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Letter to the Editor

Revisiting Galerkin's method through global piecewise-smooth functions in Christides and Barr's cracked beam theory

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1. Introduction

It has been recognized that the dynamic behavior of structural mechanical components can be altered by local stiffness or mass changes. The monitoring of the variations of dynamic parameters can be used as a useful warning in diagnosing presence, location and extent of damage. Due to several advantages that can be expected with respect to other non-destructive evaluation (NDE) techniques, an enormous amount of investigations have been carried out in the last decades [1–3]. Among all the proposed methodologies certain analytical models that try to correlate the modal data variations with local damage play a significant role. This note deals with the model of Christides and Barr [4] that was subsequently reconsidered by Shen and Pierre [5].

In particular, Christides and Barr [4] introduced a theoretical model for uniform Bernoulli–Euler beams containing pairs of symmetric cracks. Namely, the differential equation of motion and the associated boundary conditions were consistently obtained through a variational statement. Such a theory was applied to the case of a beam simply supported at both ends containing a symmetric pair of open cracks in the middle of the beam. The decrease of the fundamental frequency, as a result of the cracks, was evaluated from a numerical and experimental point of view. The close agreement between analytical predicted frequency ratio ($\text{frequency}_{\textit{damaged beam}}/\text{frequency}_{\textit{undamaged beam}}$) and experimental ratios corroborated the proposed analytical model. Christides and Barr [4] were able to resolve the analytical problem by adopting a two-term trial function in the frame of the Rayleigh–Ritz method.

The model of Christides and Barr [4] was later reconsidered by Shen and Pierre [5], who used the Christides and Barr theory to accurately evaluate natural frequencies and mode shapes of cracked beams. In particular, Shen and Pierre [5] aimed at solving the boundary value problem introduced in Ref. [4] through a Galerkin procedure [6,7] in which the cracked beam deflection was expanded in a series of functions. They, however, pointed out that the convergence of Galerkin's procedure that is very slow for this type of problem and that, therefore, a new

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technique which increased the convergence speed was needed. The technique used in Ref. [5], to increase the convergence speed, consisted of adding up a supplementary function, having the same continuity properties as the exact solution of the eigenvalue problem, to a classical set of suitable infinitely differentiable co-ordinate functions. Numerical simulations showed how the convergence could thus be improved with respect to that based on using infinitely differentiable functions. Indeed, a minor number of co-ordinate functions ($\sim\frac{1}{3}$) was needed to solve the boundary value problem at a fixed convergence speed. The minor number of co-ordinate functions did not limit the application of Christides and Barr's model that was used to investigate different test cases [5].

This note aims at illustrating that the convergence speed of Galerkin's method, applied to the boundary value problem of Christides and Barr theory, can even be improved in comparison with the technique presented by Shen and Pierre [5]. Such an increased performance is obtained through recent global piece-wise smooth functions [8,9] (GPSFs). Namely, the convergence speed of Galerkin's method can be increased by a factor >2 with respect to the convergence speed based on the Shen and Pierre technique [5] referred to the natural frequencies of a cracked beam. Therefore, this short note shows the utility of certain GPSFs, recently introduced by Messina [8,9], in all those contexts where continuous and discrete modelling aspects must coexist in the same problem.

2. Theoretical description of the boundary value problem

2.1. A brief review of the cracked beam theory

The continuous model of a transversally vibrating beam presented by Christides and Barr [4] consists of a Bernoulli–Euler model extracted from the three-dimensional theory through Hu–Washizu–Barr variational equation [10]. The variational equation [4,5,10], allowing independent assumptions regarding displacement, velocities, strain and stress fields, provides, in the case of rectangular section beams (Fig. 1), a differential equation of free motion, which, separated in the

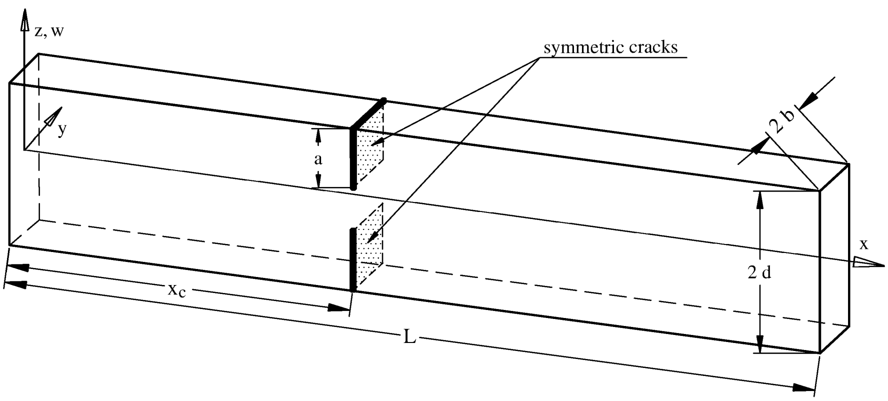

Fig. 1. Geometry and nomenclature of a cracked beam.

spatial domain (x), reduces to the following eigenvalue problem:

$$(EIQ\hat{w}'')'' - \omega^2 \rho A \hat{w} = 0 \quad (1)$$

with associated conditions at the ends. The symbols illustrated in Eq. (1) correspond to the following definitions:

$$Q(x) = [1 + (m-1)\exp(-2\alpha|x - x_c|/d)]^{-1}, \quad 1/m = (1 - CR)^3, \quad CR = a/d, \quad (2)$$

where EI, ω, ρ, A and $\hat{w}$ correspond to the bending stiffness, the circular frequency, the volumetric density, the transverse sectional area and the eigenfunctions, respectively. The term α is a dimensionless positive constant and constitutes a calibration parameter for the stress decay from the crack tip. For the purpose of the present note α is assumed to be 1.936 whilst $x_c/L = 0.5$ [5].

Eqs. (1) and (2) highlight that a uniform undamaged beam is characterized by $Q(x) = 1$ and $CR = 0$, in the sense that the presence of a crack locally modifies the bending stiffness through $Q(x)$. As it was pointed out by Shen and Pierre [5], the crack also modifies the continuity characteristics of the eigenfunctions in Eq. (1), which may be expected to be only C^2 ($\hat{w}$ is continuous up to the second derivative) in the variable space.

2.2. Galerkin solution proposed by Shen and Pierre

The continuity characteristics of the eigenfunctions associated with eigenvalue problem (1), as pointed out by Shen and Pierre [5], significantly deteriorate the convergence of the Galerkin procedure.

In particular, if the following expansion is considered in Eq. (1) and Galerkin's method applied:

$$\hat{w}(x) = \sum_{i=1}^{N} a_i \phi(x)_i \quad (3)$$

a discrete eigenvalue problem is obtained

$$\mathbf{Ka} = \omega^2 \mathbf{Ma}, \quad (4)$$

where $\mathbf{a} = (a_1, \ldots, a_N)^T$ and $\mathbf{K}$ and $\mathbf{M}$ are the stiffness and mass matrices depending on the base used in expansion (3). The eigenvalues and eigenvectors evaluated through Eq. (4) analytically furnish approximated eigenvalues and eigenfunctions through Eq. (3). Such an approximation can provide exact quantities whenever a sufficiently high number (N) of co-odinate functions $\phi(x)_i$, that are compatible with boundary conditions, is considered.

With regard to a cracked beam, simply supported at both ends (SS), the convergence speed deteriorates when the C^2-eigenfunctions ($\hat{w}$) are expanded in Eq. (3) on a base of infinitely differentiable functions (e.g. $\phi(x)_i = \sin(i\pi x/L)$).

Therefore, Shen and Pierre [5] suggested expanding the eigenfunctions by adding a two-term polynomial, having the continuity characteristics of the unknown eigenfunctions, to the base of infinitely differentiable functions (3). The two-terms polynomial proposed by Shen and Pierre [5] corresponds to the following function:

$$\phi(x)_0 = \begin{cases} (\xi_c^2 - 3\xi_c + 2)\xi + \left(1 - \dfrac{1}{\xi_c}\right)\xi^3, & 0 < \xi \leqslant \xi_c, \\ \xi^3 - 3\xi^2 + (2 + \xi_c^2)\xi - \xi_c^2, & \xi_c < \xi \leqslant 1, \end{cases} \quad (5)$$

where ξ is indicating the dimensionless x-co-ordinate (x/L). Function (5) not only fulfills the boundary conditions related to an SS beam, but it is also continuous up to the second derivative. Therefore, in the technique proposed by Shen and Pierre [5] the eigenfunctions are expanded as

$$\hat{w}(x) = a_0\phi(x)_0 + \sum_{i=1}^{N} a_i\phi(x)_i. \quad (6)$$

2.3. Galerkin solution based on GPSFs

In Ref. [8] a novel set of functions was introduced. Such an effort was prompted by the requirement to study the free vibrations of multilayered plates by modelling the relevant displacement and stress quantities in a global sense through the thickness of the plate. The performances of this set of functions were shown from a mathematical point of view (approximating certain piece-wise differentiable continuous functions in the mean) as well as in the frame of a boundary value problem (i.e., studying certain simply supported freely vibrating plates). In this latter case the use of these functions (the so-called global piece-wise smooth functions: GPSFs) allowed an efficient approach to the three-dimensional exact results. Based on the encouraging results achieved in Refs. [8,9], it is here shown how the GPSFs can serve different areas where continuous and discrete modelling aspects must coexist in the same problem.

The dimensionless domain with which this note deals with is [0, 1]. The crack is located in the middle of an SS-beam. The co-ordinate functions are herein created domain by domain, firstly in order to fulfill the boundary conditions and secondly to fulfill the continuity requirements at the cracked section of the beam. In particular, in the domain [0, 1] a polynomial base made up of co-ordinate functions constrained to fulfill the following conditions:

$$\varphi(0)_i = 0, \quad \varphi(0)_i'' = 0 \quad (7)$$

can be extracted from a classical polynomial base by eliminating constant and second order terms:

$$\varphi(\xi)_1 = \xi, \quad \varphi(\xi)_2 = \xi + \xi^3, \quad \varphi(\xi)_3 = \xi + \xi^3 + \xi^4, \quad \varphi(\xi)_4 = \xi + \xi^3 + \xi^4 + \xi^5, \ldots . \quad (8)$$

Moreover, in order to have polynomials with similar magnitude in [0 1], sequence (8) can also be made orthonormal. In this latter respect sequence (9) illustrates the first three orthogonal polynomials that were obtained through a symbolic software implementation of the Gram–Schmidt orthogonalization process:

$$\tilde{\varphi}(\xi)_1 = \sqrt{3}\xi, \quad \tilde{\varphi}(\xi)_2 = (-3\xi + 5\xi^3)\sqrt{7}/2, \quad \tilde{\varphi}(\xi)_3 = (5\xi - 35\,\xi^3 + 32\xi^4)3/2, \ldots . \quad (9)$$

which still preserves conditions (7).

The polynomials (9) can be re-adapted in the variable space [0, ξ_c] by substituting ξ with ξ/ξ_c. They (9) can also be adapted in [ξ_c, 1], as its mirror image, by substituting ξ with $(1-\xi)/(1-\xi_c)$. The mentioned local polynomials adapted in the subdomains [0, ξ_c] and [ξ_c, 1] can be joined through the scaling and graph-based process illustrated in Ref. [8] to identify independent global functional components ($\phi(\xi)_i$), which fulfill the following external and internal boundary conditions:

$$\begin{aligned} &\phi(0)_i = 0, \quad \phi(0)_i'' = 0, \\ &\phi(1)_i = 0, \quad \phi(1)_i'' = 0, \\ &\phi(\xi_c^-)_i = \phi(\xi_c^+)_i. \end{aligned} \tag{10}$$

In order to clarify the concept, the first three polynomial terms illustrated in Eq. (9) can be considered. From these three polynomials, a set of five (2(3−1)+1) GPSFs can be identified to constitute a partial base in [0, 1] as shown in Fig. 2. The selection carried out for the five functional components of Fig. 2 also corresponds to the graph of Fig. 3, which illustrates the local

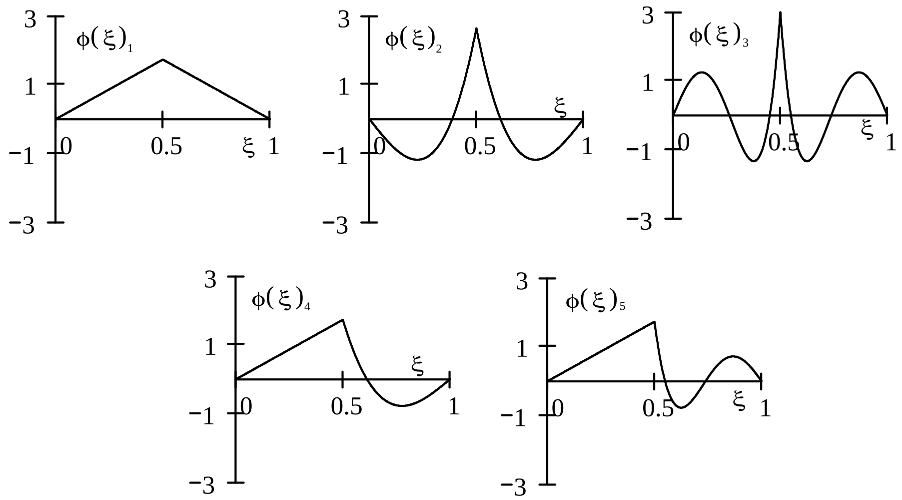

Fig. 2. A set of five linearly independent global piece-wise smooth functions: 3 functional components in 2 subdomains.

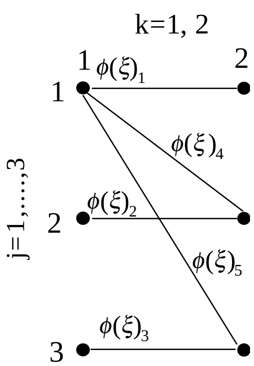

Fig. 3. Representation of extractable independent paths of GPSFs.

functional components (9) (adapted in the aforementioned two sub domains) and its joining by nodes and branches, respectively.

However, the GPSFs associated with the graph of Fig. 3 and fulfilling conditions (7) are still not an appropriate functional base (for $N \to \infty$) in the Galerkin procedure for solving eigenvalue problem (1). Indeed, the set of GPSFs still needs to be C^2 in ξ_c. Here such a requirement is carried out by adding to every GPSF, in the second subdomain (i.e. $[\xi_c, 1]$), a fourth order polynomial $y(\xi)$ that satisfies five conditions: continuity of the first and second derivative in ξ_c, $y(1) = y(\xi_c) = 0$

Table 1
Nomenclature and expansions used in Galerkin's method

Method and acronym	Expansion	Functional components $(\phi(x)_i)$
Classical sinusoidal base: S	$\hat{w}(x) = \sum_{i=1}^{N} a_i \phi(x)_i$	$\sin(i\pi x/L)$
Shen and Pierre [5]: SP	$\hat{w}(x) = a_0 \phi(x)_0 + \sum_{i=1}^{N} a_i \phi(x)_i$	$\phi(x)_0$: Eq. (5) with $\xi_c = 0.5$; $\sin(i\pi x/L)$
Present method: GPSFs	$\hat{w}(x) = \sum_{i=1}^{N} a_i \phi(x)_i$	$\phi(x)_i$: GPSFs

Table 2
Decrease of the first three natural frequencies with respect to the undamaged beam ($L/2d = 18.11$, $\alpha = 1.936$, $CR = 1/2$, $x_c = L/2$)

N	GPSFs			SP-method			S-method		
	I	II	III	I	II	III	I	II	III
9	0.9433	0.9997	0.9493	0.9443	0.9997	0.9497	0.9614	0.9997	0.9644
11	0.9354	0.9997	0.9424	0.9388	0.9997	0.9452	0.9589	0.9997	0.9622
13	0.9285	0.9997	0.9369	0.9340	0.9997	0.9414	0.9564	0.9997	0.9600
15	0.9231	0.9997	0.9326	0.9300	0.9997	0.9381	0.9538	0.9997	0.9578
17	0.9191	0.9997	0.9296	0.9267	0.9997	0.9355	0.9513	0.9997	0.9557
19	0.9165	0.9997	0.9275	0.9239	0.9997	0.9333	0.9487	0.9997	0.9535
21	0.9148	0.9996	0.9263	0.9217	0.9997	0.9316	0.9463	0.9997	0.9515
23	0.9139	0.9996	0.9255	0.9199	0.9997	0.9302	0.9439	0.9997	0.9495
25	0.9134	0.9996	0.9251	0.9185	0.9997	0.9291	0.9416	0.9997	0.9476
27	0.9131	0.9996	0.9249	0.9173	0.9997	0.9282	0.9394	0.9997	0.9458
29	0.9130	0.9996	0.9248	0.9164	0.9997	0.9275	0.9374	0.9997	0.9441
31	0.9129	0.9996	0.9248	0.9157	0.9997	0.9269	0.9355	0.9997	0.9426
33	0.9129	0.9996	0.9248	0.9152	0.9997	0.9265	0.9337	0.9997	0.9411
40				0.9138	0.9997	0.9255	0.9292	0.9996	0.9374
50				0.9132	0.9996	0.9250	0.9237	0.9996	0.9331
60				0.9130	0.9996	0.9249	0.9202	0.9996	0.9304
70				0.9129	0.9996	0.9248	0.9179	0.9996	0.9286
80				0.9129	0.9996	0.9248	0.9165	0.9996	0.9275
90							0.9155	0.9996	0.9267
100							0.9148	0.9996	0.9262
110							0.9144	0.9996	0.9259
120							0.9140	0.9996	0.9256
130							0.9138	0.9996	0.9254
140							0.9136	0.9996	0.9253
150							0.9135	0.9996	0.9252

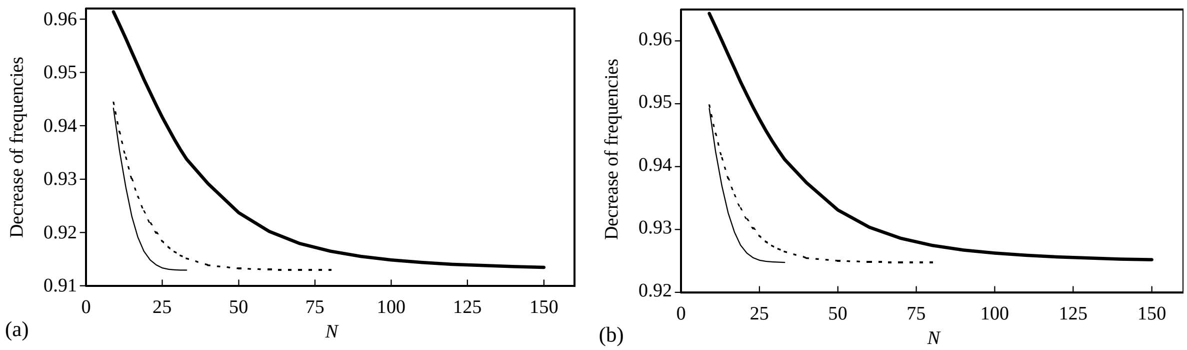

Fig. 4. Decrease of (a) the fundamental and (b) the third natural frequency: comparison of different methods (—, GPSFs, - - - SP; —, S).

and $y(1)''=0$. This assures the C^2-continuity in ξ_c for all GPSFs and preserves the external boundary conditions. However, in such a circumstance, $\phi(\xi)_1$, $\phi(\xi)_4$ and $\phi(\xi)_5$ (Fig. 2) become linearly dependent on the degree of the added polynomial $y(\xi)$ and, therefore, two of those three GPSFs were not considered with expansion (3) in Galerkin's method.

3. Numerical comparisons and closure

In the following comparisons the results were achieved through Galerkin's method as indicated in Table 1.

Table 2 shows the comparison between the aforementioned three methods. This table illustrates a convergence test on the lowest three frequencies when a mid-span symmetric crack is present. All results are presented in terms of frequency ratio ($\text{frequency}_{damaged\ beam}/\text{frequency}_{undamaged\ beam}$).

A perusal of Table 2 firstly confirms what Shen and Pierre [5] reported: the convergence speed is highly deteriorated when the S-method is considered. The SP-method is able to improve the performance of Galerkin's method with a factor around 3. Indeed 50 terms in SP-method (0.9132, 0.9996, 0.9250) provide an accuracy comparable to that achieved with the S-method with 150 terms (0.9135, 0.9996, 0.9252). However, Table 2 illustrates how the GPSFs are able to perform even better with respect to the SP-method. Indeed 31 terms of GPSFs are able to achieve identical frequency ratios (0.9129, 0.9996, 0.9248) achieved through the SP-method with 70 terms. Thus the GPSFs provide an increase in the convergence speed related to the SP-method of a factor about 2.

The frequency ratio concerning the fundamental and the third frequency (the most sensitive to a mid-span crack of an SS-beam) is also illustrated in Fig. 4. which clarifies the advantages obtained in Galerkin's method when the SP-method and, even better, when GPSFs are used.

It is concluded that the GPSFs are able to perform better than the other analyzed methods because they constitute a complete base to global approximation in the infinite functional space C^∞-functions.

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