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Journal of Sound and Vibration 263 (2003) 945–953
JOURNAL OF SOUND AND VIBRATION
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Letter to the Editor

Forced vibrations of a rigid circular plate on a tensionless Winkler edge support

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Received 5 July 2002; accepted 22 October 2002

1. Introduction

The response of circular plates supported by a Winkler foundation has been studied widely by assuming that the foundation reacts in compression as well as in tension. The assumption that the contact between the plate and its support is established continuously simplifies the problem. It is well known that in many practical cases, this assumption is questionable. When the foundation reacts only in compression, then the problem becomes a non-linear one because the boundary depends on the configuration of the loading [1]. However, there are various investigations dealing with beams and plates resting on a foundation that reacts in compression only. Various authors have studied circular plates on a tensionless foundation subjected to static loading mostly by applying approximate solutions techniques to the non-linear governing equations of the problem [2–6]. The governing equations of the unilateral contact problem can be also derived by using a variational formulation [7,8]. This may have special advantage for obtaining boundary conditions of the problem, when a two-parameter soil model is considered. However, the essential step in the solution is the selection of the numerical methodology to deal with the governing equations including the boundary conditions. Usually, the solution of the governing equations can be accomplished by applying the finite element technique or Galerkin and Ritz methods. When vibrations of the plate are investigated, the contact region of the plate becomes a function of time; the problem appears to be a moving boundary value problem. The treatment of such problems requires step-wise solutions by updating the contact regions, which changes the behaviour of the problem continuously [9]. In the present study, a rigid circular plate supported by a tensionless Winkler support along the edge of the plate is considered. The plate is assumed to be subjected to a concentrated load applied off centre and a uniformly distributed load. Numerical results are presented in figures to illustrate the behaviour of the plate under static and dynamic loadings and to determine the effects of the system parameters.

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doi:10.1016/S0022-460X(02)01472-4

2. Statement of the problem

A circular rigid plate of radius a, of mass M subjected to a concentrated load P having an eccentricity B and a uniformly distributed load Q is considered. The plate is assumed to be supported along its edge by tensionless Winkler support of modulus K_e as shown in Fig. 1. Since the loading and the configuration of the plate are symmetric with respect to one of the axes of the plate, the displacement shape of the plate $W(R, \theta, t)$ will display the same symmetry. Consequently, the equations of motion of the plate in vertical translation and rotation about a horizontal axis can be expressed as

$$M\frac{\partial^2}{\partial t^2}W(R=0,\theta,t) = \pi a^2 Q + P - 2K_e a\int_0^\pi H(\theta,t)W(R=a,\theta,t)\,\mathrm{d}\theta,$$

$$0.25Ma^2\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial R}W(R=0,\theta,t) = PB - 2K_e a^2\int_0^\pi H(\theta,t)W(R=a,\theta,t)\cos\theta\,\mathrm{d}\theta. \tag{1}$$

Since the support reacts in compression only, generally a lift-off is expected. Because the plate and the loading as well as the supporting are symmetric with respect to one of the diameters of the rigid plate, the contact and the lift-off of the plate will be established only along one part of the edge support having the same symmetry as shown in Fig. 1. The tensionless character of the edge support of the plate is taken into consideration in equations of motion (1) by introducing the contact function $H(\theta, t)$ defined as

$$H(\theta,t) = \begin{cases} 1 & \text{for } W(R=a,\theta,t) > 0, \\ 0 & \text{for } W(R=a,\theta,t) \leqslant 0. \end{cases} \tag{2}$$

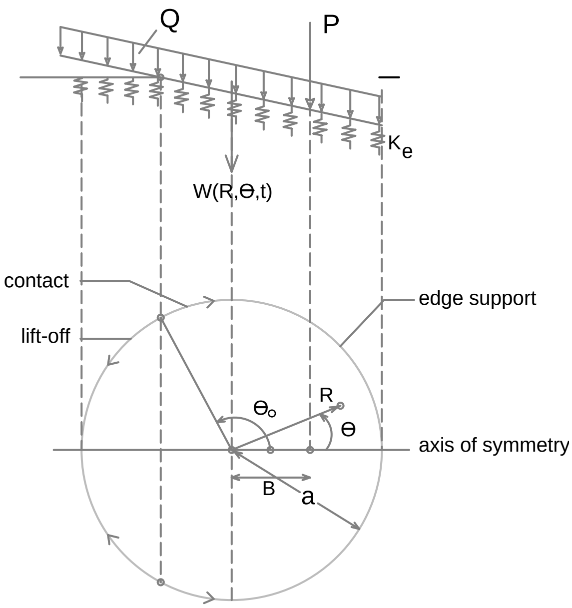

Fig. 1. Circular plate supported along its edge.

The contact function ensures that the integrations in Eq. (1) are carried along the contact curve. The translation and rotation of the rigid plate can be expressed to reflect the symmetry of the motion as

$$W(R, \theta, t) = aw(r, \theta, t) = a[R_0(t) + R_1(t) r \cos\theta], \tag{3}$$

where $r = R/a$ and $R_0(t)$ and $R_1(t)$ represent the rigid translation and rotation of the plate. The lift-off angle θ_0 is to be evaluated from

$$\theta_0 = \arccos\left(-\frac{R_0}{R_1}\right), \tag{4}$$

provided that $0 \leqslant \theta_0 \leqslant \pi$, where the cases $\theta_0 = 0$ ($R_0/R_1 \leqslant -1$) and $\theta_0 = \pi$ ($R_0/R_1 \geqslant 1$) represent the total lift-off and the total contact of the plate to the edge support, respectively. Substituting the displacement function (3) into equations of motion (1), the following system of two differential equations are obtained:

$$\mathbf{M}\ddot{\mathbf{R}} + \mathbf{K}\mathbf{R} = \mathbf{F}, \tag{5}$$

where the dots denote the differentiation with respect to the non-dimensional time τ and

$$\mathbf{R}^T = [R_0(t),\ R_1(t)], \quad \mathbf{M} = [m_{ij}], \quad \mathbf{K} = [k_{ij}(t)], \quad \mathbf{F} = [f_i(t)],$$
$$m_{11} = 1, \quad m_{12} = m_{21} = 0, \quad m_{22} = 0.25,$$

$$\begin{gathered} k_{11}(t) = 2k_e \int_0^\pi H(\theta, t)\,\mathrm{d}\theta, \quad k_{12}(t) = k_{21}(t) = 2k_e \int_0^\pi H(\theta, t) \cos\theta\,\mathrm{d}\theta, \\ k_{22}(t) = 2k_e \int_0^\pi H(\theta, t) \cos^2\theta\,\mathrm{d}\theta, \quad f_1(t) = q(t), \quad f_2(t) = p(t)b. \end{gathered} \tag{6}$$

The non-dimensional parameters introduced are defined as follows

$$\tau = t\sqrt{\frac{g}{a}}, \quad p = \frac{P}{Mg}, \quad q = \frac{\pi a^2 Q}{Mg}, \quad b = \frac{B}{a}, \quad k_e = \frac{K_e a^2}{Mg}. \tag{7}$$

Although the governing equation of problem (5) represents the small amplitude motion of the rigid plate, it is highly non-linear due to the tensionless character of the edge support that leads a time-dependent stiffness matrix **K**.

When a regular Winkler edge support is assumed, it can be shown easily that the system has two free vibration periods

$$T_0 = \sqrt{\frac{2\pi}{k_e}}, \quad T_1 = \sqrt{\frac{\pi}{k_e}}, \tag{8}$$

which correspond to the vertical and rotational vibrations of the rigid plate.

Furthermore, the case of the static equilibrium problem can be investigated by using the static version of Eq. (4)

$$\mathbf{K}\mathbf{R} = \mathbf{F}, \tag{9}$$

which yields the static configuration of the rigid plate subjected to uniformly distributed load Q and the vertical off-centre load P. Assuming that in the case of static equilibrium the contact takes place for $0 \leqslant \theta \leqslant \theta_0$, the elements of the matrix $\mathbf{K}$ can be evaluated as follows:

$$k_{11} = 2k_e\theta_0, \quad k_{12} = k_{21} = 2k_e \sin\theta_0, \quad k_{22} = k_e(\theta_0 + 0.5\sin 2\theta_0).$$

When only one of the loading cases, Q or P, is considered, it can be deduced that the R_0 and R_1 depend linearly on the loading, where the lift-off angle θ_0 does not depend on the level of the loading, as reported in various similar studies [1–3]. On the other hand, when two types of loading are present, then the lift-off angle will depend on the ratio of the loads p/q.

3. Numerical solutions and discussion

The effects of the various parameters of the system are studied by obtaining various numerical results. When partial contact takes place, the solution of the static case requires an iterative solution. On the other hand, the dynamic behaviour of the system is obtained by employing a numerical solution procedure for the governing differential equation (5) along the time axis and by assuming initial conditions for the problem. At each time step the contact angle θ_o is updated according the displacements of the plate at the previous time step and the elements of matrix $\mathbf{K}$ are evaluated accordingly.

The static configuration of the plate is obtained assuming $k_e = 1.0$ and $p = 1.0$ for various values of the eccentricity b and that of the uniformly distributed load q. Figs. 2(a)–(c) show the variations of the lift-off angle θ_0, the non-dimensional vertical displacement R_0 and the rotation R_1, respectively. As it is seen, the lift-off comes into being, when the eccentricity of the vertical load increases and when the distributed load decreases. When no lift-off is present, R_0 and R_1 are linear functions of the load q and the eccentricity b, respectively. However, the dependency becomes non-linear, when lift-off appears and the vertical displacement and rotation increase rapidly. Figs. 3(a)–(c) display similar variations for $k_e = 1.0$ and $q = 0.4$ for various values of the eccentricity b and the vertical load p. Inspection of the figures reveals that full contact is established, when the eccentricity b and the load p decrease. The linear dependency of R_0 and R_1 on p and b can be observed, when no lift-off takes place. However, large and non-linear increases in R_0 and R_1 arise due to the lift-off the plate from the support.

Although the present formulation does not have any restriction concerning the time variation of the loads as well as the initial conditions of the system for numerical evaluation, it is assumed that the system starts from rest and the loads are applied instantaneously. The dynamic behaviour of the system is represented for $p = 1.0$, $b = 0.5$ and $k_e = 1.0$ in Fig. 4, where the time variation of the lift-off angle $\theta_0(t)$, the displacement $R_0(t)$ and the rotation $R_1(t)$ are displayed for various values of the load q. As Fig. 4(a) shows, the full and partial contact cases follow each other in course of the oscillation for the present numerical combination of the parameters. As expected, the variation displays a highly non-linear behaviour. When the full contact of the plate is developed, the plate has two free vibration periods as given in Eq. (8) which are $T_0 = 2.507$ and $T_1 = 1.772$ for the given value of k_e. As Figs. 4(b) and (c) reveal, $R_0(t)$ and $R_1(t)$ exhibit oscillations similar to harmonic variations having the corresponding approximate periods. However, as lift-off appears the harmonic variation vanishes and non-linear motion takes place, while the system softens and the oscillations lengthen. Fig. 5 displays $\theta_0(t)$, $R_0(t)$ and $R_1(t)$

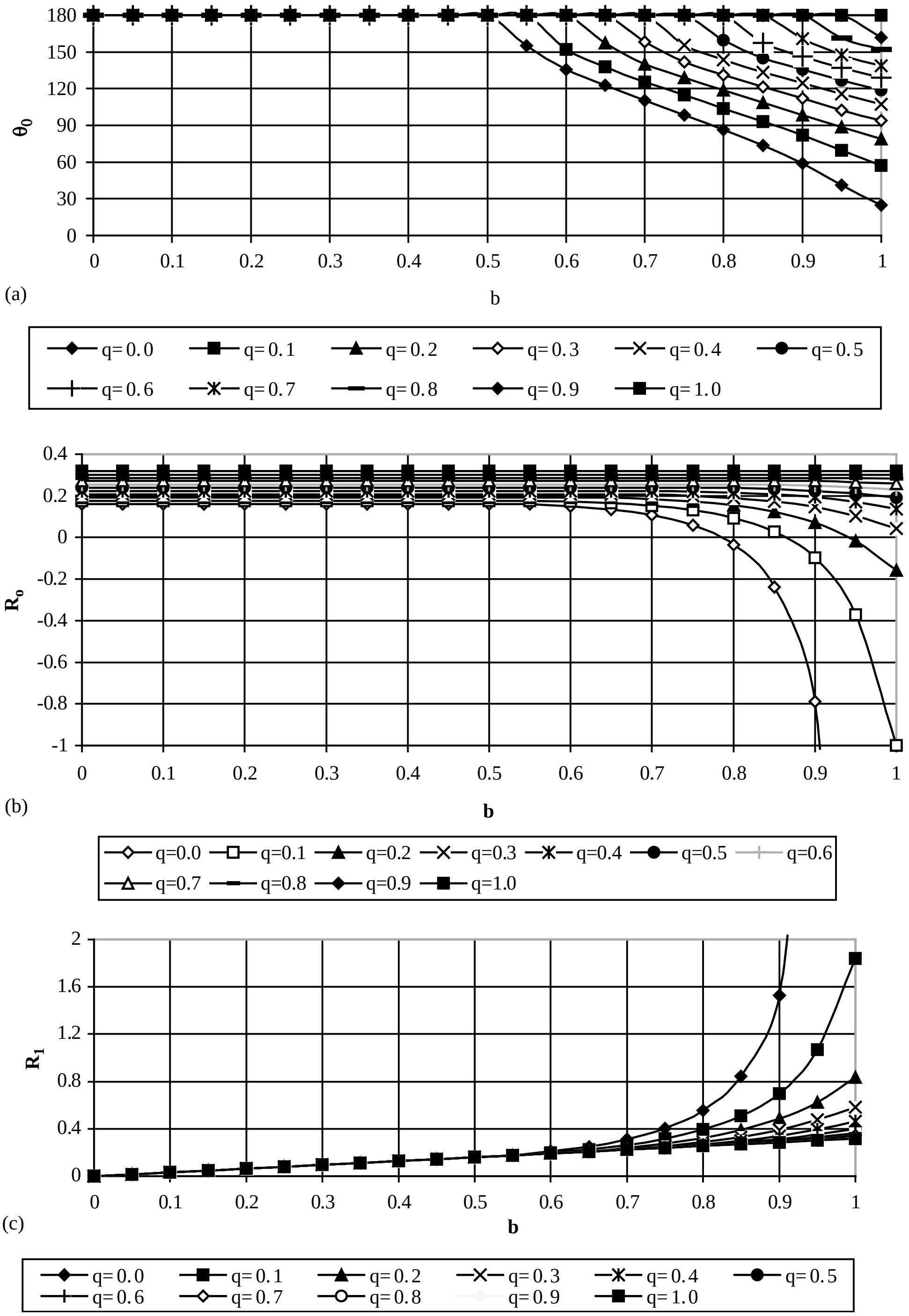

Fig. 2. Variations of (a) θ_0 lift-off angle, (b) R_0 vertical displacement and (c) R_1 rotation depending on b eccentricity for $k_e = 1.0$, $p = 1.0$ and various values of q.

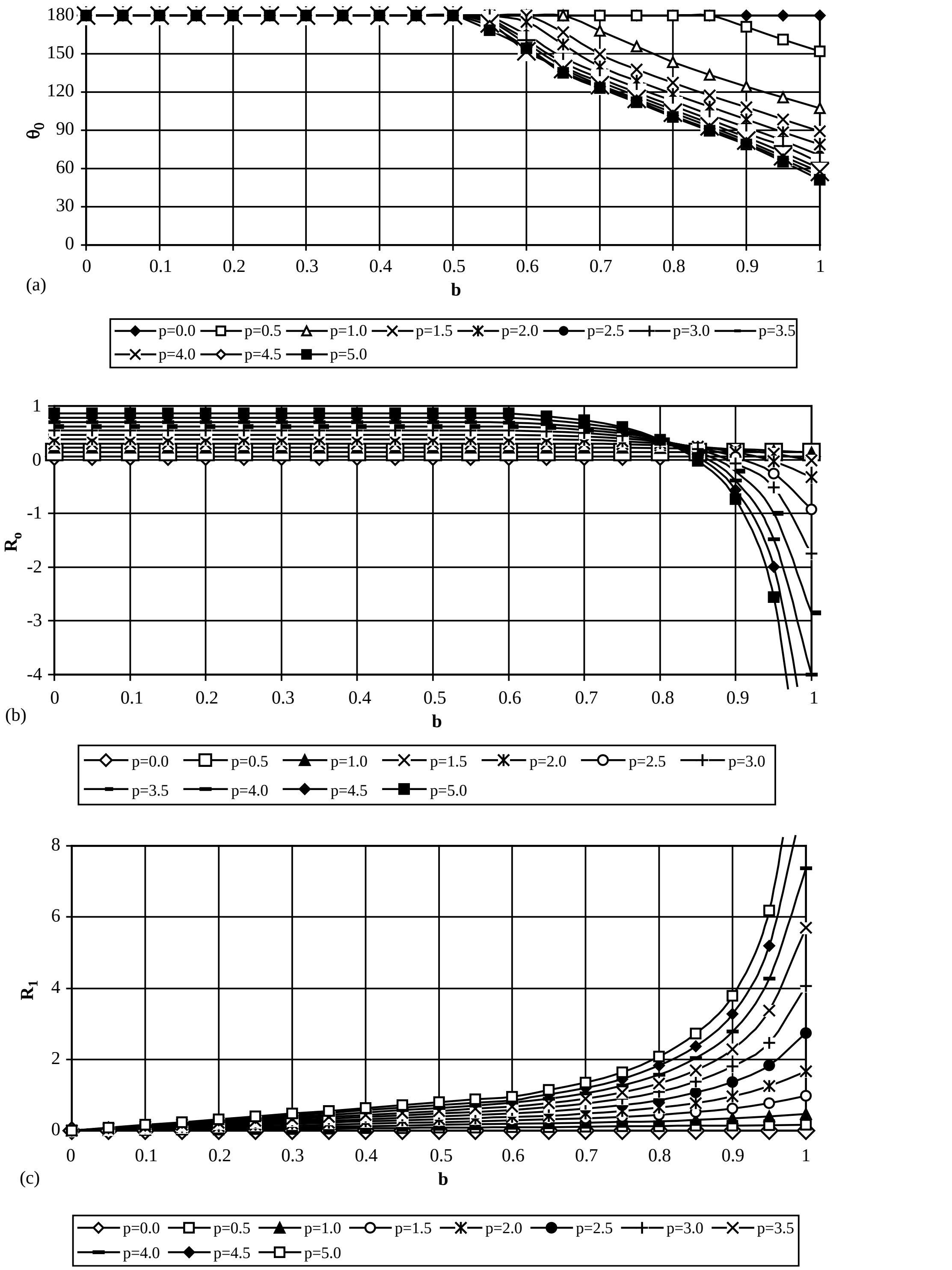

Fig. 3. Variations of (a) θ_0 lift-off angle, (b) R_0 vertical displacement and (c) R_1 rotation depending on b eccentricity for $k_e = 1.0$, $q = 0.4$ and various values of p.

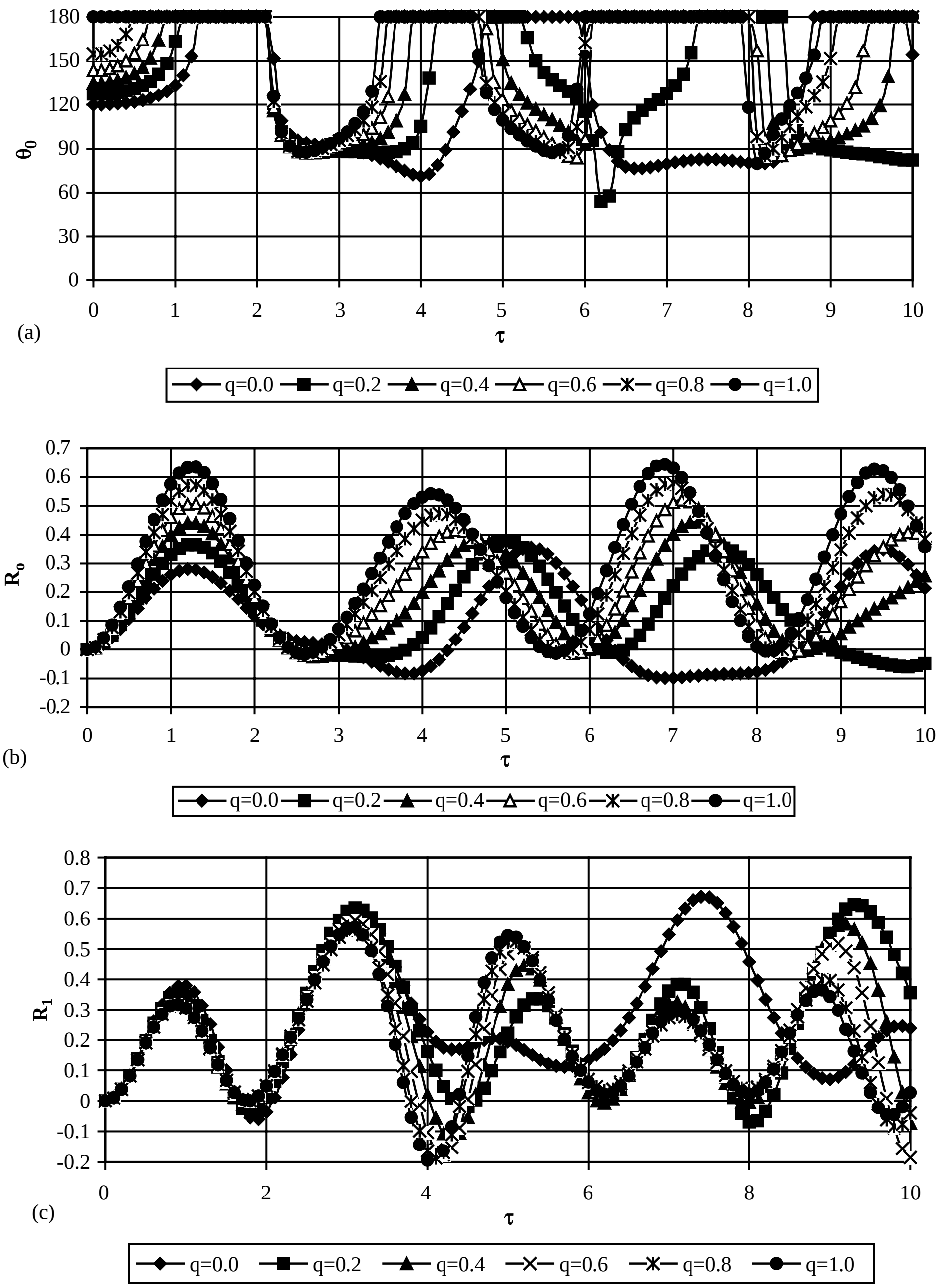

Fig. 4. Time variations of (a) $\theta_0(\tau)$ lift-off angle, (b) $R_0(\tau)$ vertical displacement and (c) $R_1(\tau)$ rotation for $k_e = 1.0$, $p = 1.0$, $b = 0.5$ and various values of q.

for $p = 1.0$, $q = 0.4$ and $k_e = 1.0$ for various values of the eccentricity b. As it is seen, the time variations of these parameters deviate from the harmonic variations as partial lift-off takes place.

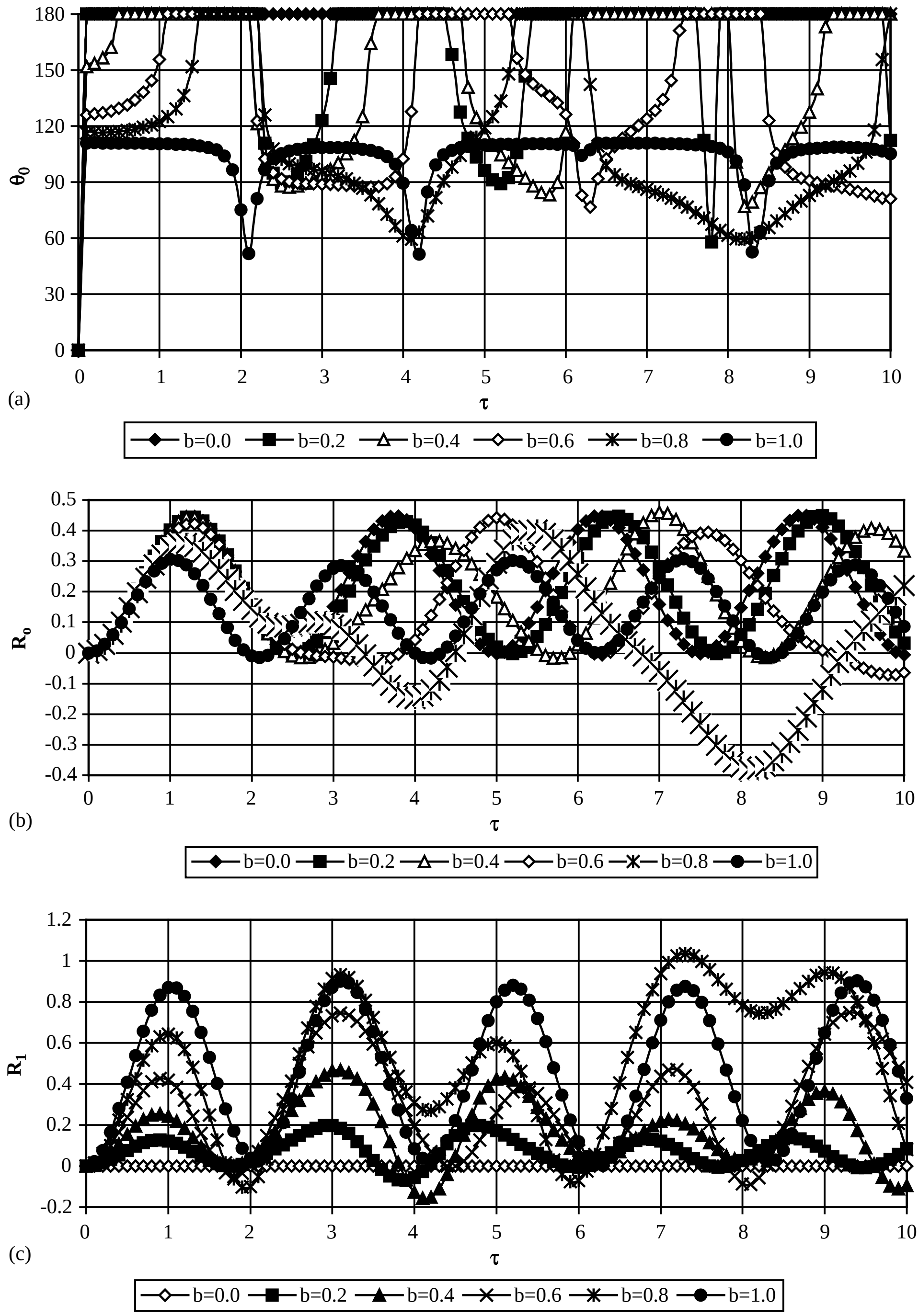

Fig. 5. Time variations of (a) $\theta_0(\tau)$ lift-off angle, (b) $R_0(\tau)$ vertical displacement and (c) $R_1(\tau)$ rotation for $k_e = 1.0$, $p = 1.0$, $q = 0.4$ and various values of b.

4. Conclusion

Forced vibrations of a rigid plate supported along its edge by unilateral elastic edge support have been studied. The plate is assumed to be subjected to uniformly distributed load and an off-centre vertical load: Although the displacements of the plate are assumed to be small, the problem appears to be a non-linear one due to the tensionless character of the edge support. The static problem is evaluated as a special case. Numerical results are obtained to determine the effects of the system parameters on the dynamic behaviour of the plate. It is seen that lift-off has a significant effect on the vibrations of the plate, the oscillations are lengthened and the amplitudes become larger, because the unilateral support model is relatively less constrained.

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