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Letter to the Editor

Large-amplitude free vibrations of uniform beams on Pasternak foundation

G. Venkateswara Rao

Structural Engineering Group, Vikram Sarabhai Space Centre, Trivandrum 695 022, India

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1. Introduction

Uniform slender beams with axially immovable edges, are widely used structural elements in many fields of engineering. These beams when subjected to severe dynamic loads, experience large amplitudes and the linear theory to predict the frequencies is not applicable. Geometrical non-linear theories have to be considered in the formulations to study them.

This was the subject matter of the classic work of Woinowsky-Krieger [1], wherein the non-linear differential equation, which contains the tension term developed because of large amplitudes, was solved. Continuum [2] and different finite-element formulations [3–5] are also available for solving the large-amplitude vibration problem of uniform slender beams.

The effect of the elastic foundation has to be considered in the formulation for the cases of rails, underground pipes, etc. While the Winkler foundation [6] model is the simplest, two-parameter foundation models [7] represent the foundation characteristics more accurately for practical purposes. Pasternak foundation model [8] is one widely used two-parameter model.

The large-amplitude vibration problem of uniform slender beams on the Pasternak foundation is studied using the conservation of total energy principle in this paper. The temporal equation governing the large-amplitude vibrations is directly obtained in this approach, by assuming a suitable admissible spatial function satisfying the boundary conditions of the beams. The temporal equation can be solved by using any standard numerical integration scheme.

The numerical results in the form of ratio of the fundamental non-linear frequency to the linear frequency for both the simply supported and clamped beams are presented in the tables for several values of the amplitude parameter and the two stiffness parameters of the Pasternak foundation.

E-mail address: gv_rao@vssc.org (G.V. Rao).

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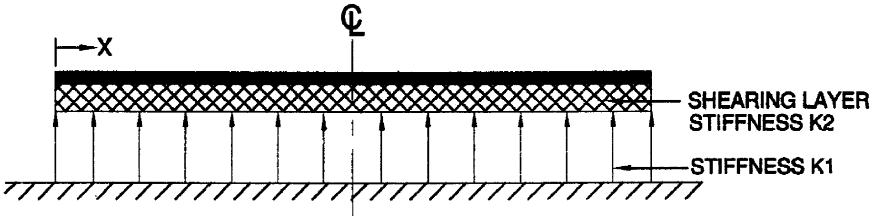

Fig. 1. A uniform beam on Pasternak foundation.

2. Formulation

For a uniform slender beam on Pasternak foundation (Fig. 1) undergoing large-amplitude vibrations, the total energy U_T at any given instant of time is

$$U_T = U_E + U_F + W_P + T = Constant. \tag{1}$$

In Eq. (1), U_E is the elastic energy, U_F is the energy stored in the foundation, W_P is the work done by the tension 'P' developed in the beam due to large amplitudes and T is the kinetic energy and these are given by

$$U_E = \frac{EI}{2}\int_0^L (w'')^2\, \mathrm{d}x, \tag{2}$$

$$U_F = \frac{k_1}{2}\int_0^L w^2\, \mathrm{d}x + \frac{k_2}{2}\int_0^L (w')^2\, \mathrm{d}x, \tag{3}$$

$$W_P = \frac{P}{2}\int_0^L \tfrac{1}{2}(w')^2\, \mathrm{d}x \tag{4}$$

and

$$T = \frac{m}{2}\int_0^L (\dot{w})^2\, \mathrm{d}x, \tag{5}$$

where E is the Young's modulus, I is the area moment of inertia, L is the length of the beam, w is the lateral displacement, x is the axial coordinate, k_1 and k_2 are the two stiffnesses of the foundation (k_1 being the usual Winkler stiffness), $(\ )'$ represents differentiation with respect to 'x' and $(\dot{\ })$ represents differentiation with respect to time 't'.

The tension 'P' [Eq. (4)] developed in the beam due to large amplitudes is [1]

$$P = \frac{EI}{2Lr^2}\int_0^L (w')^2\, \mathrm{d}x, \tag{6}$$

where r is the radius of gyration.

A variable separable solution for 'w' is assumed as

$$w(x,t) = W\, w(x), \tag{7}$$

where W is a function of time alone and $w(x)$ is the admissible spatial distribution satisfying the geometric boundary conditions of the beam. The following simple trigonometric functions

$$w(x) = \sin\frac{\pi x}{L} \tag{8}$$

and

$$w(x) = \frac{1}{2}\left(1 - \cos\frac{2\pi x}{L}\right) \tag{9}$$

satisfy all the boundary conditions, namely

$$w(0) = w''(0) = w(L) = w''(L) = 0 \tag{10}$$

and

$$w(0) = w'(0) = w(L) = w'(L) = 0 \tag{11}$$

for the cases of a simply supported beam and a clamped beam, respectively. It may be noted here that the function $w(x)$ has a maximum amplitude at the middle of the span of the beam.

Substituting Eq. (7), with an appropriate function for $w(x)$ as given in Eqs. (8) and (9) in Eq. (1), we get, after evaluating the integrals and simplification,

$$\dot{W}^2 + \frac{\alpha_1}{im}W^2 + \frac{\alpha_2}{m}\frac{1}{jr^2}W^2 = Constant = H, \tag{12}$$

where α_1 and α_2 are

$$\alpha_1 = EI\left(\frac{\pi}{L}\right)^4 + k_1 + k_2\left(\frac{\pi}{L}\right)^2 \tag{13}$$

and

$$\alpha_2 = EI\left(\frac{\pi}{L}\right)^4 \tag{14}$$

for the simply supported beam, with $i = 1$ and $j = 8$, and

$$\alpha_1 = EI\left(\frac{2\pi}{L}\right)^4 + 3k_1 + k_2\left(\frac{2\pi}{L}\right)^2 \tag{15}$$

and

$$\alpha_2 = EI\left(\frac{2\pi}{L}\right)^4 \tag{16}$$

for the clamped beam, with $i = 3$ and $j = 32$.

3. Linear free vibrations

If the third term in Eq. (12) is neglected, the governing temporal differential equation for the linear vibrations is

$$\dot{W}^2 + \frac{\alpha_1}{im}W^2 = H. \tag{17}$$

Using the condition, $\dot{W} = 0$ at $W = W_m$, where, W_m is the maximum amplitude, the value of 'H' is

$$H = \frac{\alpha_1}{im} W_m^2. \tag{18}$$

Using Eq. (18), Eq. (17) can be written as

$$\frac{dW}{dt} = \left[\frac{\alpha_1}{im}(W_m^2 - W^2)\right]^{1/2}. \tag{19}$$

Eq. (19) can be integrated as

$$\int_0^{2\pi/\omega_L} dt = \int_0^{W_m} \frac{dW}{\left[(\alpha_1/im)(W_m^2 - W^2)\right]^{1/2}}. \tag{20}$$

Assuming

$$W = W_m \sin\theta \tag{21}$$

and carrying out the integrations of Eq. (20), we obtain the time period

$$T = \frac{2\pi}{\omega_L} = \frac{2\pi}{(\alpha_1/im)^{1/2}}, \tag{22}$$

where ω_L is the linear radian frequency, or

$$\omega_L = \left(\frac{\alpha_1}{im}\right)^{1/2}. \tag{23}$$

The linear frequency parameter $\lambda_L\ (= m\omega_L^2 L^4/EI)$ can be obtained after substituting for α_1, i and m as

$$\lambda_L = [\pi^4 + \lambda_{F10} + \lambda_{F20}\,\pi^2] \tag{24}$$

for the simply supported beam and

$$\lambda_L = \tfrac{16}{3}\pi^4 + \lambda_{F10} + \tfrac{3}{4}\lambda_{F20}\pi^2 \tag{25}$$

for the clamped beam. λ_{F10} and λ_{F20} are the foundation stiffness parameters

$$\lambda_{F10} = \frac{k_1 L^4}{EI} \tag{26}$$

and

$$\lambda_{F20} = \frac{k_2 L^2}{EI}. \tag{27}$$

4. Large-amplitude (non-linear) free vibrations

The large-amplitude free vibration behaviour of beams on Pasternak foundation can be obtained by solving Eq. (12), following the same procedure followed in the case of linear

Table 1
Values of ω_{NL}/ω_L of simply supported beam on Pasternak foundation

α	λ_{F1}	λ_{F2}				
		0.0	0.5	1.0	1.5	2.0
0.5	0.0	1.0231	1.0155	1.0116	1.0093	1.0078
	0.5	1.0155	1.0116	1.0093	1.0078	1.0067
	1.0	1.0116	1.0093	1.0078	1.0067	1.0058
	5.0	1.0039	1.0036	1.0033	1.0031	1.0029
	10.0	1.0021	1.0020	1.0020	1.0019	1.0018
1.0	0.0	1.0892	1.0604	1.0457	1.0367	1.0307
	0.5	1.0604	1.0457	1.0367	1.0307	1.0264
	1.0	1.0457	1.0367	1.0307	1.0264	1.0231
	5.0	1.0155	1.0143	1.0133	1.0124	1.0116
	10.0	1.0085	1.0081	1.0078	1.0075	1.0072
2.0	0.0	1.3178	1.2219	1.1708	1.1389	1.1171
	0.5	1.2219	1.1708	1.1389	1.1171	1.1012
	1.0	1.1708	1.1389	1.1171	1.1012	1.0891
	5.0	1.0604	1.0559	1.0520	1.0486	1.0457
	10.0	1.0334	1.0320	1.0307	1.0295	1.0283
3.0	0.0	1.6257	1.4491	1.3519	1.2898	1.2466
	0.5	1.4491	1.3519	1.2898	1.2466	1.2147
	1.0	1.3519	1.2898	1.2466	1.2147	1.1902
	5.0	1.1309	1.1213	1.1132	1.1060	1.0997
	10.0	1.0736	1.0705	1.0677	1.0650	1.0627

vibration. The final expression for the non-linear radian frequency ω_{NL} is

$$\frac{1}{\omega_{NL}} = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\alpha_1/im}\left[1 + \{1/(1+\lambda_{F1}+\lambda_{F2})\}(\alpha^2/8)(1+\sin^2\theta)\right]^{1/2}} \tag{28}$$

for the simply supported beam and

$$\frac{1}{\omega_{NL}} = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\alpha_1/im}\left[1 + \{1/(1+\frac{3}{16}\lambda_{F1}+\frac{1}{4}\lambda_{F2})\}(\alpha^2/32)(1+\sin^2\theta)\right]^{1/2}} \tag{29}$$

for the clamped beam, where λ_{F1} is equal to λ_{F10}/π^4 and λ_{F2} is equal to λ_{F20}/π^2 and

$$\alpha = \frac{W_m}{r}. \tag{30}$$

Eqs. (28) and (29) can be integrated for different given values of λ_{F1}, λ_{F2} and α, using any standard numerical integration scheme. It may be noted here that $(\alpha_1/im)^{1/2}$ is the linear radian frequency.

Table 2
Values of ω_{NL}/ω_L of clamped beam on Pasternak foundation

α	λ_{F1}	λ_{F2}				
		0.0	0.5	1.0	1.5	2.0
0.5	0.0	1.0058	1.0052	1.0047	1.0042	1.0039
	0.5	1.0053	1.0048	1.0043	1.0040	1.0037
	1.0	1.0049	1.0044	1.0041	1.0037	1.0034
	5.0	1.0030	1.0028	1.0027	1.0025	1.0024
	10.0	1.0020	1.0019	1.0019	1.0018	1.0017
1.0	0.0	1.0231	1.0206	1.0185	1.0169	1.0155
	0.5	1.0212	1.0190	1.0173	1.0158	1.0146
	1.0	1.0195	1.0177	1.0162	1.0149	1.0138
	5.0	1.0120	1.0113	1.0106	1.0101	1.0096
	10.0	1.0081	1.0078	1.0075	1.0072	1.0069
2.0	0.0	1.0892	1.0797	1.0720	1.0657	1.0604
	0.5	1.0818	1.0738	1.0672	1.0616	1.0569
	1.0	1.0756	1.0687	1.0629	1.0580	1.0539
	5.0	1.0471	1.0443	1.0418	1.0396	1.0376
	10.0	1.0320	1.0307	1.0295	1.0284	1.0273
3.0	0.0	1.1902	1.1708	1.1550	1.1418	1.1308
	0.5	1.1753	1.1586	1.1449	1.1334	1.1236
	1.0	1.1625	1.1481	1.1361	1.1259	1.1171
	5.0	1.1028	1.0969	1.0916	1.0869	1.0826
	10.0	1.0705	1.0677	1.0650	1.0626	1.0604

5. Numerical results and discussion

Fig. 1 shows a schematic representation of the uniform beam resting on the two-parameter Pasternak foundation. Both the simply supported and clamped beams are considered in the present study. The two ends of the beam are assumed to be immovable axially. Using the formulation presented in the previous section, the values of ω_{NL}/ω_L are computed for different practical values of the two foundation parameters, λ_{F1} and λ_{F2} of the Pasternak foundation. The numerical results for the simply supported beam and clamped beam are presented in Tables 1 and 2, respectively. The following broad conclusions can be arrived at from these results:

- The effect of the two-parameter foundation, in general, is to reduce the non-linearity.
- For higher values of λ_{F1}, the effect of λ_{F2} on the frequency ratios is found to be small. However, for a particular λ_{F2}, as λ_{F1} is increased, the frequency ratios decrease considerably.
- The non-linearity involved in the case of the clamped beam is much less compared to the case of the simply supported beam irrespective of the foundation stiffness parameters.
- The non-linearity increases with the increase in amplitude ratio α, as anticipated, irrespective of the foundation stiffness parameters.

- All the above conclusions can be explained qualitatively because the non-linearity in the present study is mainly dependent on the rotation term w'.

It may be noted here that the results obtained presently for the simply supported and clamped beams without Pasternak foundation ($\lambda_{F1} = \lambda_{F2} = 0$) match very well with those of Refs. [1,5], respectively. The present results with $\lambda_{F2} = 0$ are for uniform beams on a simple Winkler foundation [6].

6. Concluding remarks

The conservation of total energy principle is used to study the large-amplitude vibrations of uniform, simply supported and clamped beams, with axially immovable ends, on Pasternak foundation. A governing temporal equation which can be integrated numerically is obtained from the energy principle. The ratios of non-linear to linear frequencies of the beams for specific practical values of the two foundation stiffness parameters of the Pasternak foundation are presented. Based on the numerical results, some broad interesting conclusions are made.

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