Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Sound and Vibration 263 (2003) 1017–1030
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

# Non-linear interactions in asymmetric vibrations of a circular plate

W.K. Lee*, M.H. Yeo

*School of Mechanical Engineering, Yeungnam University, Gyongsan 712-749, South Korea*

Accepted 4 February 2003

**Abstract**

In order to investigate the non-linear asymmetric vibrations of a clamped circular plate on an elastic foundation, the primary resonances of a plate with an internal resonance, in which the natural frequencies of two asymmetric modes are commensurable are considered. The response is expressed as an expansion in terms of the linear, free oscillation modes, and its amplitude is considered to be small but finite. The method of multiple scales is used to reduce the non-linear governing equations to a system of autonomous ordinary differential equations for amplitude and phase variables. For a numerical example the case of internal resonance (a commensurable relationship between natural frequencies), $\omega_{32} \approx 3\omega_{11}$, where the first subscript refers to the number of nodal diameters and the second subscript the number of nodal circles including boundary is considered. When the frequency of excitation is near $\omega_{11}$, there exist at most five stable steady-state responses. Four of them are superpositions of traveling wave components and one is a superposition of standing wave components. The result shows the interaction between modes corresponding to $\omega_{11}$ and $\omega_{32}$ by showing non-vanishing amplitudes of the mode not directly excited. When the frequency of excitation is near $\omega_{32}$, similarly the interaction between modes is shown to exist. All of the responses with non-vanishing amplitudes of modes excited indirectly, however, turn out to be unstable, which is a peculiar phenomenon.
© 2003 Elsevier Science Ltd. All rights reserved.

## 1. Introduction

Among many studies [1–5] dealing with non-linear modal interactions of vibrations of circular plates, the work done by Sridhar et al. [2] may be said to be general in the sense that it includes asymmetric vibrations as well as symmetric vibrations, and it includes all of the natural modes. In

*Corresponding author.
*E-mail address:* wklee@yu.ac.kr (W.K. Lee).

0022-460X/03/$ - see front matter © 2003 Elsevier Science Ltd. All rights reserved.
doi:10.1016/S0022-460X(03)00271-2

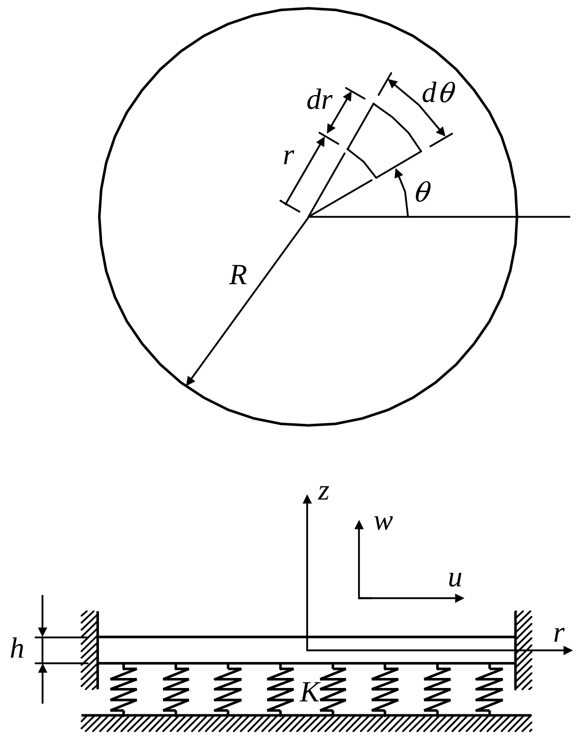


Fig. 1. A schematic diagram of a clamped circular plate on an elastic foundation.

this work they derived solvability conditions for modal interactions of clamped circular plates. Recently, Yeo and Lee [6] found that these conditions were misderived, and then corrected the conditions. They observed that in the absence of internal resonance, the steady-state response can have not only the form of a standing wave but also the form of a traveling wave. This observation is a remarkable contrast to Sridhar et al. [2], in which the steady-state response can only have the form of a standing wave.

In order to investigate modal interactions of circular plates with internal resonance, a circular plate on an elastic foundation shown in Fig. 1 is considered. In this study, the elastic foundation is considered to get a varied natural characteristic, which generates a desired commensurable relation between natural frequencies. Circular plates on an elastic foundation are also known to model some heat exchangers [7]. The dynamic analogue of von Karman equations is used to study a primary resonance of the plate. The method of multiple scales is used to reduce the non-linear governing equations to a system of autonomous ordinary differential equations for amplitude and phase variables. The equilibrium solutions of the system and their stability are examined.

## 2. Equations of motion and steady-state responses

The equations governing the free, undamped oscillations of non-uniform circular plates were derived by Efstathiades [8]. These equations are simplified to fit the special case of uniform plates, and damping and forcing terms are added. Then the non-dimensionalized equations of motion of a circular plate with an elastic foundation shown in Fig. 1 can be given as follows [2,9]:

$$\frac{\partial^2 w}{\partial t^2} + (\nabla^4 + K)w = \varepsilon\left[L(w, F) - 2c\frac{\partial w}{\partial t} + p^*(r, \theta, t)\right], \tag{1}$$

$$\nabla^4 F = \left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right)^2 - \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right), \tag{2}$$

where

$$\begin{aligned} L(w, F) = & \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) + \frac{\partial^2 F}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) \\ & - 2\left(\frac{1}{r}\frac{\partial^2 F}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial F}{\partial \theta}\right)\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right), \end{aligned} \tag{3}$$

where $\varepsilon = 12(1 - v^2)h^2/R^2$, $c$ is the damping coefficient, $p^*$ is the forcing function, $K$ is the stiffness of the foundation, $v$ is Poisson's ratio, $h$ is the thickness, $R$ is the radius, $w$ is the deflection of the middle surface, $F$ is the force function which satisfies the in-plane equilibrium conditions (in-plane inertia is neglected), and

$$\nabla^4 \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2. \tag{4}$$

The boundary conditions are developed for plates which are clamped along a circular edge. For all $t$ and $\theta$,

$$w = 0, \quad \frac{\partial w}{\partial r} = 0 \quad \text{at } r = 1, \tag{5a, b}$$

$$\frac{\partial^2 F}{\partial r^2} - v\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) = 0 \quad \text{at } r = 1, \tag{6a}$$

$$\frac{\partial^3 F}{\partial r^3} + \frac{1}{r}\frac{\partial^2 F}{\partial r^2} - \frac{1}{r^2}\frac{\partial F}{\partial r} + \frac{2 + v}{r^2}\frac{\partial^3 F}{\partial r \partial \theta^2} - \frac{3 + v}{r^3}\frac{\partial^2 F}{\partial \theta^2} = 0 \quad \text{at } r = 1. \tag{6b}$$

In addition, it is necessary to require the solution to be bounded at $r = 0$.

The forcing function $p^*$ is considered as follows:

$$p^*(r, \theta, t) = \left[\sum_{m=1}^{\infty} P_{0m}\phi_{0m} + 2\sum_{n,m=1}^{\infty} P_{nm}\phi_{nm}\cos(n\theta + \tau_{nm})\right]\cos\lambda t, \tag{7}$$

where the linear symmetric vibration modes $\phi_{nm}(r)$ corresponding to the natural frequencies $\omega_{nm}$ (see appendix). In these expressions, the first subscript $n$ refers to the number of nodal diameters and the second subscript $m$ refers to the number of nodal circles including boundary. And $\lambda$ is the excitation frequency.

To obtain the first order approximate solution of Eqs. (1)–(6), the method of multiple scales is used. $w$ and $F$ are expanded as follows:

$$w(r, \theta, t; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j w_j(r, \theta, T_0, T_1, \ldots), \tag{8}$$

$$F(r, \theta, t; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j F_j(r, \theta, T_0, T_1, \ldots), \tag{9}$$

where $T_n = \varepsilon^n t$.

Following Sridhar et al. [2], the first order solution is as follows:

$$w_0 = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \phi_{nm}(r) u_{nm}(T_0, T_1, \ldots) \mathrm{e}^{\mathrm{i}n\theta}, \quad u_{nm} = A_{nm} \mathrm{e}^{\mathrm{i}\omega_{nm} T_0} + \bar{B}_{nm} \mathrm{e}^{-\mathrm{i}\omega_{nm} T_0}, \tag{10}$$

where the $\omega_{nm}$ are linear natural frequencies, the $\phi_{nm}(r)$ are linear symmetric vibration modes (see appendix), and the responses $A_{nm}$ and $B_{nm}$ are complex functions of the all $T_k$ for $k \geqslant 1$.

For a circular plate without an elastic foundation—i.e. the case of $K = 0$, Yeo and Lee [6] had corrected the solvability conditions for the responses derived by Sridhar et al. [2]. Since the value of $K$ does not change the solvability conditions at all, Yeo and Lee [6] gives the conditions as follows:

$$\begin{aligned} & -2\mathrm{i}\omega_{kl}(\mathrm{D}_1 A_{kl} + c_{kl} A_{kl}) + A_{kl} \left\{ \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \gamma_{klnm}(A_{nm}\bar{A}_{nm} + B_{nm}\bar{B}_{nm}) - \gamma_{klkl} A_{kl} \bar{A}_{kl} \right\} \\ & \quad + 2(1 - \delta_{k0}) B_{kl} \left\{ \sum_{m=1}^{\infty} \hat{\gamma}_{klkm} A_{km} \bar{B}_{km} - \hat{\gamma}_{klkl} A_{kl} \bar{B}_{kl} \right\} + N_{kl}^A + R_{kl}^A = 0, \end{aligned} \tag{11a}$$

$$\begin{aligned} & 2\mathrm{i}\omega_{kl}(\mathrm{D}_1 \bar{B}_{kl} + c_{kl} \bar{B}_{kl}) + \bar{B}_{kl} \left\{ \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \gamma_{klnm}(A_{nm}\bar{A}_{nm} + B_{nm}\bar{B}_{nm}) - \gamma_{klkl} B_{kl} \bar{B}_{kl} \right\} \\ & \quad + 2(1 - \delta_{k0}) \bar{A}_{kl} \left\{ \sum_{m=1}^{\infty} \hat{\gamma}_{klkm} A_{km} \bar{B}_{km} - \hat{\gamma}_{klkl} A_{kl} \bar{B}_{kl} \right\} + N_{kl}^B + R_{kl}^B = 0, \end{aligned} \tag{11b}$$

where $\mathrm{D}_1 = \partial/\partial T_1$, $\delta_{k0}$ are Kronecker delta, $R_{kl}^{A,B}$ are terms due to internal resonances, if any, $N_{kl}^{A,B}$ are terms due to the external excitation, if any, and $\gamma_{klnm}$ and $\hat{\gamma}_{klkm}$ are constants given in the appendix.

In order to consider the internal resonance condition $\omega_{NM} \approx 3\omega_{CD}$ $(N = 3C)$ and the external resonance condition $\lambda \approx \omega_{GH}$ $(GH = CD$ or $NM)$, the detuning parameters, $\sigma_1$ and $\sigma_2$, are introduced as follows:

$$\omega_{NM} = 3\omega_{CD} + \varepsilon\sigma_1, \quad \lambda = \omega_{GH} + \varepsilon\sigma_2. \tag{12, 13}$$

In this case

$$R_{NM}^A = Q_{NM} A_{CD}^3 \mathrm{e}^{-\mathrm{i}\sigma_1 T_1}, \quad R_{NM}^B = Q_{NM} \bar{B}_{CD}^3 \mathrm{e}^{\mathrm{i}\sigma_1 T_1}, \tag{14a, b}$$

$$R_{CD}^A = Q_{CD} \bar{A}_{CD}^2 A_{NM} \mathrm{e}^{\mathrm{i}\sigma_1 T_1}, \quad R_{CD}^B = Q_{CD} B_{CD}^2 \bar{B}_{NM} \mathrm{e}^{-\mathrm{i}\sigma_1 T_1}, \tag{14c, d}$$

$$R_{kl}^{A,B} = 0 \quad \text{for } kl \neq CD, NM, \tag{14e}$$

$$N_{GH}^{A} = \tfrac{1}{2}P_{GH}\mathrm{e}^{\mathrm{i}(\sigma_2 T_1 + \tau_{GH})}, \quad N_{GH}^{B} = \tfrac{1}{2}P_{GH}\mathrm{e}^{-\mathrm{i}(\sigma_2 T_1 - \tau_{GH})}, \tag{15a, b}$$

$$N_{kl}^{A,B} = 0 \quad \text{for } kl \neq GH, \tag{15c}$$

where the $Q_{NM}$ and $Q_{CD}$ are constants given in the appendix. Next let

$$A_{nm} = \tfrac{1}{2}a_{nm}\mathrm{e}^{\mathrm{i}\alpha_{nm}}, \quad B_{nm} = \tfrac{1}{2}b_{nm}\mathrm{e}^{\mathrm{i}\beta_{nm}}, \tag{16a, b}$$

where the $a_{nm}$, $b_{nm}$, $\alpha_{nm}$ and $\beta_{nm}$ are real functions of $T_1$. Substituting Eqs. (14)–(16) into (11) and separating the result into real and imaginary parts, gives

$$\begin{aligned}\omega_{kl}(a'_{kl} + c_{kl}a_{kl}) &- \tfrac{1}{4}(1-\delta_{k0})b_{kl}\hat{s}^{s}_{kl} - \tfrac{1}{8}\delta_{kC}\delta_{lD}Q_{CD}a^2_{CD}a_{NM}\sin\tilde{\mu}_A \\ &+ \tfrac{1}{8}\delta_{kN}\delta_{lM}Q_{NM}a^3_{CD}\sin\tilde{\mu}_A - \tfrac{1}{2}\delta_{kG}\delta_{lH}P_{GH}\sin\mu^a_{GH} = 0,\end{aligned} \tag{17a}$$

$$\begin{aligned}\omega_{kl}(b'_{kl} + c_{kl}b_{kl}) &+ \tfrac{1}{4}(1-\delta_{k0})a_{kl}\hat{s}^{s}_{kl} - \tfrac{1}{8}\delta_{kC}\delta_{lD}Q_{CD}b^2_{CD}b_{NM}\sin\tilde{\mu}_B \\ &+ \tfrac{1}{8}\delta_{kN}\delta_{lM}Q_{NM}b^3_{CD}\sin\tilde{\mu}_B - \tfrac{1}{2}\delta_{kG}\delta_{lH}P_{GH}\sin\mu^b_{GH} = 0,\end{aligned} \tag{17b}$$

$$\begin{aligned}\omega_{kl}a_{kl}\alpha'_{kl} &+ \tfrac{1}{8}a_{kl}(s_{kl} - \gamma_{klkl}a^2_{kl}) + \tfrac{1}{4}(1-\delta_{k0})b_{kl}\hat{s}^{c}_{kl} + \tfrac{1}{8}\delta_{kC}\delta_{lD}Q_{CD}a^2_{CD}a_{NM}\cos\tilde{\mu}_A \\ &+ \tfrac{1}{8}\delta_{kN}\delta_{lM}Q_{NM}a^3_{CD}\cos\tilde{\mu}_A + \tfrac{1}{2}\delta_{kG}\delta_{lH}P_{GH}\cos\mu^a_{GH} = 0,\end{aligned} \tag{17c}$$

$$\begin{aligned}\omega_{kl}b_{kl}\beta'_{kl} &+ \tfrac{1}{8}b_{kl}(s_{kl} - \gamma_{klkl}b^2_{kl}) + \tfrac{1}{4}(1-\delta_{k0})a_{kl}\hat{s}^{c}_{kl} + \tfrac{1}{8}\delta_{kC}\delta_{lD}Q_{CD}b^2_{CD}b_{NM}\cos\tilde{\mu}_B \\ &+ \tfrac{1}{8}\delta_{kN}\delta_{lM}Q_{NM}b^3_{CD}\cos\tilde{\mu}_B + \tfrac{1}{2}\delta_{kG}\delta_{lH}P_{GH}\cos\mu^b_{GH} = 0,\end{aligned} \tag{17d}$$

where

$$s_{kl} = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\gamma_{klnm}(a^2_{nm} + b^2_{nm}), \tag{18a}$$

$$\hat{s}^{s}_{kl} = \sum_{m=1}^{\infty}\hat{\gamma}_{klkm}a_{km}b_{km}\sin(\alpha_{km} - \beta_{km} - \alpha_{kl} + \beta_{kl}), \tag{18b}$$

$$\hat{s}^{c}_{kl} = \sum_{m=1}^{\infty}(1-\delta_{ml})\hat{\gamma}_{klkm}a_{km}b_{km}\cos(\alpha_{km} - \beta_{km} - \alpha_{kl} + \beta_{kl}), \tag{18c}$$

$$\mu^a_{GH} = \sigma_2 T_1 + \tau_{GH} - \alpha_{GH}, \qquad \mu^b_{GH} = \sigma_2 T_1 - \tau_{GH} - \beta_{GH}, \tag{19a, b}$$

$$\tilde{\mu}_A = \sigma_1 T_1 - 3\alpha_{CD} + \alpha_{NM}, \qquad \tilde{\mu}_B = \sigma_1 T_1 - 3\beta_{CD} + \beta_{NM}. \tag{19c, d}$$

Each equilibrium solution of the system of autonomous ordinary differential equations to be obtained from system (17) corresponds to a steady-state response. The steady-state response to the first order approximation is given as follows:

$$w = w_{CD} + w_{NM} + O(\varepsilon), \tag{20}$$

where

$$
\begin{aligned}
w_{CD} = {} & \delta_{GC}\delta_{HD}\phi_{CD}\{a_{CD}\cos(\lambda t - \mu^a_{CD} + C\theta + \tau_{CD}) \\
& + b_{CD}\cos(\lambda t - \mu^b_{CD} - C\theta - \tau_{CD})\} \\
& + \delta_{GN}\delta_{HM}\phi_{CD}\left\{a_{CD}\cos\left(\frac{\lambda}{3}t - \frac{\mu^a_{NM}}{3} - \frac{\tilde{\mu}_A}{3} + C\theta + \frac{\tau_{NM}}{3}\right)\right. \\
& \left. + b_{CD}\cos\left(\frac{\lambda}{3}t - \frac{\mu^b_{NM}}{3} - \frac{\tilde{\mu}_B}{3} - C\theta - \frac{\tau_{NM}}{3}\right)\right\},
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
w_{NM} = {} & \delta_{GC}\delta_{HD}\phi_{NM}\{a_{NM}\cos(3\lambda t - 3\mu^a_{CD} + \tilde{\mu}_A + N\theta + 3\tau_{CD}) \\
& + b_{NM}\cos(3\lambda t - 3\mu^b_{CD} + \tilde{\mu}_B - N\theta - 3\tau_{CD})\} \\
& + \delta_{GN}\delta_{HM}\phi_{NM}\{a_{NM}\cos(\lambda t - \mu^a_{NM} + N\theta + \tau_{NM}) \\
& + b_{NM}\cos(\lambda t - \mu^b_{NM} - N\theta - \tau_{NM})\}.
\end{aligned} \tag{22}
$$

Each of the $w_{CD}$ and $w_{NM}$ is the superposition of two traveling wave components. If $a_{CD} = b_{CD}$, one obtains $a_{NM} = b_{NM}$, $\mu^a_{CD} = \mu^b_{CD}$, $\mu^a_{NM} = \mu^b_{NM}$ and $\tilde{\mu}_A = \tilde{\mu}_B$. Then Eqs. (21) and (22) can be reduced as follows:

$$
\begin{aligned}
w_{CD} = {} & 2\delta_{GC}\delta_{HD}\phi_{CD}a_{CD}\cos(\lambda t - \mu^a_{CD})\cos(C\theta + \tau_{CD}) \\
& + 2\delta_{GN}\delta_{HM}\phi_{CD}a_{CD}\cos\left(\frac{\lambda}{3}t - \frac{\mu^a_{NM}}{3} - \frac{\tilde{\mu}_A}{3}\right)\cos\left(C\theta + \frac{\tau_{NM}}{3}\right),
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
w_{NM} = {} & 2\delta_{GC}\delta_{HD}\phi_{NM}a_{NM}\cos(3\lambda t - 3\mu^a_{CD} + \tilde{\mu}_A)\cos(N\theta + 3\tau_{CD}) \\
& + 2\delta_{GN}\delta_{HM}\phi_{NM}a_{NM}\cos(\lambda t - \mu^a_{NM})\cos(N\theta + \tau_{NM}).
\end{aligned} \tag{24}
$$

Now each of the $w_{CD}$ and $w_{NM}$ becomes a superposition of two standing wave components.

## 3. Numerical example

For a numerical example the case of $K = 1032$, which gives natural frequencies $\omega_{11} = 38.52$ and $\omega_{32} = 115.58$ is considered. Then there is an internal resonance condition $\omega_{32} \approx 3\omega_{11}$ and an internal detuning parameter $\varepsilon\sigma_1 = 0.007412$. Pursuing the internal resonance condition $\omega_{NM} \approx 3\omega_{CD}$ $(N = 3C)$, gives the relations $C = 1$, $D = 1$, $N = 3$ and $M = 2$. Consider two primary resonance cases, $\lambda \approx \omega_{11}$ $(G = 1$ and $H = 1)$ and $\lambda \approx \omega_{32}$ $(G = 3$ and $H = 2)$. In Figs. 2–7 the amplitudes $a_{11}$, $b_{11}$, $a_{32}$ and $b_{32}$ are plotted as functions of detuning parameter $\varepsilon\sigma_2 = \hat{\sigma}_2$ when $\{\nu, \varepsilon, \varepsilon c, \tau_{11}, \tau_{32}\} = \{\frac{1}{3}, 0.001067, 0.01, 0.0, 0.0\}$. Solid and dotted lines denote, respectively, stable and unstable responses. The abbreviations SS, US, ST and UT denote, respectively, stable standing, unstable standing, stable traveling and unstable traveling wave components. Numerical results were verified by using a software package AUTO [10], which can perform bifurcation analysis and continuation of solutions for ordinary differential equations.

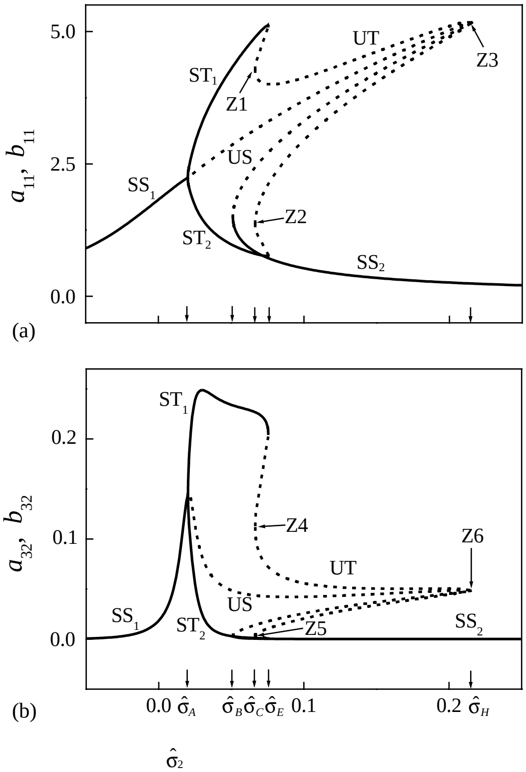


Fig. 2. Variations of the amplitudes with detuning parameter $\hat{\sigma}_2$ when $\lambda \approx \omega_{11}$ and $\varepsilon P_{11} = 4$. ——, stable; - - -, unstable.

In the case of $\lambda \approx \omega_{11}$ ($G = 1$ and $H = 1$), Fig. 2(a) and its partial enlargements, Fig. 3 show that the response curve corresponding to standing waves is similar to the response curve of the Duffing oscillator, except that the upper branch changes its stability at pitchfork bifurcation points, $\hat{\sigma}_A(0.0219)$ and $\hat{\sigma}_G(0.2144)$. Fig. 2(b) and its partial enlargements, Fig. 4 shows that the mode corresponding to $\omega_{32}$ is excited indirectly through the non-linear interaction. If there were no non-linear interaction, $a_{32}$ and $b_{32}$ would be zero. Figs. 3 and 4 show that traveling wave components change their stability at Hopf bifurcation points, $\hat{\sigma}_D(0.0678)$ and $\hat{\sigma}_F(0.2099)$. These figures show that the response curves have four saddle-node bifurcation points, $\hat{\sigma}_B(0.0512)$, $\hat{\sigma}_C(0.0665)$, $\hat{\sigma}_E(0.0755)$ and $\hat{\sigma}_H(0.2162)$. When $\hat{\sigma}_C < \hat{\sigma}_2 < \hat{\sigma}_D$, there exist five stable steady-state responses. Those are from $SS_2$, $ST_1$, $ST_2$, $ST_3$ and $ST_4$. Since the overall deflection of the plate is a superposition of two wave components, respectively, due to modes excited directly ($\omega_{11}$) and indirectly ($\omega_{23}$), it will be one of five superpositions (one superposition of standing wave components and four superpositions of traveling wave components). The initial condition determines which deflection is to be realized.

In the case of $\lambda \approx \omega_{32}$ ($G = 3$ and $H = 2$), Fig. 5 and its partial enlargements, Fig. 6 show responses of the directly excited mode, with response of $a_{11} = 0$ and $b_{11} = 0$, which means no interaction between two modes. The responses are similar with those [6] in the absence of internal resonance. The enlargements were not plotted in the previous work [6], though. The response

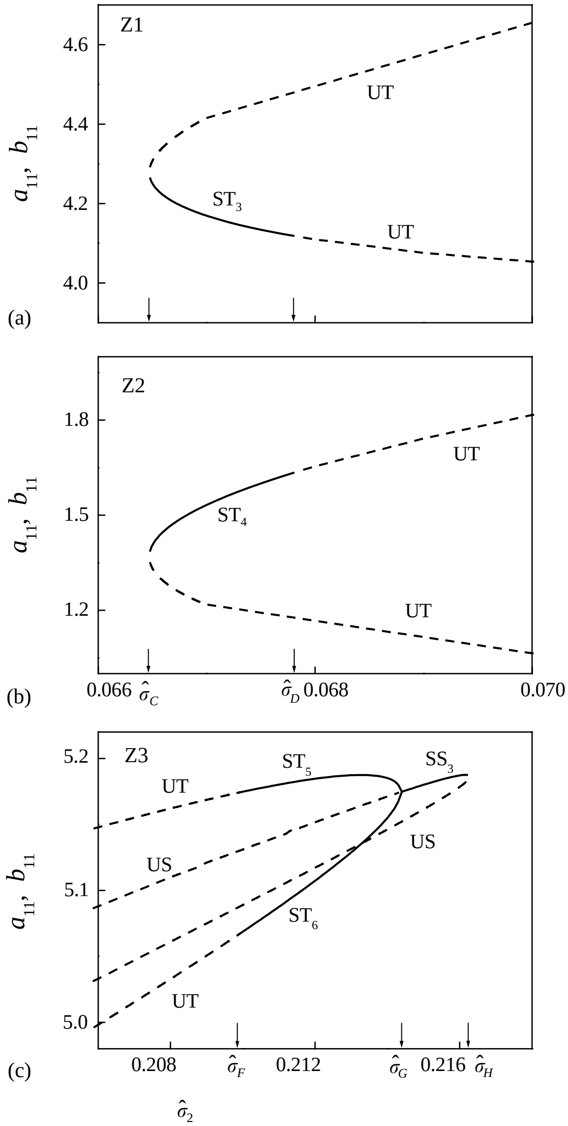


Fig. 3. Variations of the amplitudes with detuning parameter $\hat{\sigma}_2$ when $\lambda \approx \omega_{11}$ and $\varepsilon P_{11} = 4$. ——, stable; - - -, unstable. Enlargements of the Z1, Z2 and Z3 in Fig. 2(a).

curves in Figs. 5 and 6 are analogous to the response curves in Figs. 2(a) and 3. Fig. 7 shows that there exist additional steady-state responses, all of which turn out to be unstable. In other words, no stable response with non-vanishing amplitudes of a mode excited indirectly is found. It is believed that modal interaction via unstable responses is a peculiar phenomenon. Non-existence of stable steady-state responses may imply the existence of quasi-periodic response or chaos. Exploring the entity of unstable responses in Fig. 7, however, is beyond the scope of this work.

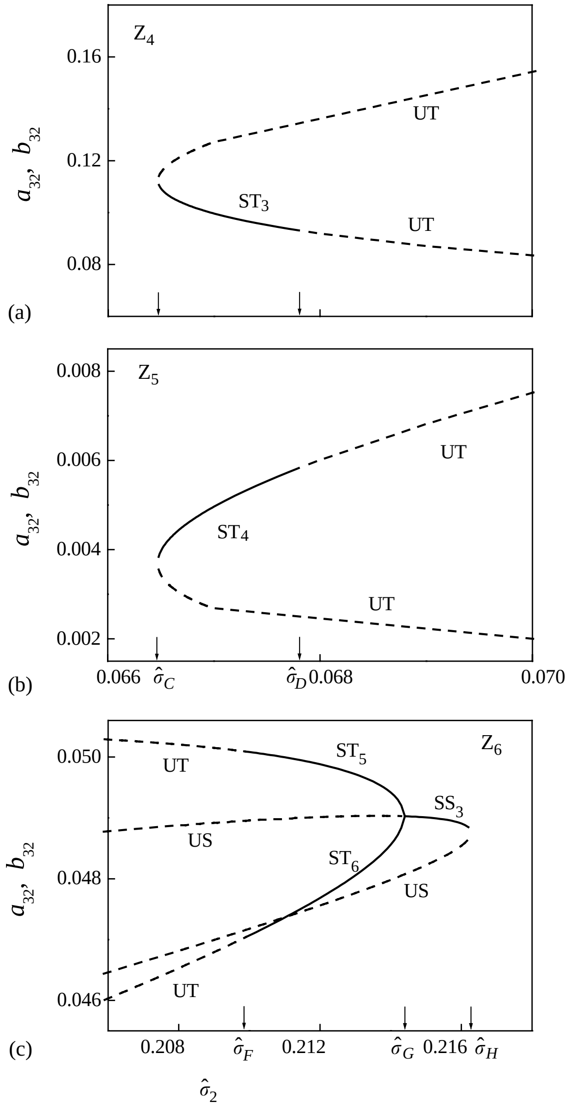


Fig. 4. Variations of the amplitudes with detuning parameter $\hat{\sigma}_2$ when $\lambda \approx \omega_{11}$ and $\varepsilon P_{11} = 4$. ——, stable; - - -, unstable. Enlargements of the Z4, Z5 and Z6 in Fig. 2(b).

## 4. Conclusions

An analysis for non-linear interaction of asymmetric vibrations of a circular plate on an elastic foundation has been presented. Two primary resonance cases with internal resonance condition $\omega_{32} \approx 3\omega_{11}$ are considered.

When the lower mode is excited ($\lambda \approx \omega_{11}$), there exists one type of response. It is the type of response with $a_{32} \neq 0$ and $b_{32} \neq 0$, meaning interaction between two modes. Among at most five such stable steady-state responses, one is a superposition of standing wave components and four of them are superpositions of traveling wave components.

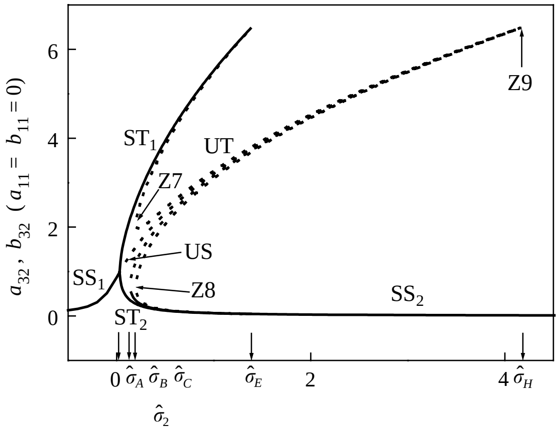


Fig. 5. Variations of the amplitudes ($a_{11} = 0$ and $b_{11} = 0$) with detuning parameter $\hat{\sigma}_2$ when $\lambda \approx \omega_{32}$ and $\varepsilon P_{32} = 15$. ——, stable; - - -, unstable.

When the higher mode is excited ($\lambda \approx \omega_{32}$), there exist two types of response. One is the type of response with $a_{11} = 0$ and $b_{11} = 0$, meaning no interaction between two modes. Among at most five such stable steady-state responses, one is a standing wave and four of them are traveling waves. The other is the type of response with $a_{11} \neq 0$ or $b_{11} \neq 0$, meaning interaction between two modes. All of these responses with non-vanishing amplitudes of mode excited indirectly, however, turn out to be unstable, which is a peculiar phenomenon.

## Acknowledgements

This work was supported by the Korea Science and Engineering Foundation under grant No. R01-2000-000-00289-0.

## Appendix

### A.1. Eq. (7)

The linear symmetric vibration modes $\phi_{nm}(r)$ corresponding to the natural frequencies $\omega_{nm}$ are given by

$$\phi_{nm} = \kappa_{nm}\left[\mathrm{J}_n(\eta_{nm}r) - \frac{\mathrm{J}_n(\eta_{nm})}{\mathrm{I}_n(\eta_{nm})}\mathrm{I}_n(\eta_{nm}r)\right]. \tag{A.1}$$

The $\kappa_{nm}$ are chosen so that

$$\int_0^1 r\phi_{nm}^2\,\mathrm{d}r = 1. \tag{A.2}$$

The function $\mathrm{J}_n$ are Bessel function of the first kind of order $n$ and the function $\mathrm{I}_n$ are modified Bessel function of the first kind of order $n$. The $\eta_{nm}$ are the roots of $\mathrm{I}_n(\eta)\mathrm{J}_n'(\eta) - \mathrm{I}_n'(\eta)\mathrm{J}_n(\eta) = 0$.

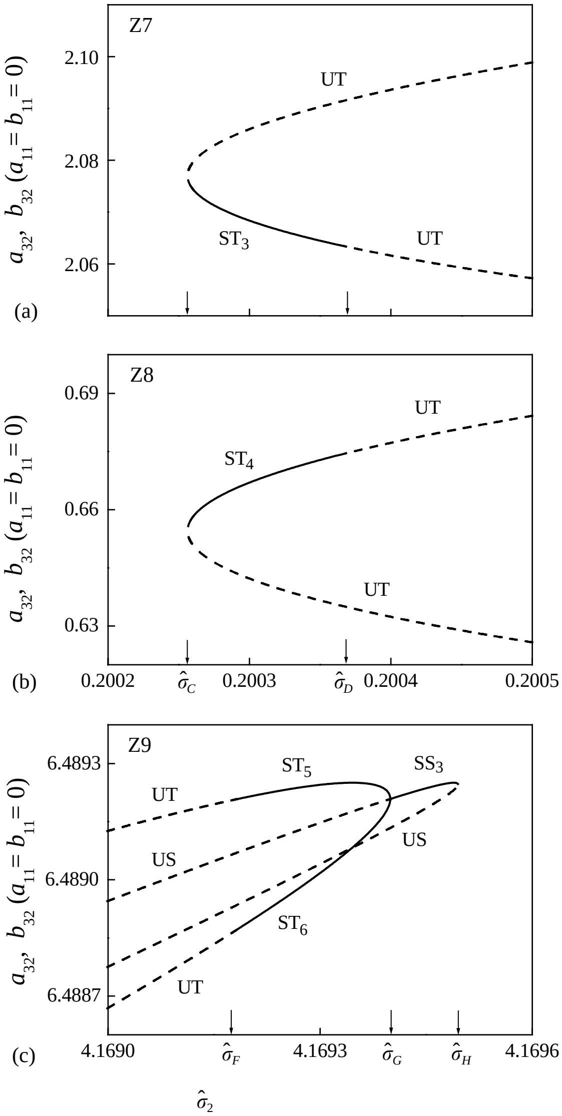


Fig. 6. Variations of the amplitudes ($a_{11} = 0$ and $b_{11} = 0$) with detuning parameter $\hat{\sigma}_2$ when $\lambda \approx \omega_{32}$ and $\varepsilon P_{32} = 15$. ——, stable; - - -, unstable. Enlargements of the Z7, Z8 and Z9 in Fig. 5.

The natural frequencies $\omega_{nm}$ are related to the eigenvalues $\eta_{nm}$ by the equation $\omega_{nm}^2 = \eta_{nm}^4 + K$. $\phi_{-nm} = \phi_{nm}$, $\omega_{-nm} = \omega_{nm}$ and $A_{-nm} = B_{nm}$.

*A.2. Eq. (11):*

$$\gamma_{klnm} = \Gamma(kl, kl, nm, -nm) + \Gamma(kl, -nm, kl, nm) + \Gamma(kl, nm, -nm, kl), \tag{A.3}$$

$$\hat{\gamma}_{klkm} = \Gamma(kl, km, km, -kl) + \Gamma(kl, -kl, km, km) + \Gamma(kl, km, -kl, km), \tag{A.4}$$

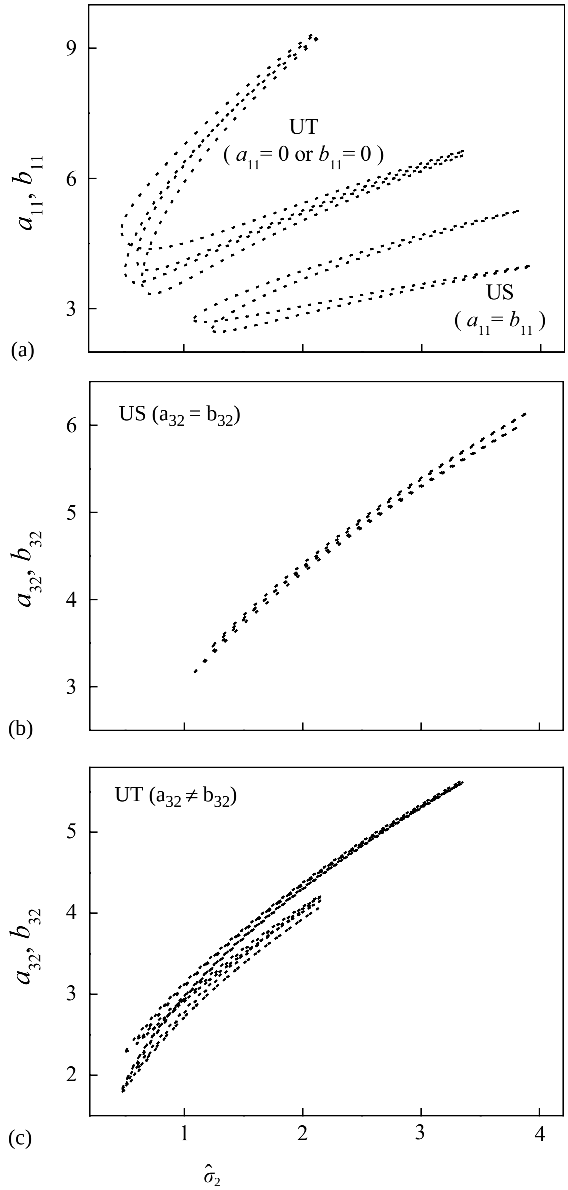


Fig. 7. Variations of the amplitudes ($a_{11} \neq 0$ or $b_{11} \neq 0$) with detuning parameter $\hat{\sigma}_2$ when $\lambda \approx \omega_{32}$ and $\varepsilon P_{32} = 15$. ——, stable; - - -, unstable.

where

$$\Gamma(kl, cd, nm, pq) = \sum_{b=1}^{\infty} G(nm, pq; ab)\hat{G}(cd, ab; kl), \tag{A.5}$$

$$a = k - c, \qquad p = k - c - n,$$

$$G(nm, pq; ab) = \xi_{ab}^{-4} \int_0^1 r\psi_{ab} E(nm, pq)\mathrm{d}r, \tag{A.6}$$

$$\hat{G}(cd, ab; kl) = \int_0^1 r\phi_{kl} \widehat{E}(cd, ab)\,\mathrm{d}r, \tag{A.7}$$

$$\begin{aligned} E(nm, pq) =& \frac{-np}{r^2}\left(\phi'_{nm} - \frac{\phi_{nm}}{r}\right)\left(\phi'_{pq} - \frac{\phi_{pq}}{r}\right) - \frac{1}{2r}(\phi'_{nm}\phi'_{pq})' \\ &+ \frac{1}{2r^2}(p^2\phi''_{nm}\phi_{pq} + n^2\phi''_{pq}\phi_{nm}), \end{aligned} \tag{A.8}$$

$$\begin{aligned} \widehat{E}(cd, ab) =& \frac{\phi''_{cd}}{r}\left(\psi'_{ab} - \frac{a^2}{r}\psi_{ab}\right) + \frac{\psi''_{ab}}{r}\left(\phi'_{cd} - \frac{c^2}{r}\phi_{cd}\right) \\ &+ \frac{2ac}{r^2}\left(\psi'_{ab} - \frac{1}{r}\psi_{ab}\right)\left(\phi'_{cd} - \frac{1}{r}\phi_{cd}\right) \end{aligned} \tag{A.9}$$

and

$$\psi_{ab} = \tilde{\kappa}_{ab}[\mathrm{J}_a(\xi_{ab}r) - \tilde{c}_{ab}\mathrm{I}_a(\xi_{ab}r)]. \tag{A.10}$$

The $\tilde{\kappa}_{ab}$ are chosen so that

$$\int_0^1 r\psi_{ab}^2\,\mathrm{d}r = 1, \tag{A.11}$$

$$\tilde{c}_{ab} = \frac{[a(a+1)(v+1) - \xi_{ab}^2]\mathrm{J}_a(\xi_{ab}) - \xi_{ab}(v+1)\mathrm{J}_{a-1}(\xi_{ab})}{[a(a+1)(v+1) + \xi_{ab}^2]\mathrm{I}_a(\xi_{ab}) - \xi_{ab}(v+1)\mathrm{I}_{a-1}(\xi_{ab})} \tag{A.12}$$

and the $\xi_{ab}$ are the roots of

$$\begin{aligned} &a^2(a+1)(v+1)[\mathrm{J}_a(\xi_{ab}) - \tilde{c}_{ab}\mathrm{I}_a(\xi_{ab})] - a^2\xi_{ab}(v+1)[\mathrm{J}_{a-1}(\xi_{ab}) - \tilde{c}_{ab}\mathrm{I}_{a-1}(\xi_{ab})] \\ &\quad + a\xi_{ab}^2[\mathrm{J}_a(\xi_{ab}) + \tilde{c}_{ab}\mathrm{I}_a(\xi_{ab})] - \xi_{ab}^3[\mathrm{J}_{a-1}(\xi_{ab}) + \tilde{c}_{ab}\mathrm{I}_{a-1}(\xi_{ab})] = 0. \end{aligned} \tag{A.13}$$

*A.3. Eq. (14):*

$$Q_{NM} = \Gamma(NM, CD, CD, CD), \tag{A.14}$$

$$\begin{aligned} Q_{CD} =& \Gamma(CD, -CD, -CD, NM) + \Gamma(CD, -CD, NM, -CD) \\ &+ \Gamma(CD, NM, -CD, -CD). \end{aligned} \tag{A.15}$$

## References


[1] S. Sridhar, D.T. Mook, A.H. Nayfeh, Non-linear resonances in the forced responses of plates. Part I: Symmetric responses of circular plates, Journal of Sound and Vibration 41 (1975) 359–373.

[2] S. Sridhar, D.T. Mook, A.H. Nayfeh, Non-linear resonances in the forced responses of plates. Part II: Asymmetric responses of circular plates, Journal of Sound and Vibration 59 (1978) 159–170.

[3] J. Hadian, A.H. Nayfeh, Modal interaction in circular plates, Journal of Sound and Vibration 142 (1990) 279–292.

[4] T.A. Nayfeh, A.F. Vakakis, Subharmonic traveling waves in a geometrically non-linear circular plate, International Journal of Non-linear Mechanics 29 (1994) 233–245.

[5] W.K. Lee, C.H. Kim, Combination resonances of a circular plate with three-mode interaction, American Society of Mechanical Engineers Journal of Applied Mechanics 62 (1995) 1015–1022.

[6] M.H. Yeo, W.K. Lee, Corrected solvability conditions for non-linear asymmetric vibrations of a circular plate, Journal of Sound and Vibration 257 (2002) 653–665.

[7] A.K. Ghosh, Axisymmetric dynamic response of a circular plate on an elastic foundation, Journal of Sound and Vibration 205 (1997) 112–120.

[8] G.J. Efstathiades, A new approach to the large-deflection vibrations of imperfect circular disks using Galerkin's procedure, Journal of Sound and Vibration 16 (1971) 231–253.

[9] A.W. Leissa, NASA-SP-160, Vibration of plates, 1969.

[10] E. Doedel, AUTO: Software for Continuation and Bifurcation Problems in Ordinary Differential Equations, California Institute of Technology, Pasadena, 1986.