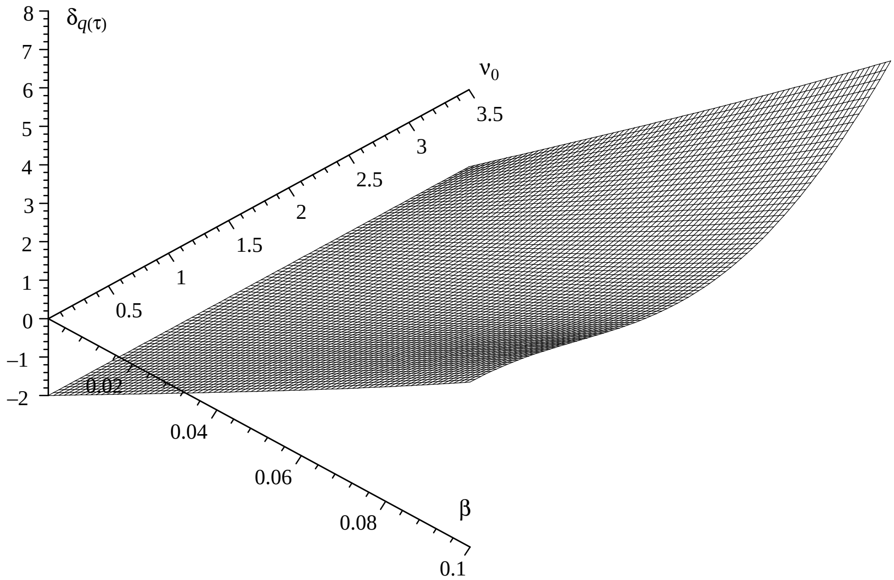

Fig. 10. Stability index: $\varepsilon_0 = 0.5$, $\omega_0 = 1.0$, $\sigma_0 = 2.0$.

stability index decreases with the increase of the amplitude of excitation ε_0. As expected, it is seen from Figs. 9 and 10 that the stability index increases with the increase of the damping parameter β_0. By comparing Figs. 9 and 10, it is also seen that the effect of parametric resonance diminishes with the increase of the noise fluctuation parameter σ_0.

It should be noted that the application of the method of regular perturbation in determining the moment Lyapunov exponent is based on the assumption that the noise fluctuation parameter σ is not small so that the infinitesimal operator $L(p)$ is not singular. Hence, the results obtained in this research cannot be used to deduce the results for the Mathieu's equation by setting σ to zero. In the case of small noise fluctuation, i.e. σ is small, a method of singular perturbation has to be employed to determine the moment Lyapunov exponent, which will be studied in future research.

Acknowledgements

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Appendix A. Solution of $L_0 T(\zeta,\varphi) = f(\zeta)g(\varphi)$

Consider the partial differential equation $L_0 T(\zeta,\varphi) = f(\zeta)g(\varphi)$, or

$$\left(\frac{\sigma^2}{2}\frac{\partial^2}{\partial\zeta^2} + v\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\varphi}\right)T(\zeta,\varphi) = f(\zeta)g(\varphi). \tag{A.1}$$

Introducing an auxiliary time t' to Eq. (A.1) leads to

$$\left(\frac{\partial}{\partial t'}+\frac{\sigma^2}{2}\frac{\partial^2}{\partial \zeta^2}+v\frac{\partial}{\partial \zeta}-\frac{\partial}{\partial \varphi}\right)T(\zeta,\varphi,t')=f(\zeta)g(\varphi). \tag{A.2}$$

Applying the transformation $\psi=\frac{1}{2}(t'+\varphi)$, $s=\frac{1}{2}(t'-\varphi)$, or $t'=\psi+s$, $\varphi=\psi-s$, Eq. (A.2) becomes

$$\left(\frac{\partial}{\partial s}+\frac{\sigma^2}{2}\frac{\partial^2}{\partial \zeta^2}+v\frac{\partial}{\partial \zeta}\right)T(\zeta,\psi,s)=f(\zeta)g(\psi-s). \tag{A.3}$$

Applying Duhamel's principle [19], the solution $T(\zeta,\psi,s)$ to Eq. (A.3) is given by

$$T(\zeta,\psi,s)=\int_0^s V(\zeta,\psi,s;r)\,\mathrm{d}r, \tag{A.4}$$

where $V(\zeta,\psi,s;r)$ is the solution of the homogeneous equation

$$\left(\frac{\partial}{\partial s}+\frac{\sigma^2}{2}\frac{\partial^2}{\partial \zeta^2}+v\frac{\partial}{\partial \zeta}\right)V(\zeta,\psi,s;r)=0 \quad \text{for } s>r,$$
$$V(\zeta,\psi,r;r)=f(\zeta)g(\psi-r) \quad \text{for } s=r. \tag{A.5}$$

To solve Eq. (A.5), consider the equation

$$\left(\frac{\partial}{\partial s}+\frac{\sigma^2}{2}\frac{\partial^2}{\partial \zeta^2}+v\frac{\partial}{\partial \zeta}\right)P(s,\zeta;t,z)=0, \quad s<t,$$
$$P(t,\zeta;t,z)=\lim_{s\uparrow t}P(s,\zeta;t,z)=\delta(z-\zeta). \tag{A.6}$$

Eq. (A.6) is Kolmogorov's backward equation for the transition probability function $P(s,\zeta;t,z)$, which is the probability density function of random variable $z(t)$ conditioned on $\zeta(s)$, $t>s$.

Eq. (7) can be integrated to yield

$$z(t)=\zeta(s)+v(t-s)+\sigma[W(t)-W(s)], \tag{A.7}$$

which Eq. (A.7) implies that, given the initial condition $\zeta(s)$, the random variable $z(t)$ is normally distributed with mean value $\mu_{z(t)}$ and variance $\sigma^2_{z(t)}$ given by

$$\mu_{z(t)}=\zeta+v\,(t-s), \quad \sigma^2_{z(t)}=\sigma^2\,(t-s). \tag{A.8}$$

Hence, the transition probability function is

$$P(s,\zeta;t,z)=\frac{1}{\sqrt{2\pi}\sigma_{z(t)}}\exp\left\{-\frac{[z-\mu_{z(t)}]^2}{2\sigma^2_{z(t)}}\right\}. \tag{A.9}$$

From Eqs. (A.5) and (A.6), the solution $V(\zeta,\psi,s;r)$ to Eq. (A.5) is given by

$$V(\zeta,\psi,s;r)=g(\psi-r)\int_{-\infty}^{\infty}f(z)P(s,\zeta;r,z)\,\mathrm{d}z, \tag{A.10}$$

where

$$E[f(z(r))]=\int_{-\infty}^{\infty}f(z)P(s,\zeta;r,z)\,\mathrm{d}z,$$

is the expected value of the random variable $f(z(r))$ with $z(r)$ being the normally distributed random variable as defined in Eqs. (A.8) and (A.9).

Combining Eqs. (A.4) and (A.10), the solution to Eq. (A.3) is given by

$$T(\zeta,\psi,s)=\int_0^s g(\psi-r)E[f(z(r))]\,\mathrm{d}r. \tag{A.11}$$

The solution $T(\zeta,\varphi)$ to Eq. (A.1) is obtained by replacing $\varphi=\psi-s$ and passing the limit $s\to-\infty$.

For the special cases when $f(\zeta)=\sin a\zeta$ or $\cos a\zeta$, one has

$$\begin{aligned}E\left[\begin{Bmatrix}\sin az(r)\\ \cos az(r)\end{Bmatrix}\right]&=\frac{1}{\sqrt{2\pi}\,\sigma_{z(r)}}\int_{\infty}^{+\infty}\begin{Bmatrix}\sin az\\ \cos az\end{Bmatrix}\exp\left\{-\frac{[z-\mu_{z(r)}]^2}{2\sigma^2_{z(r)}}\right\}\mathrm{d}z\\ &=\exp\left[-\tfrac{1}{2}a^2\sigma^2_{z(r)}\right]\begin{Bmatrix}\sin a\mu_{z(r)}\\ \cos a\mu_{z(r)}\end{Bmatrix},\end{aligned} \tag{A.12}$$

in which the integral formulas

$$\int_{-\infty}^{+\infty}\exp(-q^2x^2)\begin{Bmatrix}\sin[p(x+\lambda)]\\ \cos[p(x+\lambda)]\end{Bmatrix}\mathrm{d}x=\frac{\sqrt{\pi}}{q}\exp\left(-\frac{p^2}{4q^2}\right)\begin{Bmatrix}\sin p\lambda\\ \cos p\lambda\end{Bmatrix},$$

as given in Eqs. (1) and (2) of Ref. [20] have been employed. Substituting Eq. (A.8) into (A.12) results in

$$E\left[\begin{Bmatrix}\sin az(r)\\ \cos az(r)\end{Bmatrix}\right]=c_a(r-s)\begin{Bmatrix}\sin a\zeta\\ \cos a\zeta\end{Bmatrix}+s_a(r-s)\begin{Bmatrix}\cos a\zeta\\ -\sin a\zeta\end{Bmatrix}, \tag{A.13}$$

in which the following notations are used

$$\begin{Bmatrix}s_a(r-s)\\ c_a(r-s)\end{Bmatrix}=\exp\left[-\tfrac{1}{2}a^2\sigma^2_{z(r)}\right]\begin{Bmatrix}\sin av(r-s)\\ \cos av(r-s)\end{Bmatrix}. \tag{A.14}$$

Substituting Eqs. (A.13) into Eq. (A.11), one obtains the solution of Eq. (A.1) as, when $f(\zeta)=\sin a\zeta$,

$$T(\zeta,\psi,s)=\sin a\zeta\int_0^s g(\psi-r)c_a(r-s)\,\mathrm{d}r+\cos a\zeta\int_0^s g(\psi-r)s_a(r-s)\,\mathrm{d}r, \tag{A.15}$$

and, when $f(\zeta)=\cos a\zeta$,

$$T(\zeta,\psi,s)=\cos a\zeta\int_0^s g(\psi-r)c_a(r-s)\,\mathrm{d}r-\sin a\zeta\int_0^s g(\psi-r)s_a(r-s)\,\mathrm{d}r. \tag{A.16}$$

Appendix B. Values of $N_0^{(4)}$, $N_p^{(4)}$, and $D^{(4)}$

$$\begin{aligned}N_0^{(4)} = 8\{&4096\nu^{18} + 256(103\sigma^4 - 656)\nu^{16} + 512(133\sigma^8 - 1122\sigma^4 + 4440)\nu^{14} + 32(2819\sigma^{12}\\ &- 20\,480\sigma^8 + 147\,816\sigma^4 - 463\,360)\nu^{12} + 16(3971\sigma^{16} - 13\,216\sigma^{12} + 178\,656\sigma^8\\ &- 1\,166\,528\sigma^4 + 3\,211\,264)\nu^{10} + (20\,443\sigma^{20} + 88\,464\sigma^{16} - 63\,840\sigma^{12} - 8\,108\,032\sigma^8\\ &+ 38\,510\,592\sigma^4 - 89\,653\,248)\nu^8 - 4(61\sigma^{24} - 10\,441\sigma^{20} + 95\,508\sigma^{16} + 279\,008\sigma^{12}\\ &- 699\,392\sigma^8 + 15\,458\,304\sigma^4 - 11\,272\,192)\nu^6 - (1982\sigma^{28} + 24\,944\sigma^{24} + 53\,391\sigma^{20}\\ &- 3\,583\,616\sigma^{16} - 25\,366\,016\sigma^{12} - 73\,089\,024\sigma^8 - 136\,904\,704\sigma^4 - 79\,691\,776)\nu^4\\ &- 2(\sigma^4 + 16)^2(242\sigma^{24} + 1478\sigma^{20} + 3135\sigma^{16} + 22\,238\sigma^{12} + 130\,496\sigma^8 + 253\,440\sigma^4\\ &+ 196\,608)\nu^2 - \sigma^4(\sigma^4 + 1)(\sigma^4 + 16)^4(37\sigma^8 + 271\sigma^4 + 384)[(\sigma^2 + 2)^2 - 4\sigma^2]\},\end{aligned}$$

$$\begin{aligned}N_p^{(4)} = \sigma^4\{&-3840\nu^{16} - 5632(3\sigma^4 + 1)\nu^{14} - 32(903\sigma^8 + 400\sigma^4 + 4408)\nu^{12} - 64(363\sigma^{12}\\ &- 1281\sigma^8 - 11\,120\sigma^4 - 53\,552)\nu^{10} - (7467\sigma^{16} - 249\,280\sigma^{12} - 2\,750\,240\sigma^8\\ &- 4\,525\,056\sigma^4 + 15\,237\,120)\nu^8 + 2(426\sigma^{20} + 1\,227\,85\sigma^{16} + 1\,622\,656\sigma^{12} + 2\,767\,424\sigma^8\\ &- 7\,553\,024\sigma^4 + 12\,369\,920)\nu^6 + (1182\sigma^{24} + 102\,522\sigma^{20} + 1\,532\,477\sigma^{16} + 3\,306\,112\sigma^{12}\\ &- 13\,751\,808\sigma^8 - 4\,587\,520\sigma^4 - 18\,808\,832)\nu^4 + 2(\sigma^4 + 16)(138\sigma^{24} + 6871\sigma^{20}\\ &+ 38\,661\sigma^{16} - 67\,738\sigma^{12} - 169\,568\sigma^8 + 115\,200\sigma^4 + 253\,952)\nu^2\\ &+ (\sigma^4 + 1)^2(\sigma^4 + 16)^3(21\sigma^8 - 56\sigma^4 - 128)[(\sigma^2 + 2)^2 - 4\sigma^2]\},\end{aligned}$$

$$\begin{aligned}D^{(4)} = 2\,097\,152(\nu^2 + \sigma^4)[(\nu + 1)^2 + \sigma^4][(\nu - 1)^2 + \sigma^4][(\nu + 2)^2 + \sigma^4][(\nu - 2)^2 + \sigma^4]\\ \times [(\nu + 2)^2 + \tfrac{1}{4}\sigma^4]^3[(\nu - 2)^2 + \tfrac{1}{4}\sigma^4]^3.\end{aligned}$$

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