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Letter to the Editor

Fundamental frequencies of circular plates with internal elastic ring support

C.Y. Wang[a,*], C.M. Wang[b]

[a] Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA
[b] Department of Civil Engineering, National University of Singapore, Singapore

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1. Introduction

The vibration of circular plates is basic in structural design. Literature on the frequency of circular plates with various edge conditions has been reviewed (e.g., Refs. [1–3]). Papers have also been written on circular plates with rigid internal concentric ring supports which are added to stabilize or to increase the fundamental frequency. However, in many cases the supports are not entirely rigid, such as those made of rubber. The following concentrates on thin (Kirchhoff) circular plates with internal, concentric, elastic ring supports.

Kunukkasseril and Swamidas [4] are probably the first to consider elastic ring supports. They formulated the equations in general, but presented only the case of a circular plate with a free edge. As in Bodine [5] who studied rigid supports, a change of the fundamental mode from symmetric to asymmetric was noted in certain cases where the radius of the support is small. Later authors [6–9] tend to study only the symmetric modes.

The purpose of the present paper is to delineate the fundamental frequency (below which no vibration could occur) of circular plates with an internal elastic ring support. In particular, the interest is in the effect of the stiffness on the mode change, and the optimum location of the elastic ring. All four basic edge conditions (clamped, simply-supported, free and sliding) will be considered. The solutions are also exact in the sense that the frequencies can be obtained from a closed form characteristic equation.

2. Formulation

The general solution to the classical plate vibration equations in polar co-ordinates can be expressed as $w = u(r)\cos(n\theta)e^{i\Omega t}$, where w is the transverse displacement, n is the number of nodal

*Corresponding author. Fax: +517-432-1562.
E-mail address: cywang@mth.msu.edu (C.Y. Wang).

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diameters, Ω is the frequency, and u is a linear combination of the Bessel functions $J_n(kr), Y_n(kr), I_n(kr), K_n(kr)$ and $k = R(\rho\Omega^2/D)^{1/4}$ where R is the plate radius, ρ is the density, D is the flexural rigidity, and k is the square root of the non-dimensional frequency [2]. Let the internal elastic ring support be at the normalized radius $r = b$. Let the subscript I denote the outer region $b \leqslant r \leqslant 1$ and the subscript II denote the inner region $0 \leqslant r \leqslant b$. Considering the boundedness at the origin, the general solutions for the two regions are

$$u_I(r) = C_1 J_n(kr) + C_2 Y_n(kr) + C_3 I_n(kr) + C_4 K_n(kr), \tag{1}$$

$$u_{II}(r) = C_5 J_n(kr) + C_6 I_n(kr). \tag{2}$$

The boundary conditions are, if the outer edge is clamped

$$u_I(1) = 0, \quad u_I'(1) = 0, \tag{3a}$$

if the outer edge is simply-supported,

$$u_I(1) = 0, \quad u_I''(1) + \nu u_I'(1) = 0, \tag{3b}$$

if the outer edge is free,

$$\begin{gathered} u_I''(1) + \nu[u_I'(1) - n^2 u_I(1)] = 0, \\ u_I'''(1) - u_I'(1)[1 + \nu + n^2(2 - \nu)] + 3n^2 u_I(1) = 0, \end{gathered} \tag{3c}$$

and if the outer edge can slide vertically without rotation,

$$u_I'(1) = 0, \quad u_I'''(1) + u_I''(1) + n^2(3 - \nu)u_I(1) = 0, \tag{3d}$$

where ν is the Poisson ratio.

Except for shear, the plate is continuous in terms of displacement, slope and moment at $r = b$. The matching conditions, after some simplifications, are

$$u_I(b) = u_{II}(b), \tag{4}$$

$$u_I'(b) = u_{II}'(b), \tag{5}$$

$$u_I''(b) = u_{II}''(b), \tag{6}$$

$$u_I'''(b) = u_{II}'''(b) - \gamma u_{II}(b). \tag{7}$$

Here $\gamma = cR^3/D$ is the normalized spring constant c of the elastic ring.

Eqs. (1) and (2) are then substituted into Eqs. (3)–(7). For non-trivial solutions of the displacement u, an exact non-linear characteristic equation for the frequency k is obtained. This equation can be solved to any accuracy by a simple bisection algorithm. In all the computations, the Poisson ratio is taken to be 0.3.

3. Results

Fig. 1 shows the constant frequency lines as a function of spring constant γ and the elastic ring radius b for the clamped circular plate. It is found that the fundamental frequency is completely governed by the axisymmetric mode, $n = 0$. When b is zero or one, the fundamental frequency is

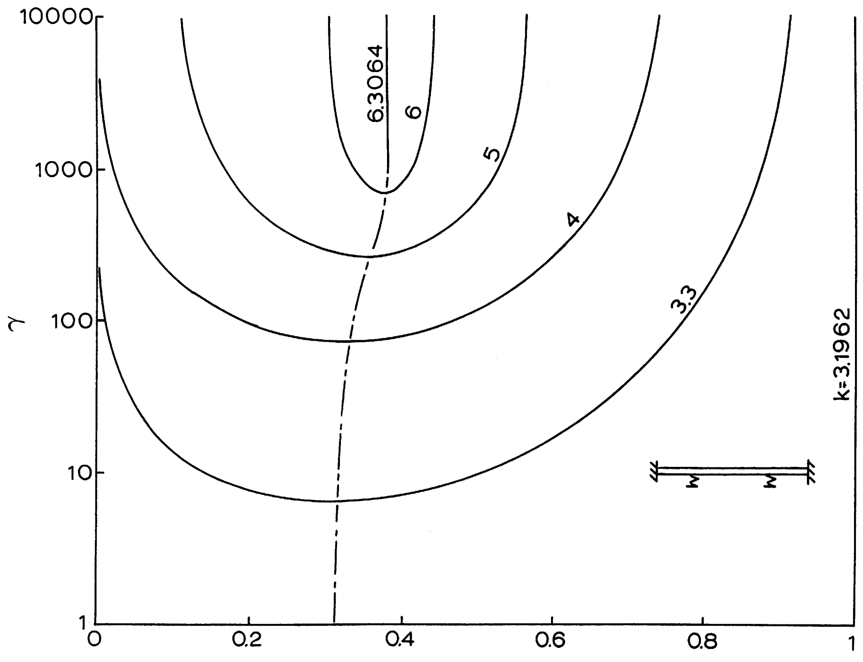

Fig. 1. Fundamental frequencies for the clamped circular plate with elastic ring support. Dash–dot line is the locus of optimum locations.

Table 1
Comparison of results for the clamped plate

γ	$b = 0.2$	$b = 0.4$	$b = 0.6$	$b = 0.8$
1000	5.1757	6.1108	4.5033	3.5392
	5.187 (A)	6.129 (A)	4.512 (A)	3.547 (A)
	4.929 (D)	6.114 (D)	4.492 (D)	3.547 (D)
10	3.3256	3.3384	3.2622	3.2041
	3.326 (A)	3.338 (A)	3.262 (A)	3.199 (A)
	3.322 (D)	3.334 (D)	3.262 (D)	3.204 (D)

(A) and (D) denote approximate results from Azimi [7] and Ding [8], respectively.

3.19623 which is the first root of

$$J_0(k)I_1(k) + I_0(k)J_1(k) = 0, \quad (8)$$

and is the same as the clamped circular plate without the ring support. The $b = 1$ case is obvious, since the clamped edge dominates the elastic support. The $b = 0$ case is not equivalent to a plate with a central spring support. This is because the spring constant γ for the elastic ring is defined as per arc length, such that the product $b\gamma$ tends to zero as $b \to 0$ for all finite γ. Comparisons of the present exact results and those of Azimi [7] and Ding [8] who used different series expansions are shown in Table 1.

For given γ, there is an optimum location where the fundamental frequency is maximized. These are listed in Table 2.

Fig. 2 shows the case when the edges are simply supported. Similar to the clamped case, the fundamental frequency is given by the axisymmetric mode. When b is zero or one, the frequency is 2.22152 which is the first zero of the simply supported plate from the characteristic equation

$$\frac{J_1(k)}{J_0(k)} + \frac{I_1(k)}{I_0(k)} = \frac{2k}{(1-\nu)}. \tag{9}$$

In this case the results of Azimi [7] are fairly close to the present exact results, especially for low spring constants (Table 3).

The optimum locations are given in Table 4.

Table 2
Optimum location b and the corresponding fundamental frequency k for the clamped plate

γ	1	10	100	1000	10000
b	0.307	0.319	0.333	0.379	0.379
k	3.213	3.349	4.199	6.3064	6.3064

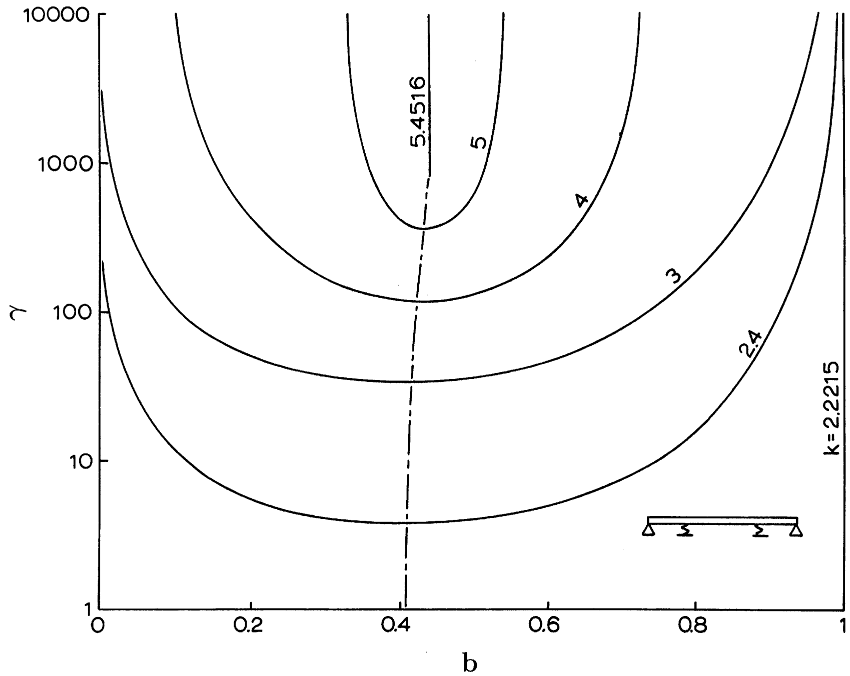

Fig. 2. Fundamental frequencies for the simply supported plate with elastic ring support. Dash–dot line is the locus of optimum locations.

Table 3
Comparison of results with those of Azimi [7], simply supported edge

γ	$b = 0.2$	$b = 0.4$	$b = 0.6$	$b = 0.8$
1000	4.2025	5.2764	4.4799	3.5323
	4.210 (A)	5.282 (A)	4.486 (A)	3.537 (A)
10	2.4607	2.5472	2.4788	2.3211
	2.461 (A)	2.547 (A)	2.479 (A)	2.321 (A)

Table 4
Optimum locations of the support for the simply supported plate

γ	1	10	100	1000	10000
b	0.405	0.408	0.421	0.442	0.442
k	2.2611	2.5473	3.7354	5.4516	5.4516

The case of a free circular plate with an elastic ring support is more complicated, as shown in Fig. 3. It can be seen that, depending on the values of γ and b, the fundamental frequency may correspond to either the axisymmetric mode ($n = 0$) or the asymmetric mode ($n = 1$). For rigid supports ($\gamma = \infty$) it has been shown [10] that the axisymmetric mode yields the fundamental frequency if $b > 0.211$ and the $n = 1$ asymmetric mode determines the fundamental frequency if $b < 0.211$. When the supports are elastic, the domain for the asymmetric mode is greatly expanded. Thus the axisymmetric frequencies (e.g., Ref. [7]) no longer represent the fundamental frequency except when b is closer to unity. The boundary for the fundamental mode switch is shown as the dotted line in Fig. 3. Kunukkasseril and Swamidas [4] found that this boundary is at $b = 0.46$ for the particular case of $k = 170$, as compared to the present exact result of $b = 0.470$. When b is zero, the frequency is also zero, which is due to an asymmetric tilt of the plate along a diameter. The asymmetric mode greatly reduces the fundamental frequency.

Of interest is the optimum location given in Table 5.

Notice that the optimum location is at the edge ($b = 1$) for $\gamma \leqslant 4.997$ and becomes constant at 0.6799 for $\gamma > 64.35$.

Lastly, the case of a plate with a sliding edge, which has not been considered before, is studied. For the special case of rigid internal support, replace Eq. (7) by $u_{II}(b) = 0$ and the characteristic equation yields the fundamental frequencies listed in Table 6.

When b becomes less than 0.0816, the mode changes from $n = 0$ to 1. The frequency decreases to zero as b becomes zero, agreeing with McCleod and Bishop [1] who studied a non-supported plate. The complex results when the support is elastic are shown in Fig. 4. It can be seen that there is a pocket of asymmetric ($n = 1$) fundamental mode. For example, when $\gamma = 100$ and the radius b is increased from zero, the fundamental frequency is first governed by the axisymmetric mode, then the asymmetric mode, then the axisymmetric mode again. Similarly two mode changes may occur for fixed b and varying stiffness. The optimum location for the ring support is given in Table 7.

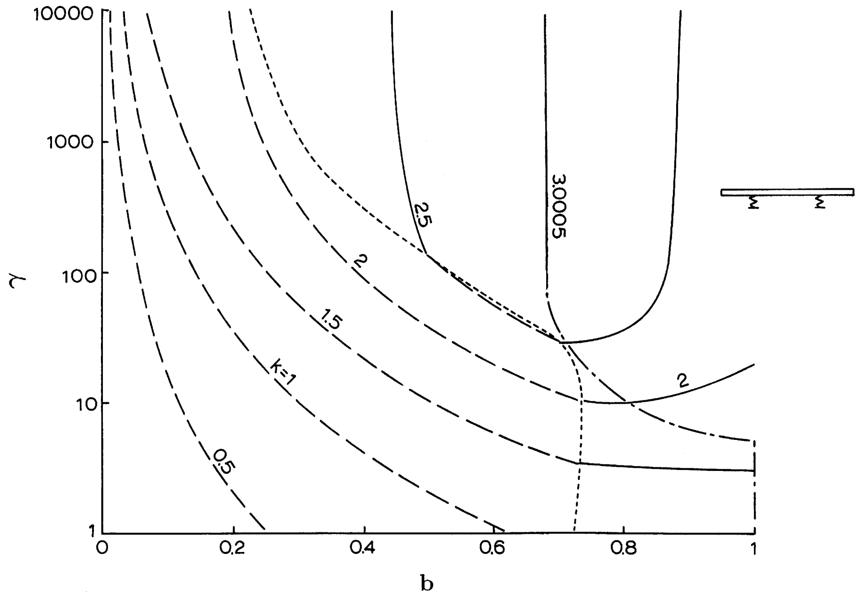

Fig. 3. Fundamental frequencies for the plate with free edge and supported by an elastic ring. Continuous lines are $n = 0$ axisymmetric mode, dashed lines are $n = 1$ asymmetric mode. The dotted line is the boundary for mode switch, and the dash–dot line is the locus of optimum locations.

Table 5
Optimum location for the elastic support of a free plate

γ	1	4.997	10	30	100	1000	10000
b	1	1	0.802	0.714	0.680	0.680	0.680
k	1.172	1.659	1.961	2.522	3.001	3.001	3.001

Table 6
Fundamental frequencies of a circular plate with movable edge and internal rigid ring support

b	0	0.001	0.0816	0.2	0.4	0.6	0.8	1
k	0	1.942	2.285	2.465	3.029	3.810	3.465	3.1692

For all these cases, one finds the existence of a minimum stiffness above which the optimum location becomes fixed and the frequency reaches a global maximum value. This phenomenon has been observed for the buckling of elastically supported columns [11], vibration of columns [12], and vibration of plates [9]. To determine the minimum stiffness accurately, the exact equations were used to find the frequency and the internal zero of the second mode for the plate without any internal support. Then the full determinant is solved for the lowest stiffness γ_{min}. The results are shown in Table 8.

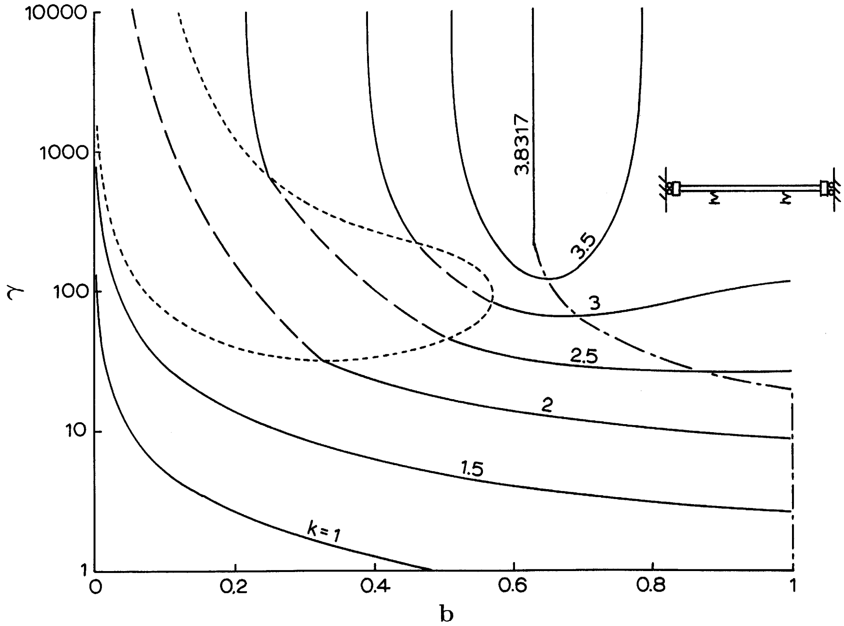

Fig. 4. Fundamental frequencies for the plate with a sliding edge and supported by an elastic ring support. Legends for the curves are same as those in Fig 3.

Table 7
Optimum locations for the ring support on a circular plate with sliding edges

γ	1	10	18.91	30	50	100	1000	10 000
b	1	1	1	0.832	0.724	0.662	0.628	0.628
k	1.186	2.059	2.356	2.549	2.842	3.306	3.832	3.832

Table 8
Minimum stiffness for maximal frequency

Edge	Clamped	Simply supported	Free	Sliding
γ_{min}	948.92	636.66	64.351	202.89
b	0.3790	0.4417	0.6799	0.6276
k	6.3064	5.4516	3.0005	3.8317

For a given amount of support material, one can fix $\Gamma = b\gamma$ and the curves in Figs. 1–4 would shift slightly while the optimum locations are unchanged. The central spring support ($b \to 0, \Gamma$ finite) can never be optimum since the maximum frequency is limited by the first asymmetric mode, where the central spring becomes ineffective. If the only choice is a central spring support, then the minimum stiffness is given in Table 9.

Table 9
Minimum stiffness for a central spring support

Edge	Clamped	Simply supported	Free	Sliding
k	4.611	3.728	0	1.756
$\Gamma_{\min}$	166.1	151.9	0	7.483

4. Conclusions

This paper is a detailed study of the fundamental frequency of the circular plate supported by an internal elastic ring. Given the edge boundary condition of the plate, the problem is governed by the interaction of two parameters: the radius and the stiffness of the supporting ring. The results show the fundamental frequency may switch between axisymmetric and asymmetric modes. Figs. 1–4 are the ‘‘frequency mosaics’’ revealed here for the first time. (The current practice of using frequency tables is inadequate in describing the complex phenomena.)

One can of course include the effects of non-negligible thickness, mass of ring support, etc. These may change the numerical values, but the basic phenomena depicted here remain unchanged. On the other hand, the present results are exact, and will serve as benchmarks for other numerical solutions.

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